# Late-time tails of self-gravitating waves

(non-rigorous quantitative analysis)

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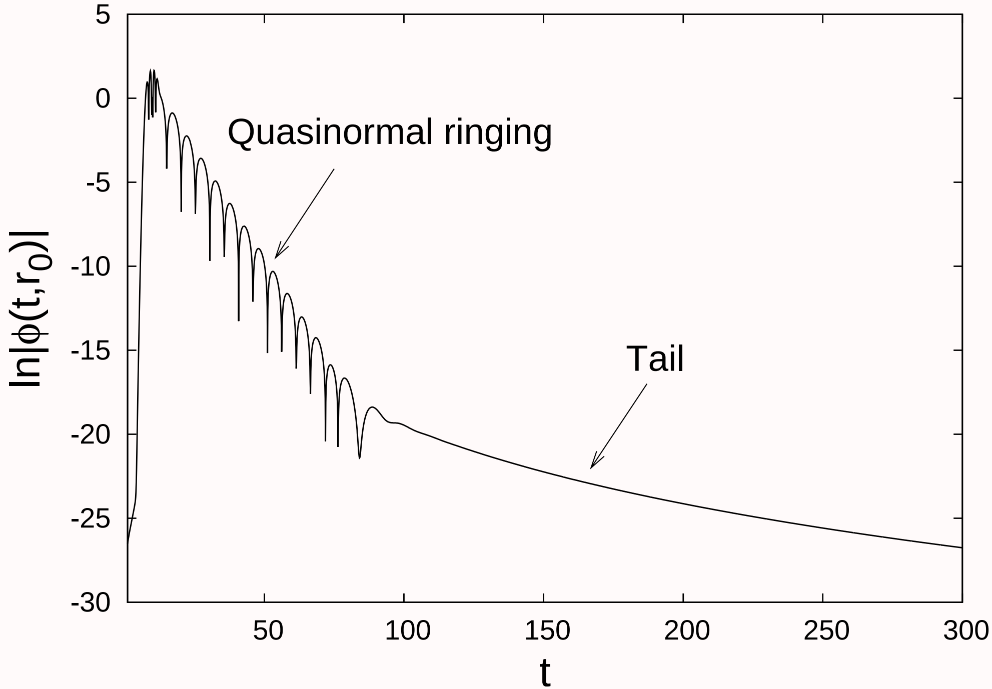
Outline:

- Motivation
- Self-gravitating scalar field
- Self-gravitating wave maps
- Linear tails on a time-dependent background

## Motivation

- Goal: understanding of convergence to equilibrium for nonlinear wave equations
- Mechanism of relaxation for conservative systems on spatially unbounded domains: dissipation by dispersion
- Example: relaxation of a perturbed soliton



## Example

Semilinear wave equation in $3+1$ dimensions

$$\ddot{\phi} - \Delta\phi + V\phi \pm |\phi|^{p-1}\phi = 0$$

- Assumption: positive small potential $V(r) \sim r^{-\alpha}$ as $r \longrightarrow \infty$ ( $\alpha > 2$)
- Well known: global existence of solutions starting from small, smooth, compactly supported initial data ($p > 1 + \sqrt{2}$)
- Not so well known: asymptotics for $t \longrightarrow \infty$ ($r = const$)

$$\phi(t,r) \sim t^{-\gamma}\,, \qquad \gamma = \min\{\alpha, p-1\}$$

- Ergo: linearization yields the sharp decay rate only if $p > \alpha + 1$.
  Otherwise, the late-time behavior is inherently nonlinear
  (a fact frequently overlooked in physics literature)

## Spherically symmetric Einstein-massless-scalar system in $3+1$ dimensions

$$G_{\alpha\beta} = 8\pi \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \left( \nabla_\mu \phi \nabla^\mu \phi \right) \right) , \qquad \nabla_\mu \nabla^\mu \phi = 0 \, .$$

- Two generic attractors (Christodoulou 1986, 1991):

  Small data $\Rightarrow$ Minkowski

  Large data $\Rightarrow$ Schwarzschild

- Rate of convergence to the attractor

  Minkowski: $|\phi(t,r)| \leq C t^{-3}$ (Christodoulou 1987)

