From radiative transfer to radiotherapy

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Motivations

□ Radiative transfer

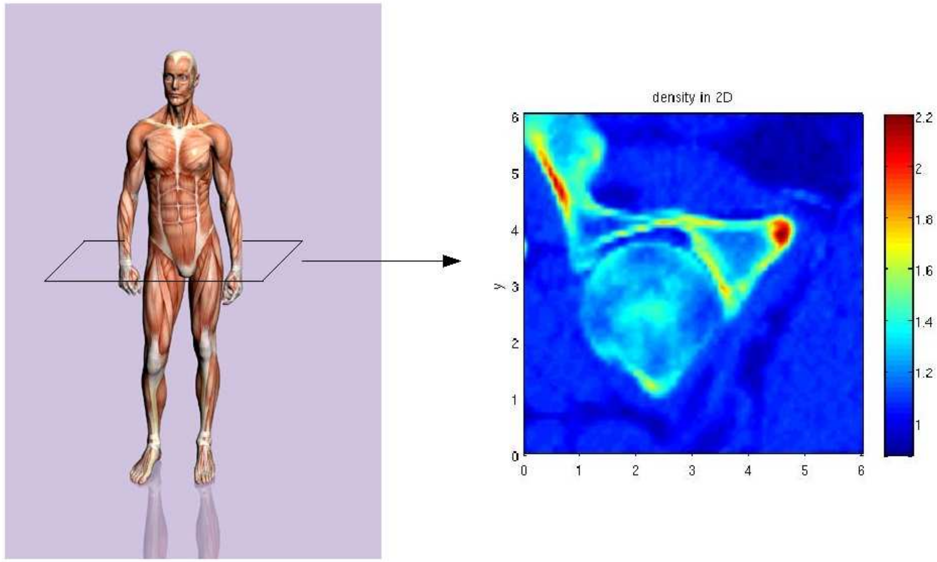
↪ Atmospheric entry

↪ Plasma flows

↪ Fluid photons interactions

□ Radiotherapy

↪ Matter electrons interactions



□ Radiative Transfer Equation (Kinetic)

$$\frac{1}{c}\partial_t I + \Omega \partial_x I = \sigma(B(T) - I)$$

- I : photon distribution
- c : speed of light
- σ : opacity
- $B(T)$: Planck's function

Prohibitive Numerical cost for our applications

The M1 model

□ The M1-model for radiative transfer (Dubroca-Feugeas 99)

$$\begin{cases} \partial_t E + \partial_x F = c\sigma^e a T^4 - c\sigma^a E \\ \partial_t F + \partial_x c^2 P = -c\sigma^f F \\ \partial_t \rho C_v T = c\sigma^a E - c\sigma^e a T^4 \end{cases} \qquad \mathbf{W} = \begin{pmatrix} E \\ F \\ T \end{pmatrix} \qquad \mathcal{F}(\mathbf{W}) \begin{pmatrix} F \\ c^2 P = c^2 E \chi(\frac{F}{cE}) \\ 0 \end{pmatrix}$$

E radiative energy

F radiative flux

P radiative pressure

T radiative temperature

and the opacities
$$\begin{cases} \sigma^a := \sigma^a(x, \mathbf{W}) \\ \sigma^e := \sigma^e(x, \mathbf{W}) \\ \sigma^f := \sigma^f(x, \mathbf{W}) \end{cases}$$

□ Numerical approximation

$\hookrightarrow$ Robustness $E > 0$ and $|F/cE| < 1$

$\hookrightarrow$ Relevant asymptotic behaviors

Outline

1. Introduction of a generic model

   $\hookrightarrow$ Application: M1-model but also Euler with friction or the Kerr-Debye model

2. A Riemann solver with stiff source terms

   $\hookrightarrow$ Free from the Godunov type scheme for the transport operator

   $\hookrightarrow$ Robustness

   $\hookrightarrow$ A systematic correction to satisfy the diffusive regime

3. Riemann solvers

   $\hookrightarrow$ Suliciu relaxation scheme

   $\hookrightarrow$ HLLC scheme

4. Electron extension : Radiotherapy

An hyperbolic model with stiff source terms

$$\partial_t \mathbf{W} + \partial_x \mathcal{F}(\mathbf{W}) = \sigma\left(\mathbf{R}(\mathbf{W}) - \mathbf{W}\right) \qquad \mathbf{W} \in \Omega \text{ convex}$$

$$\sigma := \sigma(x, \mathbf{W}) > 0$$

$$\mathbf{R}(\mathbf{W}) := \mathbf{R}(x, \mathbf{W}) \in \Omega$$

□ Telegraph equations

$$\begin{cases} \partial_t u + a\partial_x v = \sigma(v-u) \\ \partial_t v + a\partial_x u = \sigma(u-v) \end{cases} \quad \mathbf{W} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \mathbf{R}(\mathbf{W}) = \begin{pmatrix} v \\ u \end{pmatrix}$$

□ Euler with friction (Bouchut et al)

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = -\sigma\rho u \end{cases} \quad \mathbf{W} = \begin{pmatrix} \rho \\ \rho u \end{pmatrix} \quad \mathbf{R}(\mathbf{W}) = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

□ Kerr-Debye model (Aregba et al)

$$\begin{cases} \partial_t d + \partial_x h = 0 \\ \partial_t h + \partial_x \dfrac{d}{1+\chi} = 0 \\ \partial_t \chi = \sigma\left(\left(\dfrac{d}{1+\chi}\right)^2 - \chi\right) \end{cases} \quad \mathbf{W} = \begin{pmatrix} d \\ h \\ \chi \end{pmatrix} \quad \mathbf{R}(\mathbf{W}) = \begin{pmatrix} d \\ h \\ \left(\frac{d}{1+\chi}\right)^2 \end{pmatrix}$$

☐ M1-model: Source term given by

$$\mathbf{S}(\mathbf{W}) = c\begin{pmatrix} \sigma^e aT^4 - \sigma^a E \\ -\sigma^f F \\ \frac{1}{\rho C_v}(\sigma^a E - \sigma^e aT^4) \end{pmatrix} \equiv \sigma\left(\mathbf{R}(\mathbf{W}) - \mathbf{W}\right)$$

$$\underline{\text{Assumptions}} \quad \begin{cases} \sigma^f = \sigma^a + \hat{\sigma} \quad \hat{\sigma} > 0 \quad \Leftrightarrow \quad \sigma^f > \sigma^a \\ \sigma^f = \dfrac{\sigma^e aT^3}{\rho C_v} + \check{\sigma} \quad \check{\sigma} > 0 \quad \Leftrightarrow \quad \sigma^f > \dfrac{\sigma^e aT^3}{\rho C_v} \end{cases}$$

$$\text{To write} \quad \begin{cases} \sigma^e aT^4 - \sigma^a E = \sigma^f\left(\dfrac{\sigma^e aT^4 + \hat{\sigma}E}{\sigma^f} - E\right) \\ \dfrac{1}{\rho C_v}(\sigma^a E - \sigma^e aT^4) = \sigma^f\left(\dfrac{\frac{\sigma^a}{\rho C_v}E + \check{\sigma}T}{\sigma^f} - T\right) \end{cases}$$

$$\text{Hence} \quad \sigma := c\sigma^f \quad \mathbf{R}(\mathbf{W}) = \begin{pmatrix} \frac{\sigma^e aT^4 + \hat{\sigma}E}{\sigma^f} \\ 0 \\ \frac{\frac{\sigma^a}{\rho C_v}E + \check{\sigma}T}{\sigma^f} \end{pmatrix}$$

□ Diffusion regime

Large opacities $\longrightarrow \epsilon$ rescaling factor

$$\begin{cases} \epsilon\partial_t E + \partial_x F = \dfrac{1}{\epsilon}(c\sigma^e aT^4 - c\sigma^a E) \\ \epsilon\partial_t F + \partial_x c^2 P = -\dfrac{1}{\epsilon}c\sigma^f F \\ \epsilon\partial_t \rho C_v T = \dfrac{1}{\epsilon}(c\sigma^a E - c\sigma^e aT^4) \end{cases}$$

Limit $\epsilon \longrightarrow 0$

- $E \longrightarrow aT^4$ and $F \longrightarrow 0$
- T given by a diffusion equation $\partial_t \left(\rho C_v T + aT^4\right) - \partial_x \left(\dfrac{4c}{3\sigma^f}\partial_x T\right) = 0$

□ Main numerical difficulties

To preserve the relevant diffusive regimes

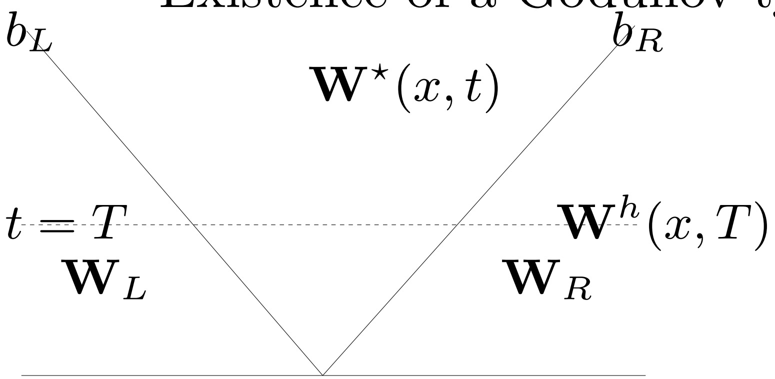
Turpault 02, Buet-Cordier 04, Buet-Després 06, CB-Dubroca-Charrier 07, Bouchut et al. 08, Coquel et al.

