Hidden symmetries and linear fields on Kerr

Lars Andersson

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joint work with Pieter Blue

Outline

Isolated systems in GR

- The Einstein equations of General Relativity

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

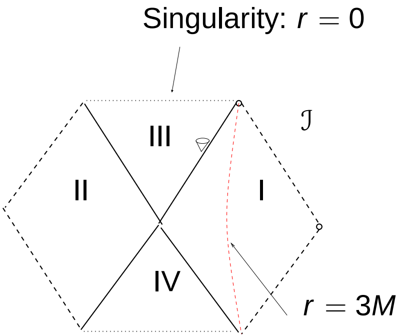
relate the Lorentzian geometry of spacetime $(\mathcal{M}, g_{\alpha\beta})$ to matter fields with energy-momentum tensor $T_{\alpha\beta}$.
  - Isolated systems in GR give an idealized picture of eg. stars, galaxies, clusters.
  - Steady states are described by geometries which admit an (asymptotically) timelike Killing field, these are *stationary spacetimes*
  - If there is a *trapped region* which is invisible to observers at infinity, the spacetime contains a *black hole*
- The Schwarzschild and Kerr spacetimes are examples of black hole spacetimes. The Schwarzschild spacetime is in addition spherically symmetric and static, while Kerr is axially symmetric.

Black hole stability

- An isolated system in GR is (expected to be) asymptotically stationary
- The Kerr solution is (expected to be) the unique stationary, asymptotically flat, vacuum spacetime.
- Kerr describes a rotating black hole
  - parameters a, M, $0 \leq |a| \leq M$. $a \leftrightarrow$ angular momentum per unit mass, setting $a = 0$ gives Schwarzschild
  - stationary (∂_t Killing)
  - axisymmetric (∂_ϕ Killing)
- To establish the astrophysical relevance of the Kerr solution, we must show it is stable — one of the main open mathematical problem concerning the Einstein equations.
- A model problem is to understand linear fields on Kerr.

Schwarzschild

$$g_{\alpha\beta}dx^\alpha dx^\beta = -fdt^2 + f^{-1}dr^2 + r^2 h_{S^2}, \quad f = 1 - \frac{2M}{r}$$



Maximally extended Schwarzschild

Schwarzschild

- For Schwarzschild, decay estimates are known for scalar waves (Blue & Sterbenz, 2006; Dafermos & Rodnianski, 2005) and Maxwell (Blue, 2008).
- Vector fields method: make use of momenta $P^\alpha(\psi, X)$ and deformation terms $\mathcal{T}_{\alpha\beta} D^\alpha X^\beta$ for cleverly chosen vector fields X (including Morawetz trapping $\mathbf{A} = \mathcal{F}\partial_r$ and conformal $\mathbf{K} = u_+^2 \partial_+ + u_-^2 \partial_-$) — need lots of symmetries
- The proofs makes use of properties of the photon sphere in Schwarzschild (codimension one), and the spherical symmetry of the spacetime.
- Get decay $|\psi| \lesssim 1/t$ in stationary regions

Kerr

$$\Delta = r^2 - 2Mr + a^2,$$
$$\Sigma = r^2 + a^2 \cos^2\theta,$$
$$\Pi = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$$

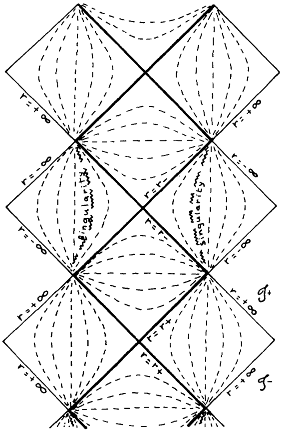
and let $r_\pm$ denote the roots of Δ,

$$r_\pm = M^2 \pm \sqrt{M^2 - a^2}$$

On the exterior region $r \geq r_+$, the Kerr metric can be written (Boyer & Lindquist, 1967)