  Schwarzschild: $|\phi(t,r)| \leq C t^{-3+\epsilon}$ (Dafermos,Rodnianski 2005)

  Numerics: for both attractors $|\phi(t,r)| \sim t^{-3}$ (Gundlach,Price,Pullin 1994)

- Price's law: solutions of $\nabla_\mu \nabla^\mu \phi = 0$ on the exterior Schwarzschild background decay as $t^{-3}$ (Price 1972,...., Kronthaler 2007, Tataru 2009)

## Evolution of opinions on tails

- *"We found that the predictions for power-law tails of perturbations of Schwarzschild spacetime hold to reasonable approximations, even quantitatively, in a variety of situations to which the predictions might seem initially not to apply."* (Gundlach,Price,Pullin 1994)
- *"Their [GPP] numerical simulations verified the existence of power law tails in the full nonlinear case, thus establishing consistency with analytic perturbative theory."*
- *"We find an excellent agreement between our numerically obtained indices and the values predicted by the linear perturbation analyses. Such an agreement is expected, even in a very nonlinear collapse problem, because of the no-hair principle."*
- In actual fact: the agreement between the decay rate of tails in nonlinear evolution and Price's law is an idiosyncratic property of Einstein's equations in four dimensions.

## Einstein-scalar system in $d+1$ dimensions

- Spherically symmetric ansatz

$$ds^2 = e^{2\alpha(t,r)}\left(-e^{2\beta(t,r)}dt^2 + dr^2\right) + r^2 d\Omega^2_{d-1}\,,$$

- Field equations

$$\left(e^{-\beta}\dot\phi\right)^{\cdot} - \frac{1}{r^{d-1}}\left(r^{d-1}e^{\beta}\phi'\right)' = 0$$

$$\begin{aligned} m' &= \kappa\, r^{d-1}e^{-2\alpha}\left(\phi'^2 + e^{-2\beta}\dot\phi^2\right)\,, \\ \dot m &= 2\kappa\, r^{d-1}e^{-2\alpha}\dot\phi\,\phi'\,, \\ \beta' &= (d-2)\frac{m}{r^{d-1}}e^{2\alpha}\,, \end{aligned}$$

where $e^{2\alpha(t,r)} = (1 - m(t,r)/r^{d-2})$ and $\kappa = 8\pi/(d-1)$.

- We consider small, smooth, compactly supported data

$$\phi(0,r) = \varepsilon f(r), \qquad \dot\phi(0,r) = \varepsilon g(r)\,.$$

## Perturbation expansion

- Up to order $\mathcal{O}(\varepsilon^3)$ we have

$$\phi = \varepsilon\phi_1 + \varepsilon^3\phi_3, \qquad m = \varepsilon^2 m_2, \qquad \beta = \varepsilon^2\beta_2,$$

- First order:

$$\Box\phi_1 = 0\,, \qquad (\phi_1, \dot\phi_1)_{t=0} = (f, g)\,.$$

- Second order:

$$\begin{aligned} m_2' &= \kappa\, r^{d-1}\left(\dot\phi_1^2 + \phi_1'^2\right)\,, \\ \beta_2' &= \frac{(d-2)m_2}{r^{d-1}}\,. \end{aligned}$$

- Third order:

$$\Box\phi_3 = 2\beta_2\ddot\phi_1 + \dot\beta_2\dot\phi_1 + \beta_2'\phi_1'\,, \qquad (\phi_3, \dot\phi_3)_{t=0} = (0, 0)\,.$$

## Tools (elementary)

- Solution of $\Box\phi = 0$ with smooth spherical data ($d = 2L + 3$)

$$\phi(t,r) = \frac{1}{r^{2L+1}} \sum_{k=0}^{L} \frac{2^{k-L}(2L-k)!}{k!(L-k)!}\, r^k \left(a^{(k)}(t-r) - (-1)^k a^{(k)}(t+r)\right)$$

- Solution of $\Box\phi = S(t,r)$ with zero initial data (Duhamel's formula)

$$\phi(t,r) = \frac{1}{2r^{L+1}} \int\limits_0^t d\tau \int\limits_{|t-r-\tau|}^{t+r-\tau} \rho^{L+1} P_L(\mu) S(\tau,\rho) d\rho\,,$$

where $\mu = \dfrac{r^2 + \rho^2 - (t-\tau)^2}{2r\rho}$.