Objective: Asymptotic preserving methods

A Riemann solver with stiff source terms

☐ Assumptions

Existence of a Godunov type Riemann solver to approximate



$$\partial_t \mathbf{W} + \partial_x \mathcal{F}(\mathbf{W}) = 0$$

b_L and b_R: velocities min and max of the approximate Riemann solver

$\mathbf{W}^\star$ approximate solution in the cone

↪ Formalism of Harten-Lax-van Leer (84)

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}})$$

$$\mathcal{F}_{i+\frac{1}{2}} = \mathcal{F}(\mathbf{W}_i^n) - \frac{\Delta x}{2\Delta t}\mathbf{W}_{i+1}^n + \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \mathbf{W}^h(x, t^n + \Delta t)dx$$

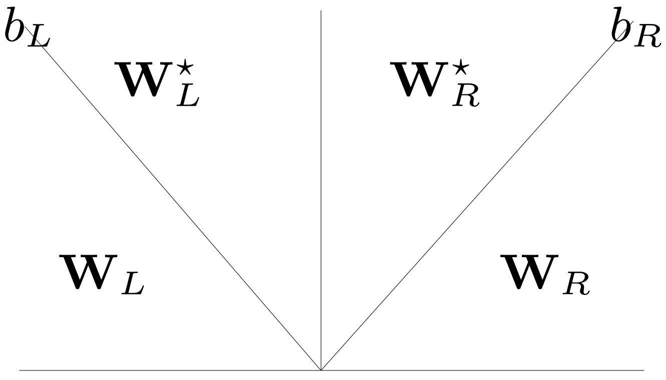
↪ CFL restriction $\quad \frac{\Delta t}{\Delta x} \max\left(|b^R_{i-\frac{1}{2}}|, |b^L_{i+\frac{1}{2}}|\right) \leq \frac{1}{2}$

Lemma

Assume $\mathbf{W}^\star \in \Omega$. As a consequence, as soon as $\mathbf{W}_i^n \in \Omega$ for all $i \in \mathbb{Z}$ then $\mathbf{W}_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$

□ Stiff source terms

Modify the Riemann solver to consider the source terms



$$\mathbf{W}_L^\star = \alpha \mathbf{W}^\star + (1-\alpha)\mathbf{R}^-(\mathbf{W}_L)$$

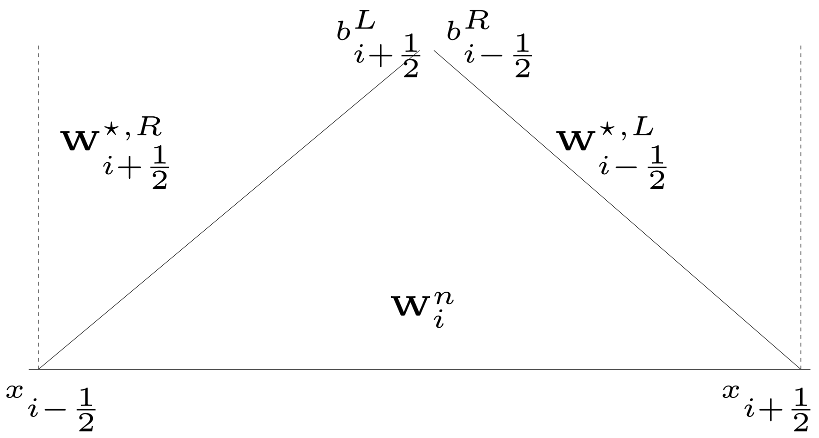
$$\mathbf{W}_R^\star = \alpha \mathbf{W}^\star + (1-\alpha)\mathbf{R}^+(\mathbf{W}_R)$$

$$\alpha = \frac{b^R - b^L}{b^R - b^L + \sigma \Delta x}$$

We note $\tilde{\mathbf{W}}^h(x,t)$ the approximate solution

$\hookrightarrow$ $\mathbf{R}^\pm(\mathbf{W})$ consistent with $\mathbf{R}(\mathbf{W})$

$\hookrightarrow$ Unchanged CFL restriction $\frac{\Delta t}{\Delta x} \max\left(|b^R_{i-\frac{1}{2}}|, |b^L_{i+\frac{1}{2}}|\right) \leq \frac{1}{2}$



$$\mathbf{W}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{W}^h(x,t^{n+1})dx$$

$$\mathbf{W}^{\star,L}_{i-\frac{1}{2}} = \alpha_{i-\frac{1}{2}}\mathbf{W}^\star_{i-\frac{1}{2}} + (1-\alpha_{i-\frac{1}{2}})\mathbf{R}^+_{i-\frac{1}{2}}(\mathbf{W}_i^n)$$

$$\mathbf{W}^{\star,R}_{i+\frac{1}{2}} = \alpha_{i+\frac{1}{2}}\mathbf{W}^\star_{i+\frac{1}{2}} + (1-\alpha_{i+\frac{1}{2}})\mathbf{R}^+_{i+\frac{1}{2}}(\mathbf{W}_i^n)$$

To obtain

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}\left(\alpha_{i+\frac{1}{2}}\mathcal{F}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}\mathcal{F}_{i-\frac{1}{2}}\right) + \Delta t\left(\frac{1}{\Delta x}(1-\alpha_{i-\frac{1}{2}})\mathbf{S}^+_{i-\frac{1}{2}} + \frac{1}{\Delta x}(1-\alpha_{i+\frac{1}{2}})\mathbf{S}^-_{i+\frac{1}{2}}\right)$$

$$\mathbf{S}^+_{i-\frac{1}{2}} = \max(0, b^R_{i-\frac{1}{2}})(\mathbf{R}^+_{i-\frac{1}{2}}(\mathbf{W}_i^n) - \mathbf{W}_i^n) - \max(0, b^L_{i-\frac{1}{2}})(\mathbf{R}^+_{i-\frac{1}{2}}(\mathbf{W}_i^n) - \mathbf{W}_{i-1}^n) + \mathcal{F}(\mathbf{W}_i^n)$$

$$\mathbf{S}^-_{i+\frac{1}{2}} = \min(0, b^R_{i+\frac{1}{2}})(\mathbf{R}^-_{i+\frac{1}{2}}(\mathbf{W}_i^n) - \mathbf{W}_{i+1}^n) - \min(0, b^L_{i+\frac{1}{2}})(\mathbf{R}^-_{i+\frac{1}{2}}(\mathbf{W}_i^n) - \mathbf{W}_{i-1}^n) - \mathcal{F}(\mathbf{W}_i^n)$$

Lemma Robustness

Assume the initial Godunov type scheme is robust: Ω stays invariant by the scheme

For all $\mathbf{W}_L$ and $\mathbf{W}_R$ in Ω, assume

$$\alpha\mathbf{W}^\star + (1-\alpha)\mathbf{R}^-(\mathbf{W}_L) \in \Omega$$

$$\alpha\mathbf{W}^\star + (1-\alpha)\mathbf{R}^+(\mathbf{W}_R) \in \Omega$$

Then, with the same CFL restriction, the full scheme with stiff source term preserve the robustness property

$$\mathbf{W}_i^n \in \Omega \quad \text{for all} \quad i \in \mathbb{Z} \Rightarrow \mathbf{W}_i^{n+1} \in \Omega \quad \text{for all} \quad i \in \mathbb{Z}$$

□ Standard approach (Telegraph equations)

Similar to Buet-Cordier 04 but for fixed numerical cone

Involving our formalism we have $\mathbf{R}(\mathbf{W}) = \begin{pmatrix} v \\ u \end{pmatrix}$

Assume $b^L < 0 < b^R$

$$\begin{cases} u_i^{n+1} = u_i^n - \dfrac{\Delta t}{\Delta x}(\alpha_{i+\frac{1}{2}} f^u_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}} f^u_{i-\frac{1}{2}}) + \Delta t S_i^u \\ v_i^{n+1} = v_i^n - \dfrac{\Delta t}{\Delta x}(\alpha_{i+\frac{1}{2}} f^v_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}} f^v_{i-\frac{1}{2}}) + \Delta t S_i^v \end{cases}$$

where

$$\alpha_{i+1\frac{1}{2}} = \frac{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}}}{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}}\Delta x} \qquad \begin{cases} f^u_{i+\frac{1}{2}} = a\dfrac{v^n_{i+1} + v^n_i}{2} - \dfrac{a}{2}(u^n_{i+1} - u^n_i) \\ f^v_{i+\frac{1}{2}} = a\dfrac{u^n_{i+1} + u^n_i}{2} - \dfrac{a}{2}(v^n_{i+1} - v^n_i) \end{cases}$$