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\Pi \sin^2\theta}{\Sigma} d\phi^2$$

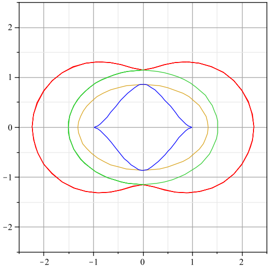
Kerr



Maximally extended Kerr

Kerr: ergoregion

Ergoregion $\Rightarrow$ no globally timelike Killing field $\Rightarrow$ no positive definite conserved energy



Kerr: Hidden symmetry

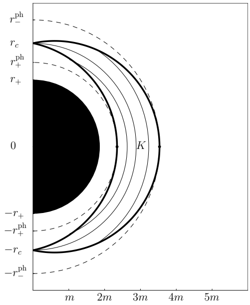
- Minkowski:
  - Killing fields: Poincaré Lie algebra $\mathfrak{so}(3,1) \ltimes \mathbf{R}^4$,
  - Conformal symmetries: dilation, **K**
- Schwarzschild: ∂_t, $\mathfrak{so}(3)$ — conserved quantities $\mathcal{E}, \mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$
- Kerr: $\partial_t, \partial_\phi$ — conserved quantities $\mathcal{E}, \mathcal{L}_z, \mathcal{Q} = Q_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta$.
- $\mathcal{Q}$ is not related to a Killing field.
- The presence of $\mathcal{Q}$ allows the geodesic equations on Kerr to be separated.

Kerr: Hidden symmetry

- $Q_{\alpha\beta}$ is a *Killing tensor*, $Q_{\alpha\beta} = Q_{(\alpha\beta)}$, $\nabla_{(\alpha} Q_{\beta\gamma)} = 0$.
- $Q_{\alpha\beta}$ is related to
  - Killing-Yano 2-form $K_{\alpha\beta} = K_{[\alpha,\beta]}$, $\nabla_{(\alpha} K_{\beta)\gamma} = 0$
  - Killing spinor K_{AB}, $\nabla_{A(A'} K_{BC)} = 0$.
- A Killing field ξ is a symmetry of the wave operator $[\mathcal{L}_\xi, \Box_g] = 0$
- Q, K are related to symmetry operators:
  - $Q = \nabla_\alpha Q^{\alpha\beta} \nabla_\beta$ with $[\Box_g, Q] = 0$,
  - $K = i\gamma_5 \gamma^\mu (K_\mu{}^\nu \nabla_\nu - \frac{1}{6} \gamma^\nu \gamma^\lambda \nabla_\lambda K_{\mu\nu})$, with $[D, K]_+ = 0$
  - The Carter constant $\mathcal{Q} = Q_{\alpha\beta} \dot\gamma^\alpha \dot\gamma^\beta$ does not correspond to a Killing field $\Rightarrow Q$ is a hidden symmetry

Kerr: photon region

Photon region $\Rightarrow$ trapping for null geodesics and waves



Location of photon orbits depend on the conserved quantities via $\mathcal{L}_z/\mathcal{E}, \mathcal{Q}/\mathcal{E}^2$

Remarks

- Kerr has a complicated photon sphere, and is only axi-symmetric
- Kerr has an ergoregion outside the horizon, where ∂_t is spacelike $\Rightarrow$ there is no positive definite conserved energy.
- Kerr has only two Killing fields $\partial_t, \partial_\phi$ but has a hidden symmetry related to the Carter constant – the geodesic equation on Kerr is separable

Each of these features form an obstacle to estimates for solutions of the wave equation on Kerr.