- Remark on notation: we use the letter $L$ on purpose

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right)\phi = \frac{1}{r^L}\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r + \frac{L(L+1)}{r^2}\right)(r^L\phi)$$

- $d = 3$

$$\phi_3(t,r) = \frac{t}{(t^2 - r^2)^2}\left[\Gamma_0 + \mathcal{O}\left(\frac{t}{t^2 - r^2}\right)\right]$$

- $d = 2L + 3 \geq 3$

Tail at timelike infinity ($r = const, t \to \infty$)

$$\phi_3(t,r) = \frac{1}{t^{3L+3}}\left[\Gamma_L + \mathcal{O}\left(\frac{1}{t}\right)\right]$$

Tail along future null infinity ($v = \infty, u \to \infty$)

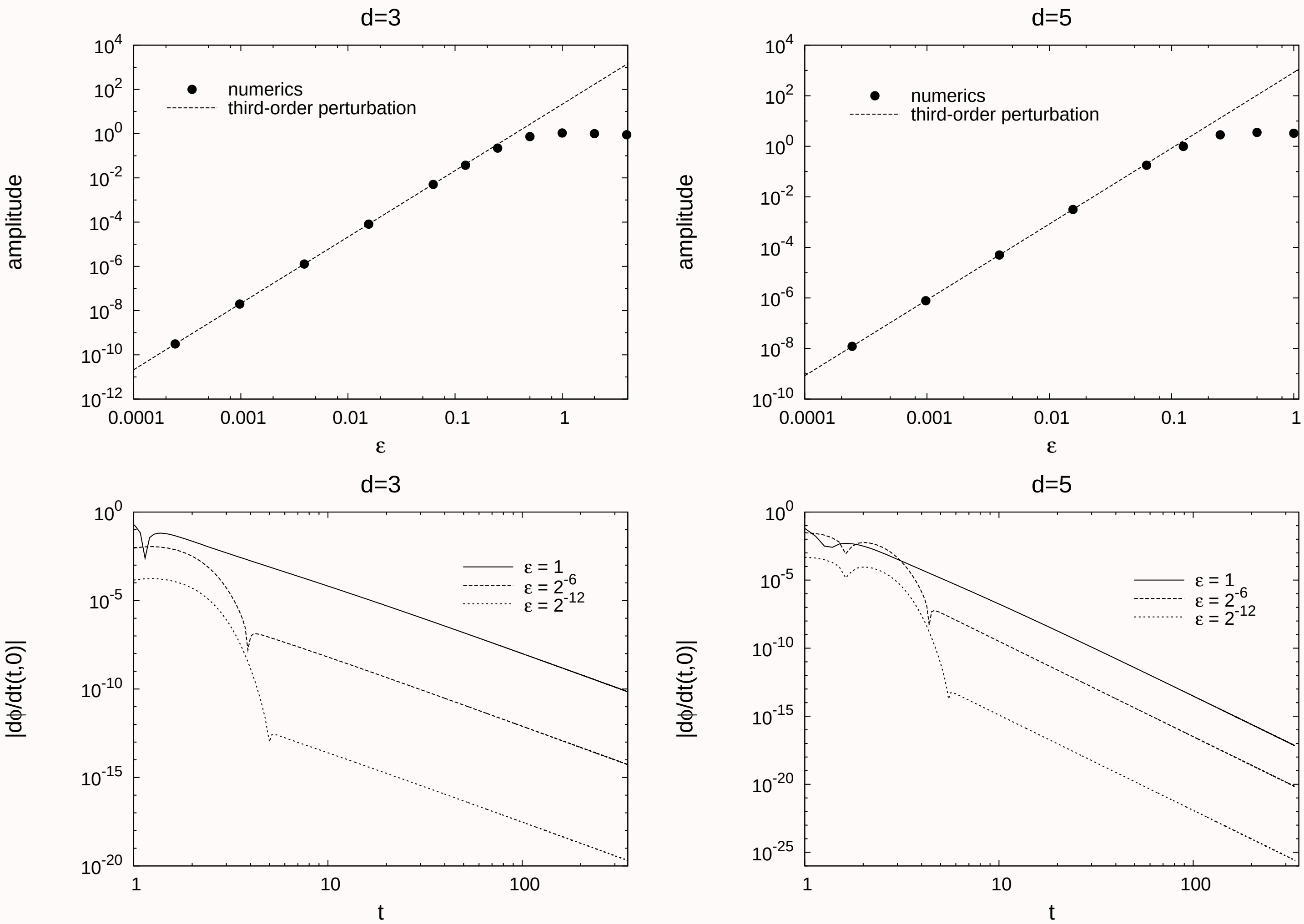
$$(r^{L+1}\phi_3)(v = \infty, u) = \frac{1}{u^{2L+2}}\left[\frac{(2L+1)!(2L+1)!!}{2(3L+2)!}\Gamma_L + \mathcal{O}\left(\frac{1}{u}\right)\right]$$