The source term is given by

$$S_i^u = \frac{\sigma_{i-\frac{1}{2}}}{b^R_{i-\frac{1}{2}} - b^L_{i-\frac{1}{2}} + \sigma_{i-\frac{1}{2}}\Delta x}(b^R_{i-\frac{1}{2}}(v_i^n - u_i^n) + a v_i^n) + \frac{\sigma_{i+\frac{1}{2}}}{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}}\Delta x}(-b^L_{i+\frac{1}{2}}(v_i^n - u_i^n) - a v_i^n)$$

$$S_i^v = \frac{\sigma_{i-\frac{1}{2}}}{b^R_{i-\frac{1}{2}} - b^L_{i-\frac{1}{2}} + \sigma_{i-\frac{1}{2}}\Delta x}(b^R_{i-\frac{1}{2}}(u_i^n - v_i^n) - a u_i^n) + \frac{\sigma_{i+\frac{1}{2}}}{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}}\Delta x}(-b^L_{i+\frac{1}{2}}(u_i^n - v_i^n) + a u_i^n)$$

$\hookrightarrow$ Enforce $\sigma = 0$ to get the correct transport regime

$\hookrightarrow$ Asymptotic regime: $\Delta t \leftrightarrow \frac{\Delta t}{\epsilon}$ and $\sigma \leftrightarrow \frac{\sigma}{\epsilon}$

$$\partial_t(u+v) = \partial_x\left(\frac{a^2}{2\sigma}\partial_x(u+v)\right)$$

Basic choice: $b^R = -b^L = a$ and $\sigma = cste$

$$(u+v)_i^{n+1} = (u+v)_i^n - \frac{\Delta t}{\Delta x^2}\frac{a^2}{\sigma}\left((u+v)_{i+1}^n - 2(u+v)_i^n + (u+v)_{i+1}^n\right)$$

Wrong asymptotic regime

□ Numerical asymptotic regime

$$\partial_t \mathbf{W} + \partial_x \mathcal{F}(\mathbf{W}) = \sigma \left(\mathbf{R}(\mathbf{W}) - \mathbf{W}\right)$$

Godunov type scheme + Source term $\Leftrightarrow$ Asymptotic regime reached

□ Introduction of a correction

$\bar{\sigma} > 0$ arbitrary

$$\begin{aligned}
\partial_t \mathbf{W} + \partial_x \mathcal{F}(\mathbf{W}) &= \left(\sigma \mathbf{R}(\mathbf{W}) + \bar{\sigma}\mathbf{W}\right) - \left(\bar{\sigma}\mathbf{W} + \sigma \mathbf{W}\right) \\
&= (\sigma + \bar{\sigma}) \left(\left(\frac{\sigma}{\sigma + \bar{\sigma}} \mathbf{R}(\mathbf{W}) + \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} \mathbf{W} \right) - \mathbf{W} \right) \\
&= (\sigma + \bar{\sigma}) \left(\bar{\mathbf{R}}(\mathbf{W}) - \mathbf{W} \right)
\end{aligned}$$

The same formalism but for

$$\bar{\mathbf{R}}(\mathbf{W}) = \frac{\sigma}{\sigma + \bar{\sigma}} \mathbf{R}(\mathbf{W}) + \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} \mathbf{W} \quad \in \Omega$$

$\bar{\sigma}$ will be a relevant correction to satisfy the diffusive regime

□ Stiff source term correction

$$\begin{cases} \partial_t u + a\partial_x v = (\sigma+\bar\sigma)(\dfrac{\sigma v+\bar\sigma u}{\sigma+\bar\sigma} - u) \\ \partial_t v + a\partial_x u = (\sigma+\bar\sigma)(\dfrac{\sigma u+\bar\sigma v}{\sigma+\bar\sigma} - v) \end{cases}$$

Here we have $\sigma \leftrightarrow \sigma+\bar\sigma$ and $\mathbf{R}(\mathbf{W}) \leftrightarrow \begin{pmatrix} \dfrac{\sigma v+\bar\sigma u}{\sigma+\bar\sigma} \\ \dfrac{\sigma u+\bar\sigma v}{\sigma+\bar\sigma} \end{pmatrix}$

The corrected scheme reads with $b^R = -b^L = a$ and $\sigma = cste$

$$\alpha_{i+\frac12} = \frac{2a}{2a+(\sigma+\bar\sigma)\Delta x}$$

$$S_i^u = 2a\frac{\sigma+\bar\sigma}{2a+(\sigma+\bar\sigma)\Delta x}\left(\frac{\sigma v_i^n+\bar\sigma u_i^n}{\sigma+\bar\sigma} - u_i^n\right)$$

$$S_i^v = 2a\frac{\sigma+\bar\sigma}{2a+(\sigma+\bar\sigma)\Delta x}\left(\frac{\sigma u_i^n+\bar\sigma v_i^n}{\sigma+\bar\sigma} - v_i^n\right)$$

$\hookrightarrow$ With $\sigma = 0$, we preserve the transport regime

$\hookrightarrow$ Asymptotic regime given by

$$\partial_t(u+v) = \partial_x \left(\frac{a^2}{2\sigma} \partial_x (u+v) \right)$$

Numerically we obtain

$$(u+v)_i^{n+1} = (u+v)_i^n - \frac{\Delta t}{\Delta x^2} \frac{a^2}{\sigma + \bar{\sigma}} \left((u+v)_{i+1}^n - 2(u+v)_i^n + (u+v)_{i-1}^n \right)$$

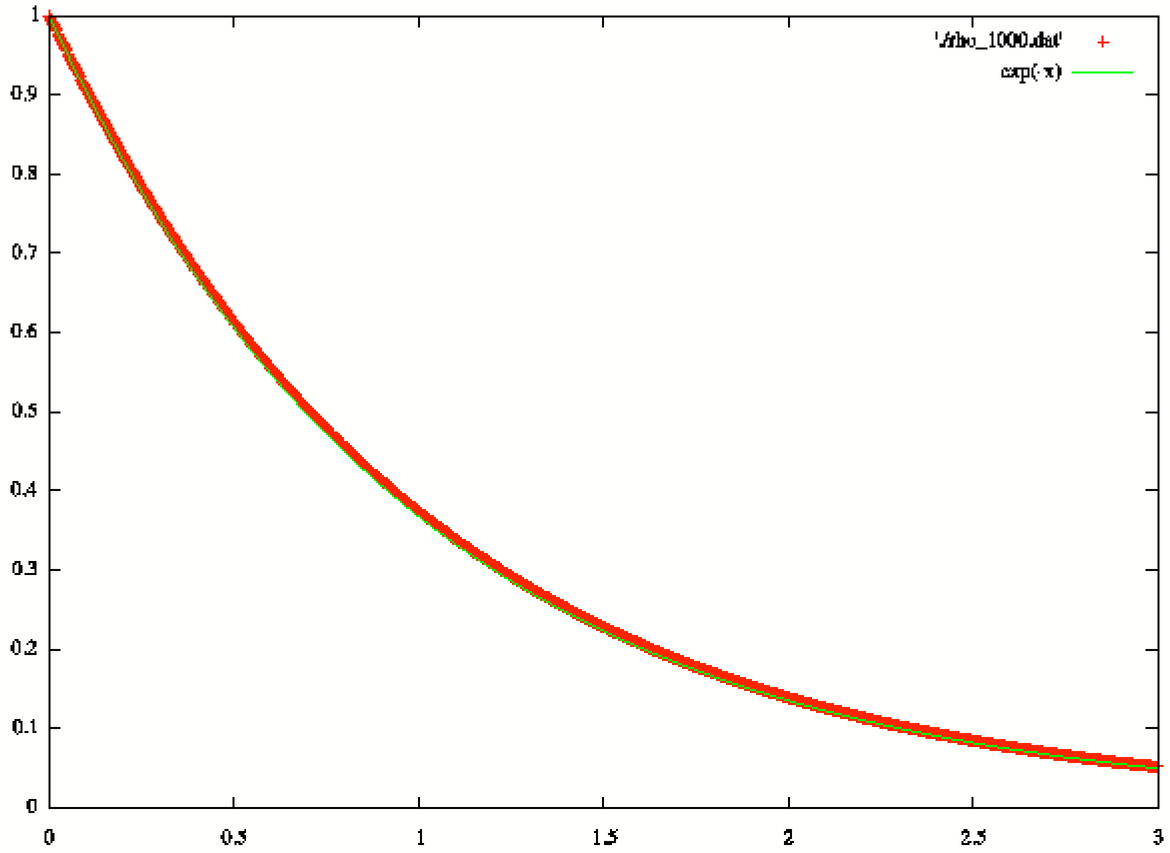
Enforce $\bar{\sigma} = \sigma$ to obtain the expected asymptotic regime

Euler with friction

Impose $\sigma(x) = e^{-x} + \kappa\gamma e^{-\gamma x}$

Exact stationnary density solution $\rho(x) = e^{-x}$

HLL scheme with asymptotic preserving correction



□ The numerical procedure

1. Chose a Godunov type scheme to approximate the transport regime $\Rightarrow \ \mathcal{F}_{i+\frac{1}{2}}$

2. Give the numerical scheme with stiff source terms

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}(\alpha_{i+\frac{1}{2}}\mathcal{F}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}\mathcal{F}_{i-\frac{1}{2}}) + \Delta t\ \mathbf{S}_i^n$$