- Recent work using Fourier techniques provides decay estimates (Tataru & Tohaneanu, 2008; Dafermos & Rodnianski, 2008; Tataru, 2009)
- Results presented here make use of "physical space" techniques, cf. also (Dafermos & Rodnianski, 2009)

Main result

Theorem (Andersson & Blue, 2009)

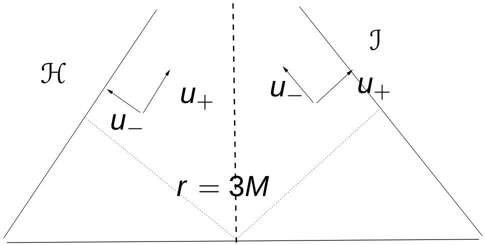
Let ψ solve $\Box_g \psi = 0$ on the exterior region $\{r \geq r_+\}$ of the Kerr black hole spacetime, with inital data $[\psi^0]$ at t_0. There is an $a_0 > 0$ such that for $|a| \in [0, a_0]$

1. *there is a norm $||[\psi^0]||_E$ on initial data and a constant c_E such that for all t, the energy bound*
$$||\psi(t)||_{H^1} + ||\dot{\psi}(t)||_{L^2} \leq c_E ||[\psi^0]||_E$$
holds on $\{r \geq r_+\}$,
2. *there is a norm $||[\psi^0]||_B$ on initial data and constants c_B, c_K such that for all t, the decay estimate*
$$||\psi(t)||_{L^\infty;loc} \leq c_B (1 + |t|)^{-1+c_K|a|} ||[\psi^0]||_B \, .$$
holds on $\{r \geq r_+\}$.

Remarks

More detailed analysis shows that we have decay at the horizon $r = r_+$ and at infinity as in Schwarzschild, with a loss as above, for example:

- Near decay, $|\psi| \lesssim (|u_+| + 1)^{-1+C_K|a|}$
- Far decay: $|\psi| \lesssim r^{-1}(|u_-| + 1)^{-1/2+C_K|a|}$ (for $r > 4M$, $t/2 < r < t$)



- (Tataru, 2009) shows t^{-3} decay for local energy – this leads to decay w/o loss (Dafermos & Rodnianski, 2009)

The Kerr wave operator

$$\Box = \Sigma \Box_g = \partial_r \Delta \partial_r + \frac{1}{\Delta}\mathcal{R}$$

where $\Box_g = \nabla^\alpha \nabla_\alpha$, and $\mathcal{R} = \mathcal{R}(r, \mathcal{E}, \mathcal{L}_z, \mathcal{Q}) = \mathcal{R}(r, \partial_t, \partial_\phi, Q)$ is

$$\mathcal{R} = -(r^2 + a^2)^2 \partial_t^2 - 4aMr\partial_t\partial_\phi + \Delta Q + (\Delta - a^2)\partial_\phi^2 \qquad (1)$$

Here Q is the (modified) Carter operator:

$$Q = \left(\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta \right) + \cot^2\theta \partial_\phi^2 + a^2 \sin^2\theta \partial_t^2$$

See from this: $\partial_t, \partial_\phi, Q$ commute with $\Box$

Canonical analysis

- Let $\mu = \sin\theta$. Then $\sqrt{-g} = \Sigma\mu$, and

$$\Box = \frac{\Sigma}{\sqrt{-g}}\partial_\alpha\left(g^{\alpha\beta}\Sigma\frac{\sqrt{-g}}{\Sigma}\partial_\beta\right) = \frac{1}{\mu}\partial_\alpha(\mathcal{G}^{\alpha\beta}\mu\partial_\beta)$$

with $\mathcal{G}^{\alpha\beta} = g^{\alpha\beta}\Sigma$

- $\Box\psi = 0$ is the E-L equation for $S = \int \mathcal{L} d\mu = \frac{1}{2}\int \mathcal{G}^{\alpha\beta}\psi_\alpha\psi_\beta d\mu$
- Canonical energy-momentum tensor:

$$\mathcal{T}^\alpha{}_\beta = \frac{\partial\mathcal{L}}{\partial(u_\alpha)}\psi_\beta - \mathcal{L}\delta^\alpha{}_\beta = \psi^\alpha\psi_\beta - \frac{1}{2}\psi^\gamma\psi_\gamma\delta^\alpha{}_\beta$$

