A relaxation method for conservation laws via the Born-Infeld system

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Outline

A class of scalar conservation laws

- Some simplified two-phase flow models can be reduced to the equation

$$\partial_t u + \partial_x (u(1-u)g(u)) = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

where $u(t,x) \in [0,1]$ and $g \in \mathscr{C}^1([0,1];\mathbb{R})$.

- The unknown u represents a volume- or mass-fraction, while

$$w(u) = (1-u)g(u) \qquad \text{and} \qquad z(u) = -ug(u) \tag{2}$$

play the role of convective phase velocities, since

$$\partial_t(u) \qquad + \partial_x(u \cdot w(u)) \qquad = 0, \tag{3a}$$

$$\partial_t(1-u) + \partial_x((1-u) \cdot z(u)) = 0. \tag{3b}$$

- The slip velocity$g(u) = w(u) - z(u)$ is assumed to keep a constant sign, i.e., $0 \notin g(]0,1[)$.

Need for a new relaxation scheme

- The scalar conservation law (1) is embedded in a larger system, which contains additional equations of the type

$$\partial_t \alpha_k + \partial_x(\alpha_k \cdot w(u)) = 0, \tag{4a}$$

$$\partial_t \beta_\ell + \partial_x(\beta_\ell \cdot z(u)) = 0, \tag{4b}$$

  where the α_k's and β_ℓ's denote the species of the mixture.
- The phase velocities w and z have to be always well-defined. But standard numerical methods for (1), such as the semi-linear relaxation, do not guarantee this property.
- Design a suitable relaxation method for this problem, based (surprisingly) on a system called Born-Infeld. Can be studied *per se* and has interesting extensions [M3AS **19** (2009), 1–38].

Outline

A general-purpose relaxation strategy

- Consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{5}$$

where $u(t,x) \in [0,1]$ and $f(.) \in \mathscr{C}^1([0,1];\mathbb{R})$ is a nonlinear flux function.

- To construct admissible weak solutions and to design robust numerical schemes, Jin and Xin (1995) proposed the semi-linear relaxation

$$\partial_t U^\lambda + \quad \partial_x F^\lambda = 0, \tag{6a}$$

$$\partial_t F^\lambda + a^2 \partial_x U^\lambda = \lambda [f(U^\lambda) - F^\lambda], \tag{6b}$$

where F^λ is a full-fledged variable, maintained close to $f(U^\lambda)$ by choosing large λ.

Diagonal form of the relaxation system

- The relaxation system (6) is linear with eigenvalues $\pm a$.

$$\partial_t\left(\frac{U^\lambda}{2}-\frac{F^\lambda}{2a}\right)-a\,\partial_x\left(\frac{U^\lambda}{2}-\frac{F^\lambda}{2a}\right)=\lambda\left[\left(\frac{U^\lambda}{2}-\frac{f(U^\lambda)}{2a}\right)-\left(\frac{U^\lambda}{2}-\frac{F^\lambda}{2a}\right)\right]$$
$$\partial_t\left(\frac{U^\lambda}{2}+\frac{F^\lambda}{2a}\right)+a\,\partial_x\left(\frac{U^\lambda}{2}+\frac{F^\lambda}{2a}\right)=\lambda\left[\left(\frac{U^\lambda}{2}+\frac{f(U^\lambda)}{2a}\right)-\left(\frac{U^\lambda}{2a}+\frac{F^\lambda}{2a}\right)\right]$$

- It can be given a kinetic interpretation

$$\partial_t K^\lambda(t,x,\xi)+\xi\,\partial_x K^\lambda(t,x,\xi)=\lambda\left[k\left(\int K^\lambda(t,x,\zeta)\mathrm{d}\zeta,\,\xi\right)-K^\lambda(t,x,\xi)\right],$$

with $\xi\in\{-a,+a\}$ and a Maxwellian $k(.,.)$ such that

$$u=\int_{\{-a,+a\}}k(u,\xi)\,\mathrm{d}\xi \quad \text{and} \quad f(u)=\int_{\{-a,+a\}}\xi k(u,\xi)\,\mathrm{d}\xi. \tag{8}$$

Approximation properties

- Inserting the Chapman-Enskog expansion

$$F^\lambda = f(U^\lambda) + \frac{1}{\lambda} F_1^\lambda + O\left(\frac{1}{\lambda^2}\right) \tag{9}$$

into the relaxation system, we obtain the equivalent equation

$$\partial_t U^\lambda + \partial_x f(U^\lambda) = \frac{1}{\lambda} \partial_x \{ (a^2 - [f'(U^\lambda)]^2) \partial_x U^\lambda \}. \tag{10}$$

- For dissipativeness, we require the subcharacteristic condition

$$f'(u) \in [-a, +a], \qquad \forall u \in [0,1]. \tag{11}$$

- Under the subcharacteristic condition and suitable assumptions on the initial data, the sequence U^λ can be shown to converge (in L^∞ weak$*$) to the entropy solution of the original scalar conservation law as $\lambda \to +\infty$.