3. Introduce the diffusion correction $\bar{\sigma}$

$$\alpha_{i+\frac{1}{2}} = \frac{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}}}{b^R_{i+\frac{1}{2}} - b^L_{i+\frac{1}{2}} + (\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}})\Delta x} \qquad \bar{\mathbf{R}}(\mathbf{W}) = \frac{\sigma}{\sigma+\bar{\sigma}}\mathbf{R}(\mathbf{W}) + \frac{\bar{\sigma}}{\sigma+\bar{\sigma}}\mathbf{W}$$

4. Fix $\bar{\sigma}$ to enforce the relevant diffusion regime

With the same CFL restriction, this procedure preserves the robustness and the stability of the Godunov type scheme

A Suliciu relaxation scheme

☐ M1-model versus Euler equations (Després 06)

$$\begin{cases} \partial_t E + \partial_x F = 0 \\ \partial_t F + \partial_x c^2 P = 0 \end{cases} \qquad \begin{array}{lll} F & \longrightarrow & \text{a flux} \;\equiv\; \rho v \\ E & \longrightarrow & \text{total energy} \;\equiv\; \rho e \end{array}$$

$$\text{define} \qquad \begin{array}{l} \beta \;\text{ by }\; \dfrac{F}{cE} = \dfrac{4c\beta}{3c^2+\beta^2} \\ q \;\text{ by }\; c^2 P = \beta F + q \end{array}$$

$$\begin{cases} \partial_t \rho v + \partial_x(\rho v \beta + q) = 0 \\ \partial_t \rho e + \partial_x(\rho e + q)\beta = 0 \end{cases} \longrightarrow \begin{cases} \partial_t \rho v + \partial_x(\rho v v + p) = 0 \\ \partial_t \rho e + \partial_x(\rho e + p) v = 0 \end{cases}$$

We propose to consider the *Euler* like equations

$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0 \\ \partial_t \rho v + \partial_x(\rho v \beta + q) = 0 \\ \partial_t \rho e + \partial_x(\rho e + q)\beta = 0 \end{cases}$$

q stands for the pressure

β stands for the contact velocity

Involving arbitrary density, the numerical results are relevant

□ Suliciu relaxation model (Bouchut 04, Coquel-Perthame 98 ...)

We approximate the weak solution of our model by the weak solution of a model with singular perturbation

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t \rho e + \partial_x (\rho e + p) u = 0 \end{cases}$$

Relaxation model where a is a parameter

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + \Pi) = 0 \\ \partial_t \rho e + \partial_x (\rho e + \Pi) u = 0 \\ \partial_t \Pi + u \partial_x \Pi + \frac{a^2}{\rho} \partial_x u = \frac{1}{\epsilon} (p - \Pi) \end{cases}$$

<u>Nonlinearities</u>: q and β

$$\begin{cases} q \quad \text{relaxed by} \quad \Pi \quad \text{(similar to a pressure)} \\ \beta \quad \text{relaxed by} \quad u \quad \text{(similar to a contact velocity)} \end{cases}$$

The proposed model $\equiv$ Suliciu relaxation model for 2D Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \Pi) = \dfrac{1}{\epsilon}(\beta - u) \\ \partial_t \rho v + \partial_x(\rho u v + c^2 \Pi) = \dfrac{1}{\epsilon}(\beta - u) \\ \partial_t \rho e + \partial_x(\rho e + \Pi)u = 0 \\ \partial_t \Pi + u \partial_x \Pi + \dfrac{a^2}{\rho}\partial_x u = \dfrac{1}{\epsilon}(q - \Pi) \end{cases}$$

Existence of a tangential velocity $V = v - c^2 u$

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \Pi) = \dfrac{1}{\epsilon}(\beta - u) \\ \partial_t \rho V + \partial_x \rho u V = 0 \\ \partial_t \rho e + \partial_x(\rho e + \Pi)u = 0 \\ \partial_t \Pi + u \partial_x \Pi + \dfrac{a^2}{\rho}\partial_x u = \dfrac{1}{\epsilon}(q - \Pi) \end{cases}$$

a and ρ relaxation parameters

Lemma

The relaxation model is hyperbolic. All the fields are linearly degenrate

The Riemann problem is easily solved

In addition, assume the parameter a large enough, then we have

$$\rho e = E > 0 \quad \text{and} \quad \left|\frac{\rho v}{c\rho e}\right| = \left|\frac{F}{cE}\right| < 1$$

The proposed Suliciu model is considered to derive an approximate Riemann solver

☐ Relaxation scheme: 2 step scheme

- Evolution (based on the Riemann solution)
- Relaxation step

☐ Time t^n : Piecewise constant approximation

Equilibrium state $\qquad \mathbf{U}_i^n = (\mathbf{W}_i^n, \Pi_i^n = q_i^n, u_i^n = \beta_i^n)$

$\rho_i^n > 0$ stay arbitrary

☐ Evolution

Godunov scheme to evolve in time the relaxation model with $\frac{1}{\epsilon} = 0$

$\Rightarrow \mathbf{U}_i^{n+1,-}$ with the CFL condition $\displaystyle \frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| u_i^n \pm \frac{a_{i+\frac{1}{2}}}{\rho_i^n} \right| \right) \le \frac{1}{2}$

☐ Relaxation $\epsilon \to 0$: To enforce $\Pi \to q$ and $u \to \beta$

$$\begin{cases} \partial_t \mathbf{U} = \begin{pmatrix} 0 \\ \frac{1}{\epsilon}(\beta - u) \\ 0 \\ 0 \\ \frac{1}{\epsilon}(p - \Pi) \end{pmatrix} \\ \mathbf{U}(x,0) = \mathbf{U}_i^{n+1,-} \end{cases} \Rightarrow \begin{cases} \mathbf{W}_i^{n+1} = \mathbf{W}_i^{n+1,-} \\ \Pi_i^{n+1} = q_i^{n+1} \\ u_i^{n+1} = \beta_i^{n+1} \end{cases}$$

$$\Rightarrow \qquad \mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x} \Big(\mathcal{F}(\mathbf{W}_i^n, \mathbf{W}_{i+1}^n) - \mathcal{F}(\mathbf{W}_{i-1}^n, \mathbf{W}_i^n) \Big)$$

□ Robustness

Assume the relaxation parameters $a_{i+\frac{1}{2}}$ large enough

Assume the relaxation parameters $\rho^{L,R}_{i+\frac{1}{2}}$ small enough

Then

$$E_i^n > 0 \text{ for all } i \in \mathbb{Z} \quad \Rightarrow \quad E_i^{n+1} > 0 \text{ for all } i \in \mathbb{Z}$$

$$\left| \frac{F_i^n}{cE_i^n} \right| < 1 \text{ for all } i \in \mathbb{Z} \quad \Rightarrow \quad \left| \frac{F_i^{n+1}}{cE_i^{n+1}} \right| < 1 \text{ for all } i \in \mathbb{Z}$$

□ Accuracy

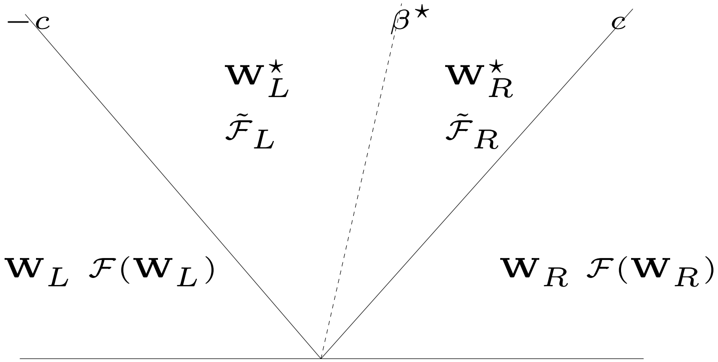
The 2D extension preserves the stationary contact discontinuities