- Momentum vector:

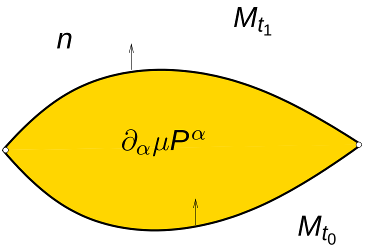
$$P^\alpha = \mathcal{T}^\alpha{}_\beta X^\alpha + q\psi^\alpha\psi - q^\alpha\psi^2/2$$

(q-terms control $\psi^\gamma\psi_\gamma$ in bulk $\partial_\alpha\mu P^\alpha$)

Canonical analysis

- Energy:

$$E(t) = -\int_{M_t} P^0 d\mu$$



- Conservation law:

$$E(t_1) - E(t_0) = -\int_{M\times[t_0,t_1]} \partial_\alpha(\mu P^\alpha)$$

- Bulk:

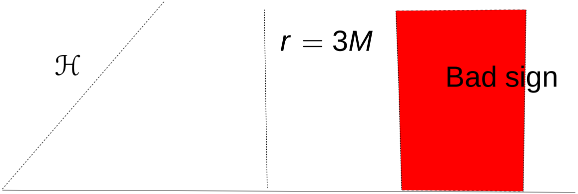
$$\frac{1}{\mu}\partial_\alpha(\mu P^\alpha) = \Box\psi\psi_\beta\xi^\beta + \mathcal{T}^\mu{}_\nu \xi^\nu{}_\mu - \frac{1}{2}\frac{1}{\mu}\xi^\gamma\partial_\gamma(\mu\mathcal{G}^{\alpha\beta})\psi_\alpha\psi_\beta$$
$$+ q\psi\Box\psi + q\psi^\alpha\psi_\alpha - \Box q\psi^2/2$$

Energy

- ∂_t fails to be timelike in ergoregion $r_+ \leq r < r_e$.
- Use a time-like "almost Killing field"

$$T_\chi = \partial_t + \chi\partial_\phi$$

where χ is a cutoff function. Want T_χ is timelike for $r > r_+$, and tangent to the horizon.



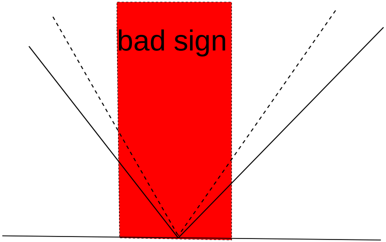
Conformal energy

- Tortoise coordinate $dx = \frac{(r^2+a^2)}{\Delta} dr, \quad x|_{r=3M} = 0$
- **K** vector field

$$\mathbf{K} = \frac{1}{2}(t^2 + x^2 + 1)T_\perp + tx\widetilde{N}^2 \partial_x$$

where $T_\perp \cong \partial_t + \omega\partial_\phi$, $\omega = \frac{2aMr}{\Pi}$, $\widetilde{N}^2 = \frac{(r^2+a^2)^2}{\Pi}$

- **K** bulk term has bad sign in a region $r_0 \leq r \leq r_1$

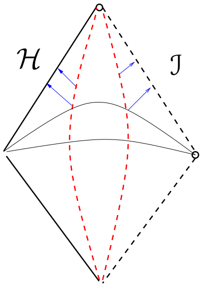


Remarks

- The bulk terms from the energy and conformal energy have regions with bad sign.
- These are controlled using an integrated energy estimate – proved by constructing a Morawetz vector field $\mathbf{A} = \mathcal{F}\partial_r$ whose bulk term has good sign.
- Cutoff, higher order Morawetz $\mathbf{A}_2 = t^2\chi(t/x)\mathbf{A}$ controls $\mathbf{K}$ bulk
- In order to construct $\mathbf{A}$ must analyze trapping of null geodesics.