First-order explicit scheme

- Splitting between differential and source terms.
  - $\lambda = \infty$. Set the data at time n to equilibrium, i.e.,

$$F_i^n = f(u_i^n). \tag{12}$$

  - $\lambda = 0$. Solve the Riemann problem associated with the relaxation system at each edge $i+1/2$. The intermediate state (U^*, F^*) is subject to

$$aU^* - F^* = au_L - F_L, \tag{13a}$$

$$aU^* + F^* = au_R + F_R. \tag{13b}$$

- Update formulae

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[F^*(u_i^n, u_{i+1}^n) - F^*(u_{i-1}^n, u_i^n)], \tag{14}$$

with

$$F^*(u_L, u_R) = \frac{f(u_L) + f(u_R)}{2} - a\frac{u_R - u_L}{2}. \tag{15}$$

Maximum principles

$$U^*(u_L,u_R) = \frac{u_L+u_R}{2} - \frac{f(u_R)-f(u_L)}{2a} \tag{16}$$

- satisfies the local maximum principle $U^* \in \lfloor u_L, u_R \rceil$ provided that

$$a \geq \left| \frac{f(u_R)-f(u_L)}{u_R-u_L} \right|, \tag{17}$$

  which is implied by the subcharacteristic condition;

- satisfies the global maximum principle $U^* \in [0,1]$ provided that

$$a \geq \max\left\{ \frac{f(u_R)-f(u_L)}{u_R+u_L}, -\frac{f(u_R)-f(u_L)}{(1-u_R)+(1-u_L)} \right\}; \tag{18}$$

  for $f(u) = u(1-u)g(u)$, a simpler and sufficient condition is

$$a \geq \max\{|g(u_L)|, |g(u_R)|\}. \tag{19}$$

Troubleshoots

- Since $F^* \neq f(U^*) = U^*(1-U^*)g(U^*)$, the intermediate velocities

$$W^* = \frac{F^*}{U^*} \qquad Z^* = -\frac{F^*}{1-U^*} \tag{20}$$

  may become unbounded as u_L and u_R go to 0 or 1.

- However, we need W^* and Z^* in order to discretize the additional equations

$$\partial_t \alpha_k + \partial_x(\alpha_k \cdot w(u)) = 0, \tag{21a}$$

$$\partial_t \beta_\ell + \partial_x(\beta_\ell \cdot z(u)) = 0. \tag{21b}$$

- In order to achieve some maximum principle on w and z, we have to take advantage of the form

$$f(u) = u(1-u)g(u). \tag{22}$$

Outline

Change of variables

- Equilibrium variables

$$w(u) = (1-u)g(u) = \frac{f(u)}{u}, \qquad u = \frac{z(u)}{z(u)-w(u)}, \tag{23a}$$

$$z(u) = -ug(u) = -\frac{f(u)}{1-u}, \qquad f(u) = \frac{w(u)z(u)}{z(u)-w(u)}. \tag{23b}$$

The pair (w,z) belongs to either $\{w \geq 0, z \leq 0\}$ or $\{w \leq 0, z \geq 0\}$.

- Relaxation variables

$$W(U,F) = (1-U)G = \frac{F}{U}, \qquad U(W,Z) = \frac{Z}{Z-W}, \tag{24a}$$

$$Z(U,F) = -UG = -\frac{F}{1-U}, \qquad F(W,Z) = \frac{WZ}{Z-W}. \tag{24b}$$

The pair (W,Z) belongs to the same quarter-plane as (w,z). The pair (U,F) is not subject to the constraint $F = f(U)$.

Definition of the relaxation system

- The Born-Infeld relaxation system for the scalar conservation law (1) is defined as

$$\partial_t W^\lambda + Z^\lambda \partial_x W^\lambda = \lambda [w(U(W^\lambda, Z^\lambda)) - W^\lambda], \qquad (25a)$$
$$\partial_t Z^\lambda + W^\lambda \partial_x Z^\lambda = \lambda [z(U(W^\lambda, Z^\lambda)) - Z^\lambda], \qquad (25b)$$

  where $\lambda > 0$ is the relaxation coefficient.
- When $\lambda = 0$, the above system coincides with a reduced form of the Born-Infeld equations, or more accurately, with a plane-wave subset of the augmented Born-Infeld system by Brenier (2004).
- The eigenvalues (Z^λ, W^λ) of (25), both linearly degenerate, are respectively associated with the strict Riemann invariants W^λ and Z^λ.

From the diagonal to the conservative form

- For all $\lambda > 0$, we recover

$$\partial_t U(W^\lambda, Z^\lambda) + \partial_x F(W^\lambda, Z^\lambda) = 0 \tag{26}$$

  by means of a nonlinear combination.