□ The asymptotic preserving correction can be adopted

HLLC scheme

□ The linearized variables

$$\begin{cases} \partial_t E + \partial_x F = 0 & F = \beta(E+\Pi) \\ \partial_t F + \partial_x c^2 P = 0 & c^2 P = \beta F + c^2 \Pi \end{cases}$$



$$\beta_L^\star = \beta_R^\star = \beta^\star \qquad \Pi_L^\star = \Pi_R^\star = \Pi^\star$$

$$F_{LR}^\star = \tilde{F}_{LR} = \beta^\star (E_{LR}^\star + \Pi^\star)$$

$$c^2 \tilde{P}_{LR} = \beta^\star \tilde{F}_{LR} + c^2 \Pi^\star$$

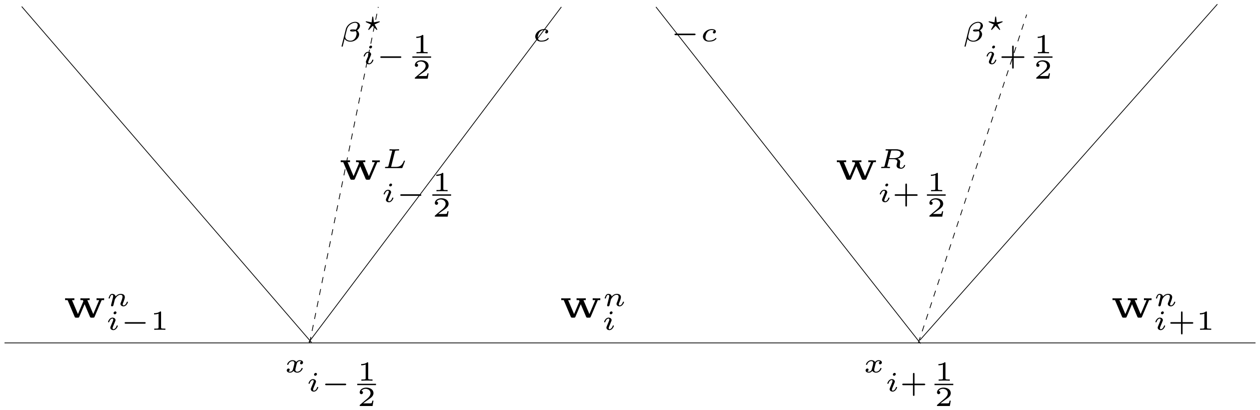
Lemma Le be given $\mathbf{W}_L$ and $\mathbf{W}_R$ two admissible states. Assume the Rankine-Hugoniot relations

$$\begin{cases} -c(E_L^\star - E_L) = \tilde{\mathbf{F}}_L - \mathbf{F}_L \\ c(E_R^\star - E_R) = \tilde{\mathbf{F}}_R - \mathbf{F}_R \end{cases} \qquad \begin{cases} -c(F_L^\star - F_L) = c^2(\tilde{P}_L - P_L) \\ c(F_R^\star - F_R) = c^2(\tilde{P}_R - P_R) \end{cases}$$

then there exist a unique $\mathbf{W}_L^\star$ and $\mathbf{W}_R^\star$ such that

$$E_{LR}^\star > 0 \qquad \text{and} \qquad \left| \frac{F_{LR}^\star}{c E_{LR}^\star} \right| < 1$$

☐ The HLLC scheme



CFL restriction $c\dfrac{\Delta t}{\Delta x} \le \dfrac{1}{2}$

$$\mathbf{W}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \mathbf{W}^h(x, t^n + \Delta t)dx$$
$$= \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+\frac12} - \mathcal{F}_{i-\frac12})$$

Lemma

Assume $E_i^n > 0$ and $\left|\frac{F_i^n}{cE_i^n}\right| < 1$ for all $i \in \mathbb{Z}$

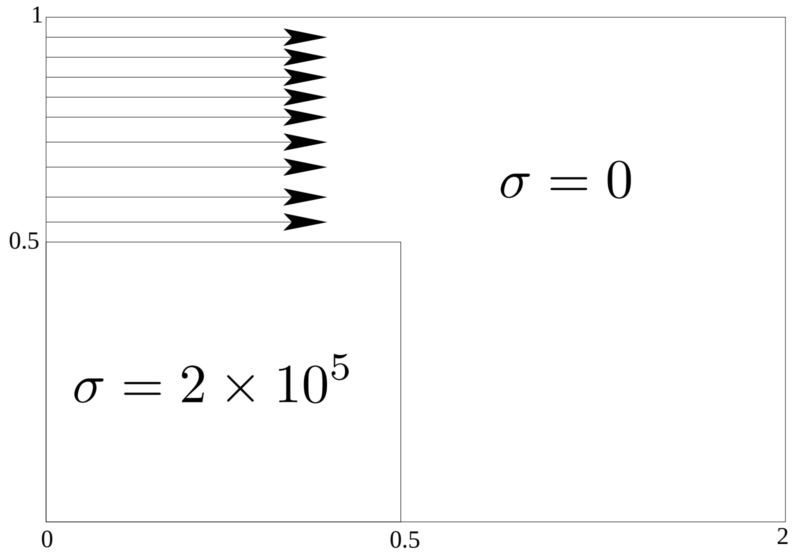
Tehn $E_i^{n+1} > 0$ and $\left|\frac{F_i^{n+1}}{cE_i^{n+1}}\right| < 1$ for all $i \in \mathbb{Z}$

☐ Accuracy

The 2D extension preserves the stationary contact discontinuities

☐ The asymptotic preserving correction can be adopted

2D case: shadow cone



Temperature on the left side of the transparent region $T = 5.8\ 10^6$

Initial temperature in the dense region $T = 1$

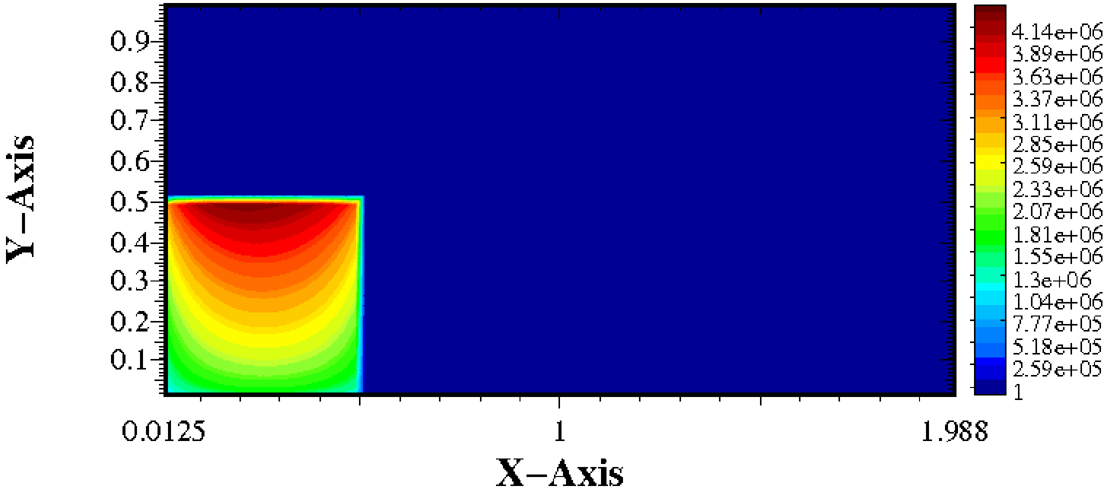
Initial temperature in the transparent region $T = 300$

Expected solution

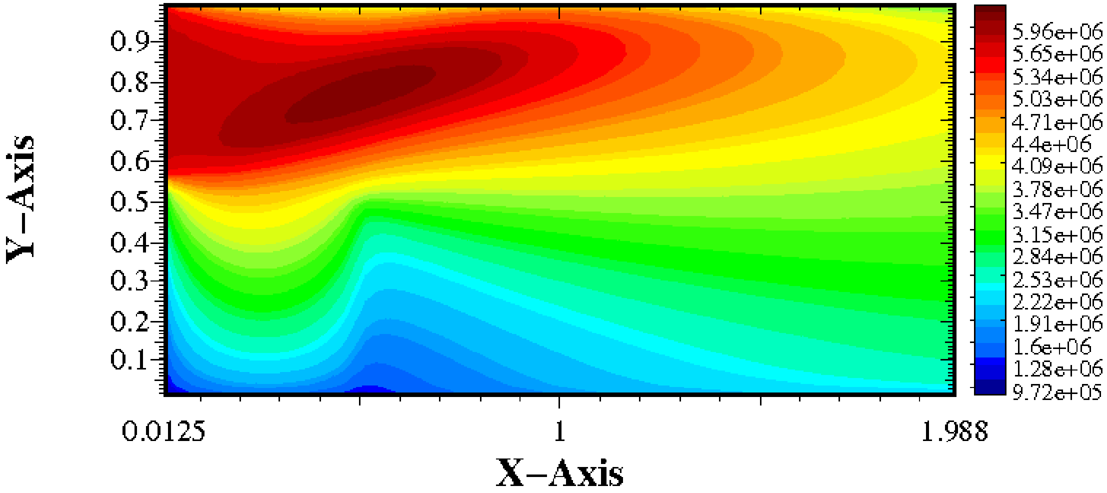
- Upper part: free steaming
- Lower part: constant solution, no photon enters this area
- $y = 0.5$: stationary contact discontinuity
- Temperature in the dense region $T = 1$