Remarks

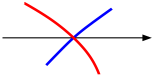
- **K**-energy controls $t^2 E_{T_\chi}$ in the interior of the light cone $|x| \leq t/2$ so bounding **K**-energy gives $1/t$ decay of local energy.
- Integrate along horizon $\mathcal{H}$ and null infinity $\mathcal{I}$ penetrating foliations to show that local decay implies global decay.



Trapping in Kerr

- $\mathcal{R}$ is the potential for null geodesics
- The photon sphere is given by

$$\mathcal{R} = 0, \quad \frac{\partial \mathcal{R}}{\partial r} = 0$$



- Bulk term for **A** must be positive. Contains terms of the form $\mathcal{F}\mathcal{R}'$, need $-\mathcal{F}\frac{\partial \mathcal{R}}{\partial r} \geq 0 \Rightarrow \mathcal{F}$ changes sign at photon orbits
- But $\mathcal{R} = \mathcal{R}(r, \mathcal{E}, \mathcal{L}_z, \mathcal{Q}) \Rightarrow$

<u>location of photon orbits depends on $\mathcal{E}, \mathcal{L}_z, \mathcal{Q}$</u>

- $\Rightarrow$ **A** must be generalized vector field

$$\mathbf{A} = \mathcal{F}(r, E, L_z, Q)\partial_r$$

Generalized momentum

- $\mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, Q\}$ — 2nd order symmetry operators.
- If $\Box\psi = 0$, then $\Box S_{\underline{a}}\psi = 0$ for $S_{\underline{a}} \in \mathbb{S}_2$
- Given $\mathcal{T}[u]^\alpha{}_\beta$, by polarization define $\mathcal{T}[\psi_1, \psi_2]^\alpha{}_\beta$.
- Generalized vector field $X^{\underline{ab}\beta} = X^{(\underline{ab})\beta}$
- Generalized momentum

$$P^\alpha[\psi, X] = \mathcal{T}^\alpha{}_\beta[S_{\underline{a}}\psi, S_{\underline{b}}\psi]X^{\underline{ab}\beta} + q^{\underline{ab}}(S_{\underline{a}}\psi)^\alpha(S_{\underline{b}}\psi) - \frac{1}{2}(q^{\underline{ab}})^\alpha(S_{\underline{a}}\psi)(S_{\underline{b}}\psi)$$

Generalized Morawetz

- $\mathbf{A} = S_{\underline{a}}\mathcal{F}^{\underline{ab}}\partial_r S_{\underline{b}}, \quad S_{\underline{a}} \in \mathbb{S}_2$
- If $\Box\psi = 0$, for suitably chosen $\mathcal{F}$, q,

$$\frac{1}{\mu}(\partial_\alpha \mu P^\alpha) = \mathcal{A}^{\underline{ab}}(S_{\underline{a}}\psi)'(S_{\underline{b}}\psi)' + \mathcal{U}^{\underline{ab}\alpha\beta}(S_{\underline{a}}\psi)_\alpha(S_{\underline{b}}\psi)_\beta + \mathcal{V}^{\underline{ab}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi)$$

where with $\mathcal{L} = \partial_t^2 + \partial_\phi^2 + Q$,

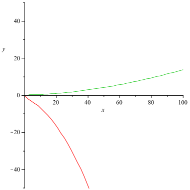
$$\mathcal{A}^{\underline{ab}} = \mathcal{A}^{(\underline{a}}\mathcal{L}^{\underline{b})}$$
$$\mathcal{U}^{\underline{bc}\alpha\beta} = \mathcal{U}^{\underline{a}(\underline{b}}\mathcal{L}^{\underline{c})}S_{\underline{a}}^{\alpha\beta}$$
$$\mathcal{V}^{\underline{ab}} = \mathcal{V}^{(\underline{a}}\mathcal{L}^{\underline{b})}$$

Generalized Morawetz

- Positivity of the **A** bulk reduces showing that the ODE

$$v'' + \frac{9x^2 - 34x - 2}{6x^2(x+2)^2} v = 0$$

has a positive solution on $x > 0$. This is Gauss' hypergeometric equation of the first type!