The coefficient

$$\Gamma_L = (-1)^{L+1} 2^{3L+5} \pi \int\limits_{-\infty}^{+\infty} a^{(L)}(u) \int\limits_{u}^{+\infty} (a^{(L+1)}(s))^2 \, ds \, du$$

is the only trace of initial data.

# Numerical verification



## Price's law in higher dimensions ($d = 2L+3$)

$$g = -\left(1 - \frac{M}{r^{d-2}}\right) dt^2 + \left(1 - \frac{M}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-1}$$

- In terms of the tortoise coordinate $x$ defined by $dr/dx = 1 - M/r^{2L+1}$ and $\psi(x) = r^{L+1}\phi(r)$, the radial wave equation $\Box_g \phi = 0$ takes the form

$$\ddot{\psi} - \psi'' + V(x)\psi = 0\,, \qquad V(x) = \left(1 - \frac{M}{r^{2L+1}}\right)\left(\frac{L(L+1)}{r^2} + \frac{M(L+1)^2}{r^{2L+3}}\right)$$

For $x \gg M$:

$$V(x) = \begin{cases} \dfrac{M}{x^3} + \dfrac{3M^2 \ln(x/M)}{x^4} + \mathcal{O}\left(\dfrac{M^2}{x^4}\right) & L = 0 \\[2ex] \dfrac{L(L+1)}{x^2} + \dfrac{(2L+1)^2(L+1)(4L+3)}{4L(4L+1)} \dfrac{M^2}{x^{4L+4}} + \mathcal{O}\left(\dfrac{M^3}{x^{6L+5}}\right) & L > 0 \end{cases}$$

- Tail for

$$V(x) = \frac{L(L+1)}{x^2} + U(x)\,, \qquad U(x) \sim x^{-\alpha}$$

Ching et al. (1995): If $\alpha > 2$ is not an odd integer less than $2L+3$, then

$$\psi(t,x) \sim t^{-(2L+\alpha)}$$

- For Schwarzschild $\alpha = 3$ (if $L = 0$) and $\alpha = 4L+4$ (if $L > 0$), thus

$$\psi(t,x) \sim t^{-3} \quad (L=0)\,, \qquad \psi(t,x) \sim t^{-(6L+4)} \quad (L>0)$$