Shadow cone with a standard HLL scheme

Matter temperature

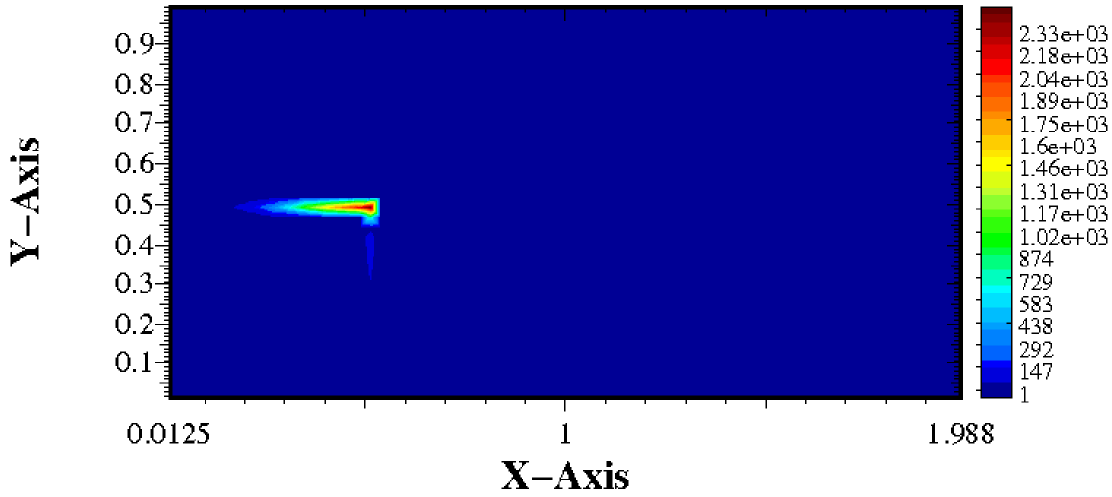


Radiative temperature

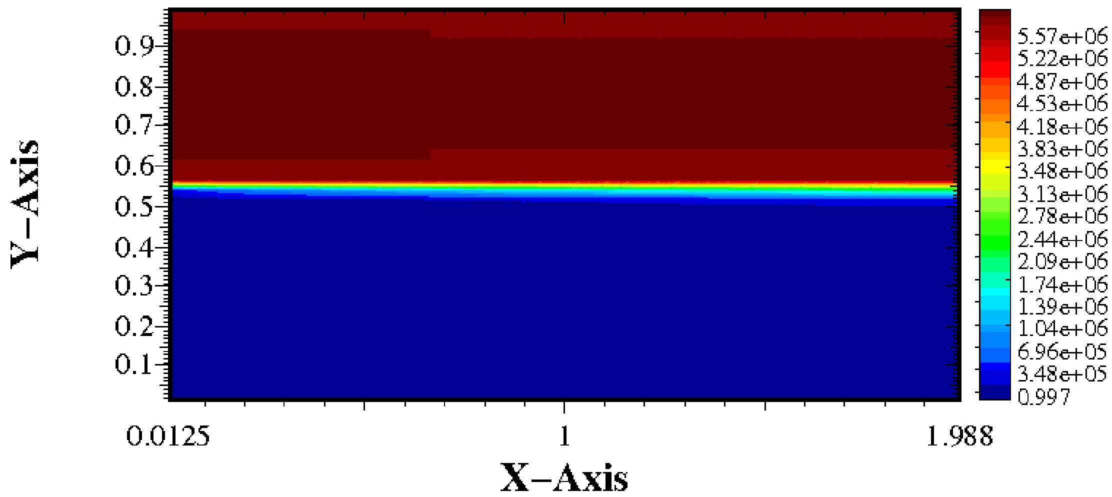


Shadow cone with the HLLC scheme

Matter temperature



Radiative temperature



Material temperature in the dense region

Exact solution is $T = 1$

Schemes	max material T		average material T	
	HLL	HLLC	HLL	HLLC
b_R, b_L csts	4300000	17000	350000	840
b_R, b_L variables	3700000	170000	290000	11000
b_R, b_L csts + AP	3600000	9000	43000	40
b_R, b_L variables + AP	740000	110000	5200	690
b_R, b_L csts + Minmod	3600000	16000	240000	340
b_R, b_L csts + Minmod + AP	2700000	6400	34000	26
b_R, b_L csts + Superbee + AP	1600000	2400	11000	6.1

From ERT to Radiotherapy

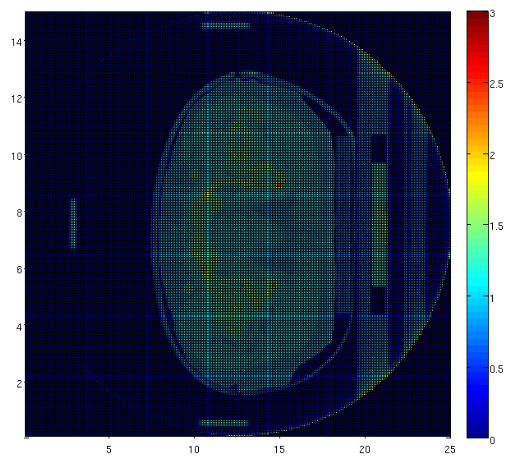
□ Modelisation of electron flows

↪ Use of ionizing radiation to treat cancer

↪ Aim: destroy the cancer cells and preserve healthy cells

↪ One or several beams sent into the body of the patient

↪ Linear accelerator: x-rays of very high energy



□ Extension: the M1 model to electrons

$$\partial_t E + \partial_x F = \partial_\varepsilon \rho(x) S(\varepsilon) E$$

$$\partial_t F + \partial_x E\chi(\frac{F}{E}) = \partial_\varepsilon \rho(x) S(\varepsilon) F - \rho(x) T(\varepsilon) F$$

$\varepsilon > 0$ an energy

Numerical simulations $\Leftrightarrow$ Numerical approximations of the M1 model

□ Extension: the M1 model to electrons

$$\partial_t \Psi^0 + \partial_x \Psi^1 = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^0$$

$$\partial_t \Psi^1 + \partial_x \Psi^0 \chi(\frac{\Psi^1}{\Psi^0}) = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^1 - \rho(x) T(\varepsilon) \Psi^1$$

$\varepsilon > 0$ the energy

Numerical simulations $\Leftrightarrow$ Numerical approximations of the M1 model

□ Extension: the M1 model to electrons

$$\partial_t \Psi^0 + \partial_x \Psi^1 = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^0$$

$$\partial_t \Psi^1 + \partial_x \Psi^0 \chi(\frac{\Psi^1}{\Psi^0}) = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^1 - \rho(x) T(\varepsilon) \Psi^1$$

$\varepsilon > 0$ the energy

□ Numerical evaluation of the dose

$$D \quad = \int_0^{+\infty} S(\varepsilon) \Psi^0 d\varepsilon$$

$\hookrightarrow$ Quadrature $\Rightarrow$ Evaluation of $(\Psi^0(x, t, \varepsilon^p))_p$ with p large

M1 model must be solved p times

☐ Extension: the M1 model to electrons

$$\partial_t \Psi^0 + \partial_x \Psi^1 = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^0$$

$$\partial_t \Psi^1 + \partial_x \Psi^0 \chi\left(\frac{\Psi^1}{\Psi^0}\right) = \partial_\varepsilon \rho(x) S(\varepsilon) \Psi^1 - \rho(x) T(\varepsilon) \Psi^1$$

$\varepsilon > 0$ the energy

☐ Numerical evaluation of the dose

$$D(x) = \int_0^{+\infty} S(\varepsilon) \Psi^0 d\varepsilon$$

$\hookrightarrow$ Quadrature $\Rightarrow$ Evaluation of $(\Psi^0(x,t,\varepsilon^p))_p$ with p large

M1 model must be solved p times

$\hookrightarrow$ Ψ^0 given by the stationary regime $t \to +\infty$

☐ Objective: FAST

$\hookrightarrow$ Robustness

$\hookrightarrow$ $\rho(x)$ is a given stiff function

Reformulation

☐ Stationary model

$$\left\{\begin{array}{l} \partial_x \Psi^1 = \partial_\varepsilon \rho S \Psi^0 \\ \partial_x \Psi^0 \chi = \partial_\varepsilon \rho S \Psi^1 - \rho T \Psi^1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \partial_\varepsilon \rho S \Psi^0 - \partial_x \Psi^1 = 0 \\ \partial_\varepsilon \rho S \Psi^1 - \partial_x \Psi^0 \chi = \rho T \Psi^1 \end{array}\right.$$

ε as a pseudo time

☐ Backwards model

$$\left\{\begin{array}{l} \lim\limits_{\varepsilon \to \infty} \Psi^0 = 0 \\ \lim\limits_{\varepsilon \to \infty} \Psi^1 = 0 \end{array}\right. \quad \text{with fast decay} \quad D(x) \sim \int_0^{\varepsilon_{\max}} S(\varepsilon) \Psi^0 d\varepsilon$$

$\hookrightarrow$ Approximation of “initial” data

$$\Psi^0(x, \varepsilon_{\max}) = 0 \qquad \Psi^1(x, \varepsilon_{\max}) = 0$$

$\varepsilon_{\max}$ large enough

Evaluation of $\Psi^0(x, \varepsilon)$ and $\Psi^1(x, \varepsilon)$ with $\varepsilon \in (0, \varepsilon_{\max})$

Changes of variables

□ Distortion of the phase space: $\rho(x)$ and $S(\varepsilon)$

$$\begin{cases} \partial_\varepsilon \rho S \Psi^0 - \partial_x \Psi^1 = 0 \\ \\ \partial_\varepsilon \rho S \Psi^1 - \partial_x \Psi^0 \chi(\dfrac{\Psi^1}{\Psi^0}) = \rho T \Psi^1 \end{cases}$$