Higher spin fields

Integer spin field equations:

- spin-0: scalar waves

$$\Box\psi = 0$$

- spin-1: Maxwell

$$F_{\alpha\beta} = F_{[\alpha\beta]}, \quad \nabla^\alpha F_{\alpha\beta} = 0, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0$$

  NP spin scalars ϕ_0, ϕ_1, ϕ_2

- spin-2: Gravity: $R_{\alpha\beta} = 0$ implies

$$\nabla^\alpha W_{\alpha\beta\gamma\delta} = 0$$

  NP spin scalars: $\Psi_0, \ldots, \Psi_4$.
  Kerr: $\Psi_2 = M\rho^3$ is only nonzero spin scalar.

Higher spin fields

- Linearized gravity:

$$\nabla_{\text{background}} W' = \Gamma' W_{\text{background}}, \quad R' = 0$$

not the spin-2 field equation if $W_{\text{background}} \neq 0$!
- Linearized NP scalars: Ψ_{iB}, $i = 0, \ldots, 4$
- Minkowski space (Penrose, 1965):
  - The massless spin-s field equations, including linearized gravity can be reduced to a scalar wave equation.
  - There is a scalar, complex, potential for the spin-s field.

Higher spin fields on Kerr

- Linearized gravity on Kerr: Only extreme the spin scalars Ψ_{0B}, Ψ_{4B} are gauge invariant.
- (Teukolsky, 1972): Extreme spin scalars ϕ_0, ϕ_2 and Ψ_{0B}, Ψ_{4B} *decouple* and solve the spin-s equation

$$\Box_s \psi^{(s)} := \left[(\nabla^\mu + s\Gamma^\mu)(\nabla_\mu + s\Gamma_\mu) + 4s^2 \Psi_{2A} \right] \psi^{(s)} = 0$$

  (here Γ^μ is a certain "connection vector").
- ϕ_0 or ϕ_2 is a potential for Maxwell
- Ψ_{0B} or Ψ_{4B} is a potential for linearized gravity.

Higher spin fields on Kerr

- $(\Sigma\Box_s)\psi^{(s)} = 4\pi\Sigma T$ is the Teukolsky Master Equation (TME$_s$).
- $\Box\psi = 0$ is precisely the $s = 0$ vacuum TME.
- The Carter constant allows to separate the TME into *radial* and *angular* equations for *separated wave forms*

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$$

- Whiting (Whiting, 1989) found a conserved energy for the separated wave forms.
- The TME$_s$ has a *long-range potential* $\rightarrow$ difficult to analyze

Potentials on Schwarzschild

- Maxwell:
  - The spin weight zero scalar ϕ_1 solves the wave equation

$$\left(\Box_g + \frac{2M}{r^3}\right)(r\phi_1) = 0$$

    (Blue, 2008) proved boundedness and decay for Maxwell starting from this equation. ϕ_1 is a *potential* for Maxwell.
- Linearized gravity:
  - Regge-Wheeler axial potential $\mathbf{y} \sim \Im\Psi_{2B}$ (spin weight 0)

$$\left(\Box_g + \frac{8M}{r^3}\right)\mathbf{y} = 0$$

  - Zerilli-Moncrief polar potential $\mathbf{x}$

$$\left(\Box_g + \frac{8M}{r^3}\mathcal{B}[\Delta_{S^2}, r]\right)\mathbf{x} = 0$$

Concluding remarks

- Significant progress has been made in numerical and analytical studies of the global properties of black hole spacetimes
- However, the main problems are still open:
  - Cosmic censorship
  - Uniqueness of Kerr
  - Stability of Kerr
- We expect the methods presented here generalize to
  - Maxwell
  - Linearized gravity

  on Kerr, and to be useful for analyzing stability of Kerr under small nonlinear perturbations

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