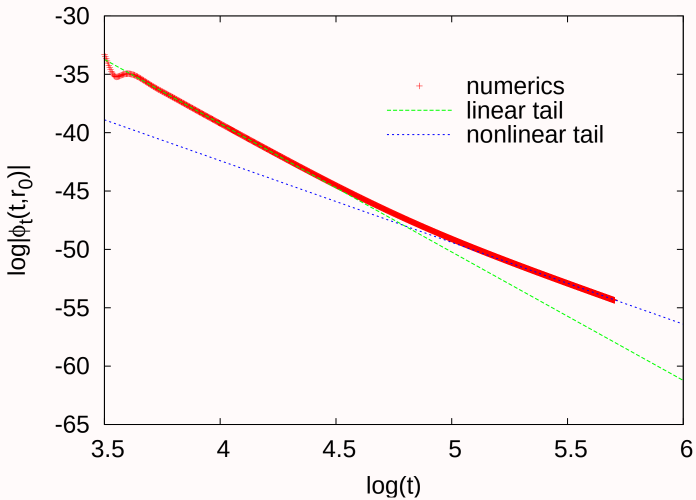
- Comparison of linear and nonlinear tails

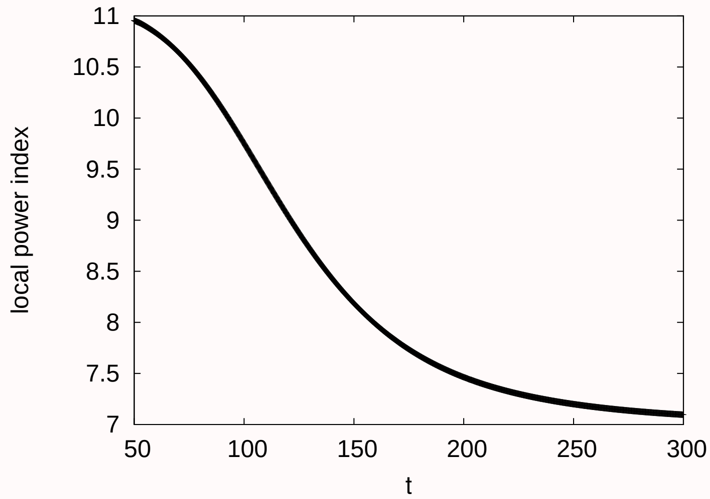
| | Linear | Nonlinear |
|---|---|---|
| $L=0$ | $t^{-3}$ | $t^{-3}$ |
| $L>0$ | $t^{-(6L+4)}$ | $t^{-(3L+3)}$ |

## Nonlinear perturbations of Schwarzschild in $5+1$ dimensions

$$\phi(t,r) \sim \frac{A\varepsilon}{t^{10}} + \frac{B\varepsilon^3}{t^6}$$

Crossover from linear to nonlinear tail at $t \sim \varepsilon^{1/2}$



## Self-gravitating wave maps

Let $U : \mathcal{M} \to \mathcal{N}$ be a map from $(\mathcal{M}, g_{\mu\nu})$ into a Riemannian manifold $(\mathcal{N}, G_{AB})$.

- $(U, g_{\mu\nu})$ is called a wave map coupled to gravity if it is a critical point of the action

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} \left(R(g) - 4\lambda S_{\mu\nu} g^{\mu\nu}\right) \sqrt{-g}\, dx\,, \qquad S_{\mu\nu} := \frac{\partial U^A}{\partial x^\mu} \frac{\partial U^B}{\partial x^\nu} G_{AB}$$

- Let $\mathcal{M}$ be a 4-dimensional spherical spacetime and $\mathcal{N} = S^3$ with metrics

$$\begin{aligned} g &= e^{2\alpha(t,r)} \left(-e^{2\beta(t,r)} dt^2 + dr^2\right) + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\,, \\ G &= dF^2 + \sin^2 F\, (d\Theta^2 + \sin^2\Theta\, d\Phi^2)\,. \end{aligned}$$

- We assume that the wave map $U$ is spherically $\ell$-equivariant:

$$F = F(t,r), \qquad (\Theta, \Phi) = \chi_\ell(\vartheta, \varphi)\,,$$

  where $\chi_\ell : S^2 \to S^2$ is a harmonic eigenmap with eigenvalue $\ell(\ell+1)$.

- The energy-momentum $2\pi T_{\mu\nu} = \lambda \left(S_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S^\alpha_\alpha\right)$ does not depend on angles!