Changes of variables

□ Distortion of the phase space: $\rho(x)$ and $S(\varepsilon)$

$$\begin{cases} \partial_\varepsilon S\Psi^0 - \dfrac{1}{\rho}\partial_x\Psi^1 = 0 \\ \partial_\varepsilon S\Psi^1 - \dfrac{1}{\rho}\partial_x\Psi^0\chi(\dfrac{\Psi^1}{\Psi^0}) = T\Psi^1 \end{cases}$$

$$S\Psi^0 = \hat{\Psi}^0 \qquad S\Psi^1 = \hat{\Psi}^1$$

Changes of variables

□ Distortion of the phase space: $\rho(x)$ and $S(\varepsilon)$

$$\begin{cases} \partial_\varepsilon \hat{\Psi}^0 - \frac{1}{\rho}\partial_x \frac{\hat{\Psi}^1}{S} = 0 \\ \partial_\varepsilon \hat{\Psi}^1 - \frac{1}{\rho}\partial_x \frac{\hat{\Psi}^0}{S} \chi(\frac{\hat{\Psi}^1}{\hat{\Psi}^0}) = \frac{T}{S}\hat{\Psi}^1 \end{cases}$$

$$S\Psi^0 = \hat{\Psi}^0 \qquad S\Psi^1 = \hat{\Psi}^1$$

Changes of variables

☐ Distortion of the phase space: $\rho(x)$ and $S(\varepsilon)$

$$\begin{cases} S(\varepsilon)\partial_\varepsilon \hat{\Psi}^0 - \dfrac{1}{\rho(x)}\partial_x \hat{\Psi}^1 = 0 \\ \\ S(\varepsilon)\partial_\varepsilon \hat{\Psi}^1 - \dfrac{1}{\rho(x)}\partial_x \hat{\Psi}^0 \chi(\dfrac{\hat{\Psi}^1}{\hat{\Psi}^0}) = T\hat{\Psi}^1 \end{cases}$$

$$S\Psi^0 = \hat{\Psi}^0 \qquad S\Psi^1 = \hat{\Psi}^1$$

Changes of variables

☐ Distortion of the phase space: $\rho(x)$ and $S(\varepsilon)$

$$\begin{cases} \partial_{\tilde{\varepsilon}} \tilde{\Psi}^0 - \partial_{\tilde{x}} \tilde{\Psi}^1 = 0 \\ \partial_{\tilde{\varepsilon}} \tilde{\Psi}^1 - \partial_{\tilde{x}} \tilde{\Psi}^0 \chi(\dfrac{\tilde{\Psi}^1}{\tilde{\Psi}^0}) = \tilde{T}(\tilde{\varepsilon}) \tilde{\Psi}^1 \end{cases}$$

$$S\Psi^0 = \hat{\Psi}^0(x,\varepsilon) = \tilde{\Psi}^0(\tilde{x},\tilde{\varepsilon}) \qquad S\Psi^1 = \hat{\Psi}^1(x,\varepsilon) = \tilde{\Psi}^1(\tilde{x},\tilde{\varepsilon})$$

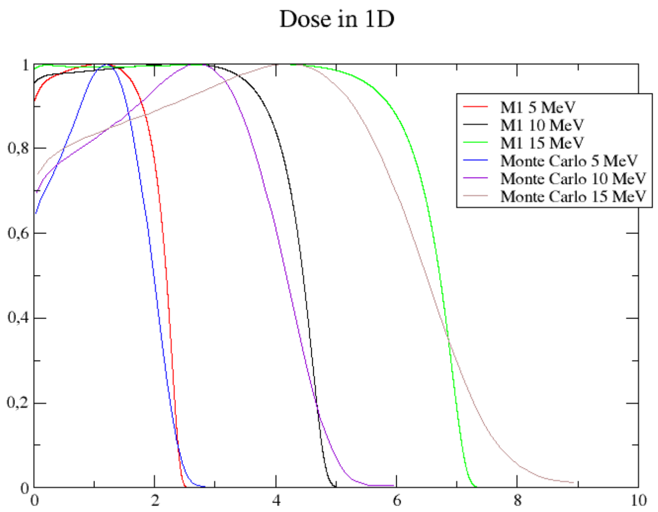
with

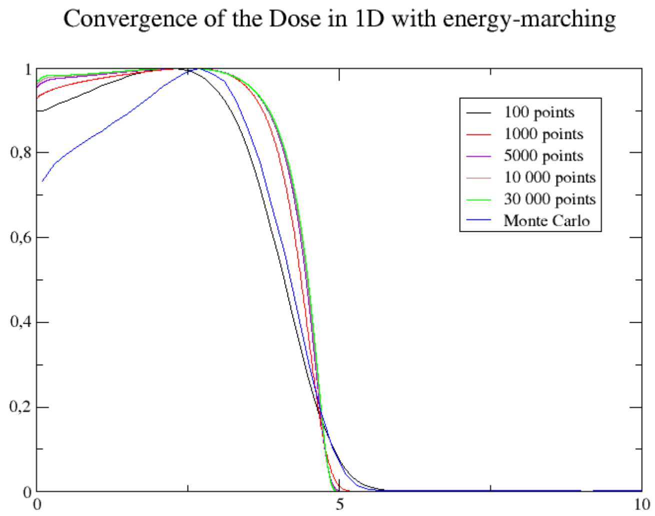
$$\tilde{x}(x) = \int_0^x \rho(t)dt \qquad \tilde{\varepsilon}(\varepsilon) = \int_0^\varepsilon \frac{1}{S(t)} dt$$

☐ Numerical approximation

$\hookrightarrow$ HLL scheme

1D valadation tests

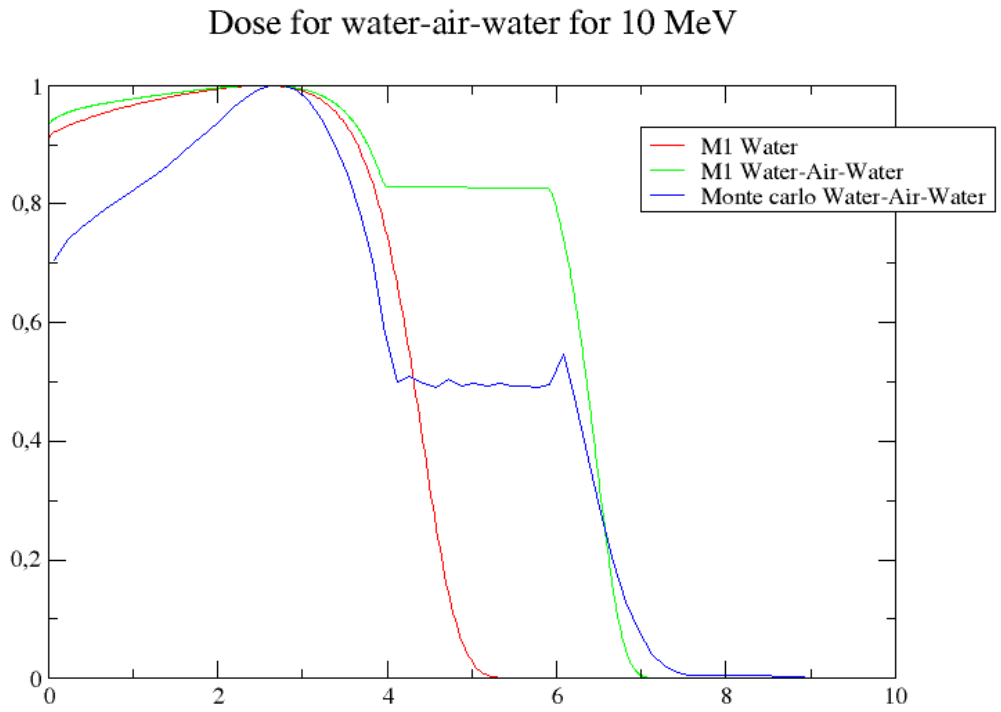




CPU time

- M1 model (5000 nodes) $\simeq$ 30 secondes
- Monte Carlo method $\simeq$ 10 minutes

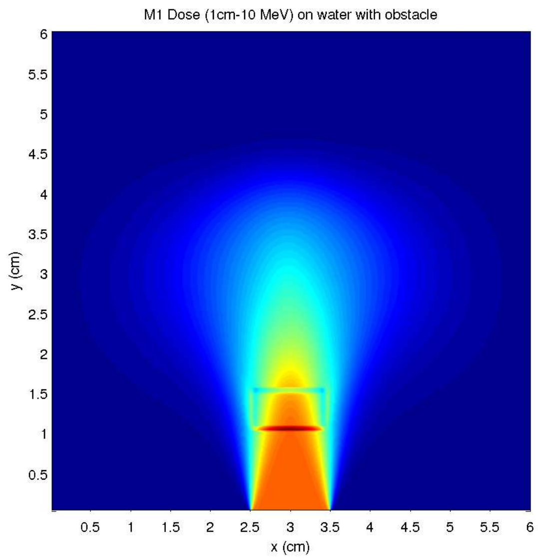
Discontinuous density

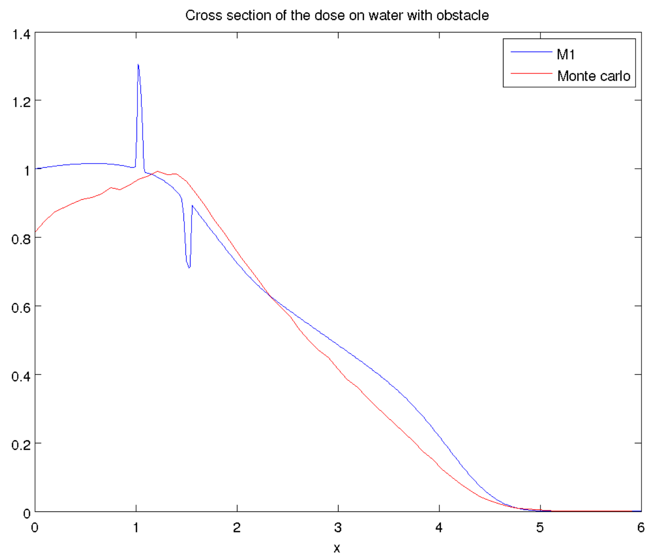


2D extension

It does not exist glogal 2D change of variables $(\tilde{x}, \tilde{y})$

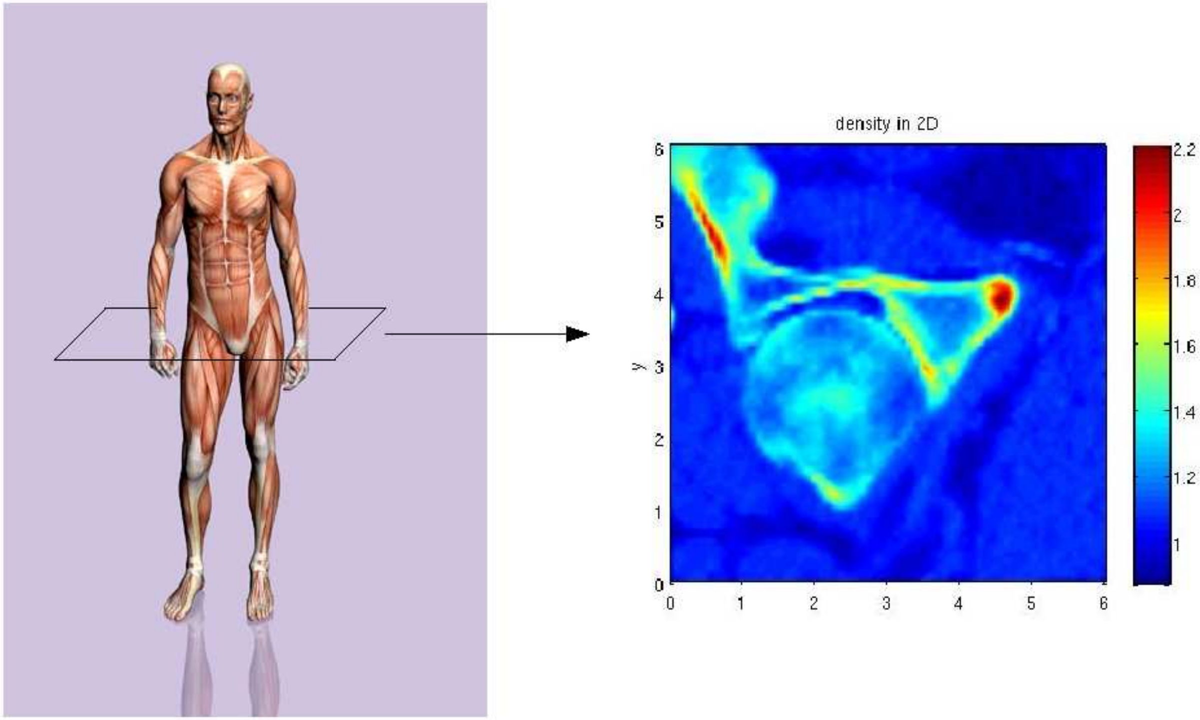
- ☐ Algorithm based on local space distortions



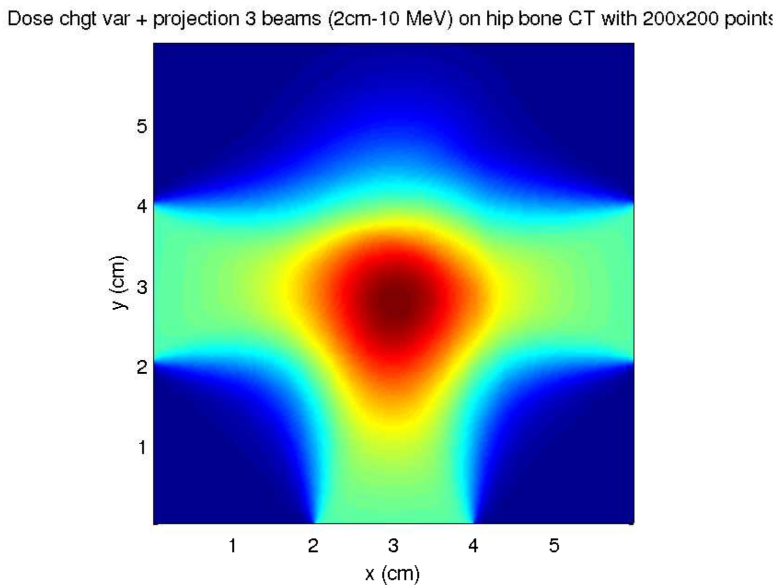


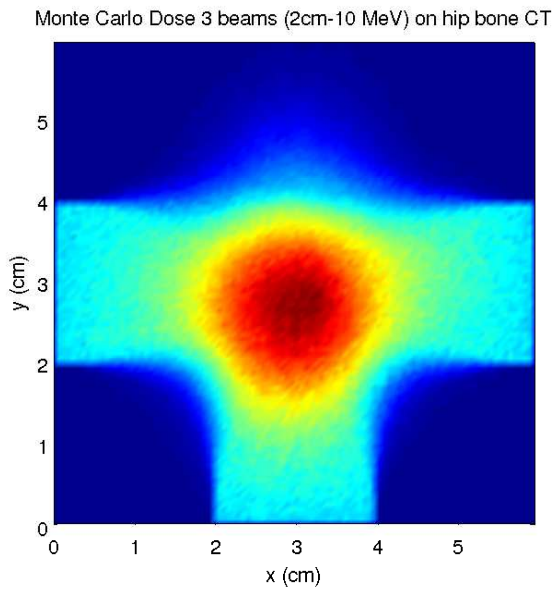
CT scan

Scanner at hip level of a man,



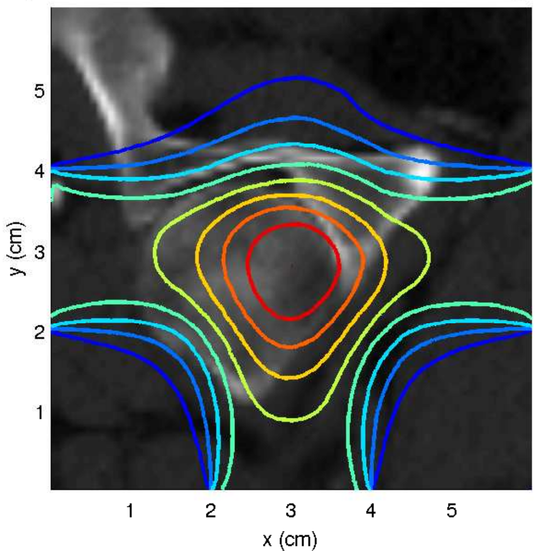
Cut an interesting part with different tissues.



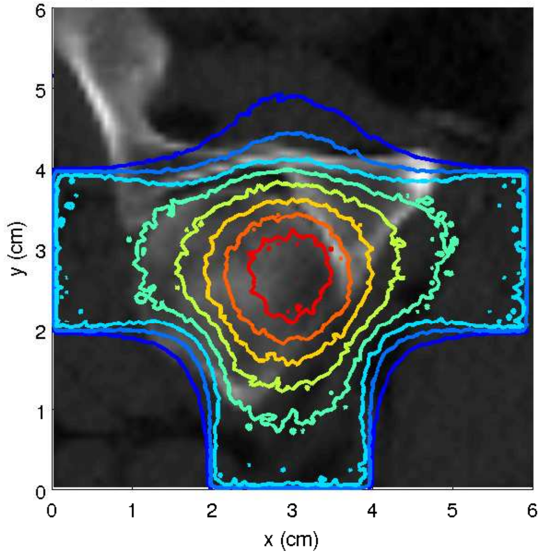


- M1 model (200x200) $\simeq$ 1 hour/ray
- Monte Carlo method $\simeq$ 23 hours/ray

Hip bone CT and contour of the 3 M1 dose with 200x200 points



Hip bone CT and contour of the Monte Carlo dose



Thanks for your attention