- Field equations

$$
\begin{aligned}
m' &= \lambda\, r^2 e^{-2\alpha} \left(F'^2 + e^{-2\beta} \dot{F}^2\right) + \lambda\, \ell(\ell+1) \sin^2 F\,, \\
\dot{m} &= 2\lambda\, r^2 e^{-2\alpha} \dot{F}\, F'\,, \\
\beta' &= \frac{2m}{r^2} e^{2\alpha} - 2\lambda\, \ell(\ell+1)\, e^{2\alpha} \frac{\sin^2 F}{r}\,,
\end{aligned}
$$

$$
\left(e^{-\beta}\dot{F}\right)^{\cdot} - \frac{1}{r^2}\left(r^2 e^{\beta} F'\right)' + \ell(\ell+1) e^{\beta+2\alpha} \frac{\sin 2F}{2r^2} = 0\,.
$$

- For $\ell \geq 1$ solutions starting from small, smooth, compactly supported data $(F, \dot{F})_{t=0} = \varepsilon(f(r), g(r))$ decay for large retarded times as

$$
F(t,r) \simeq \varepsilon^3 F_3(t,r) = \frac{\varepsilon^3 r^\ell}{(t^2-r^2)^{\ell+1}} \left[\lambda A_\ell + B_\ell + \mathcal{O}\left(\frac{t}{t^2-r^2}\right)\right],
$$

where $A_\ell$ and $B_\ell$ are explicitly determined by initial data.

## Price's law for higher harmonics

- Decomposition into spherical harmonics

$$\psi(t,r,\vartheta,\varphi) = \sum_{\ell\geq 0, |m|\leq \ell} \psi_{\ell m}(t,r)\, Y_\ell^m(\vartheta,\varphi)$$

$$\left(e^{-\beta}\dot{\psi}_{\ell m}\right)^{\cdot} - \frac{1}{r^2}\left(r^2 e^{\beta}\psi'_{\ell m}\right)' + e^{\beta}\frac{\ell(\ell+1)}{r(r-2m)}\,\psi_{\ell m} = 0$$

- Static asymptotically flat spherical background: $m = M, \beta \sim -2M/r$

$$\psi_{\ell m}(t,r) \sim \frac{C_\ell\, r^\ell t}{(t^2-r^2)^{\ell+2}} \qquad \text{(Price 1972, Poisson 2002)}$$

At timelike infinity this gives

$$\psi_{\ell m}(t,r) \sim C_\ell\, r^\ell t^{-(2\ell+3)}$$

Recently Donninger, Schlag, Soffer (2009) proved that $\psi_{\ell m}(t,r) \leq C_\ell\, t^{-(2\ell+2)}$

## Linear tails on a time-dependent background

- Consider (after Gundlach, Price, Pullin) a nonspherical test massless scalar field $\psi$ propagating on the time-dependent spherically symmetric spacetime generated by a self-gravitating massless scalar field $\phi$.
- Using perturbation expansion $\psi_{\ell m} = \psi_0 + \varepsilon^2 \psi_2 + \ldots$ we get

$$\psi_{\ell m}(t, r) \sim \begin{cases} \dfrac{\varepsilon^2 B_0 \, t}{(t^2 - r^2)^2} & \ell = 0\,, \\[2ex] \dfrac{\varepsilon^2 B_\ell \, r^\ell}{(t^2 - r^2)^{\ell+1}} & \ell > 0\,. \end{cases}$$

- For $\ell \geq 1$ the decay of tails on time-dependent backgrounds is slower than that on a stationary background (Price's tail).

## Concluding remarks

- We showed that in general it is not correct to draw conclusions about the late-time asymptotics of the nonlinear problem on the basis of linearization.

- Our results by no means diminish the importance recent flurry of results on late-time asymptotics for the linear wave equation $\Box_g \phi = 0$ on a fixed background (e.g., Kerr). Having a good hold on the linear problem is a necessary first step in proving nonlinear stability of the background spacetime.

- Viewing the dimension of a spacetime as a parameter may help understand which features of general relativity depend crucially on our world being four dimensional and which ones are general. We seem to live in the most challenging dimension.

- In higher dimensions black holes get bald very fast, hence the proof of nonlinear stability of Kerr should be easier.