- For all $\lambda > 0$, the Born-Infeld relaxation system (25) is equivalent to the system

$$\partial_t U^\lambda + \partial_x (U^\lambda (1 - U^\lambda) G^\lambda) = 0, \tag{27a}$$

$$\partial_t G^\lambda + (G^\lambda)^2 \partial_x U^\lambda = \lambda [g(U^\lambda) - G^\lambda]. \tag{27b}$$

  The second equation can transformed into the conservative form

$$\partial_t((1-2U^\lambda)G^\lambda) - \partial_x(U^\lambda(1-U^\lambda)(G^\lambda)^2) = \lambda(1-2U^\lambda)[g(U^\lambda) - G^\lambda].$$

Chapman-Enskog analysis

- Inserting the formal expansion

$$G^\lambda = g(U^\lambda) + \lambda^{-1} g_1^\lambda + O(\lambda^{-2}) \tag{28}$$

into the relaxation system yields the equivalent equation

$$\partial_t U^\lambda + \partial_x f(U^\lambda) = \frac{1}{\lambda} \partial_x \{ -[f'(U^\lambda) - w(U^\lambda)][f'(U^\lambda) - z(U^\lambda)] \partial_x U^\lambda \}.$$

- A sufficient condition for this to be a dissipative approximation to the original equation is that the subcharacteristic condition

$$f'(u) \in \lfloor w(u), z(u) \rceil, \tag{29}$$

holds true for all u in the range of the problem at hand.

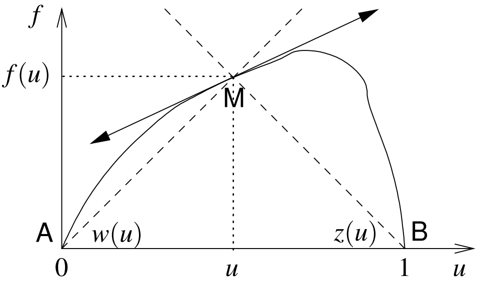
- Notation: $\lfloor \mathfrak{a}, \mathfrak{b} \rceil = \{ r\mathfrak{a} + (1-r)\mathfrak{b},\, r \in [0,1] \}$.

Geometric interpretation

The subcharacteristic condition

$$f'(u) \in \lfloor w(u), z(u) \rceil = \left\lfloor \frac{f(u)-f(0)}{u-0}, \frac{f(1)-f(u)}{1-u} \right\rceil \tag{30}$$

can be seen as a comparison between the slopes of 3 lines.



Eligible flux functions

- Two practical criteria
  - The subcharacteristic condition (29) is satisfied at $u \in]0,1[$ if and only if the functions $w(.)$ and $z(.)$ are
    - decreasing at u in the case $g(.) > 0$,
    - increasing at u in the case $g(.) < 0$.
  - The subcharacteristic condition (29) is satisfied at $u \in]0,1[$ if and only if

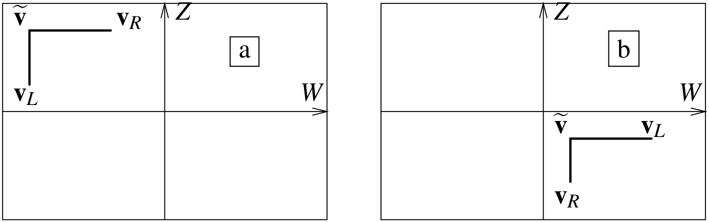
$$g'(u) \in \left[-\frac{g(u)}{u}, \frac{g(u)}{1-u} \right]. \tag{31}$$

- Examples
  - convex or concave functions f with $f(0) = f(1) = 0$ and $0 \notin f(]0,1[)$;
  - for $n \geq 2$, the function

$$f(u) = u(1-u) \left[1 + \frac{\sin(2\pi n u)}{2\pi n} \right] \tag{32}$$

is admissible, although f'' does not have a constant sign.

Solving Riemann problems



Starting from $\mathbf{v}_L = (W_L, Z_L)$ and $\mathbf{v}_R = (W_R, Z_R)$, we have

$$\mathbf{v}(t,x) = \mathbf{v}_L 1_{\{x<s_L t\}} + \widetilde{\mathbf{v}} 1_{\{s_L t<x<s_R t\}} + \mathbf{v}_R 1_{\{x>s_R t\}}, \tag{33}$$

with

$$s_L = \min(W_L, Z_L) \leq 0, \tag{34a}$$

$$s_R = \max(W_R, Z_R) \geq 0, \tag{34b}$$

$$\widetilde{\mathbf{v}} = (\widetilde{W}, \widetilde{Z}) = (W_L^- + W_R^+, Z_L^- + Z_R^+). \tag{34c}$$

Deducing maximum principles

- The intermediate velocities $\widetilde{\mathbf{v}} = (\widetilde{W}, \widetilde{Z})$ obeys the local maximum principle

$$\widetilde{W} \in \lfloor W_L, W_R \rceil, \qquad \widetilde{Z} \in \lfloor Z_L, Z_R \rceil \tag{35}$$

unconditionally.

- The intermediate fraction

$$\widetilde{U} = U(\widetilde{W}, \widetilde{Z}) = \frac{\widetilde{Z}}{\widetilde{Z} - \widetilde{W}} \tag{36}$$

obeys the local maximum principle $\widetilde{U} \in \lfloor u_L, u_R \rceil$ provided that

$$[w(u_R) - w(u_L)][z(u_R) - z(u_L)] \geq 0, \tag{37}$$

which is implied by the subcharacteristic condition;

- The intermediate fraction $\widetilde{U}$ obeys the global maximum principle $\widetilde{U} \in [0, 1]$ unconditionally.

First-order explicit scheme

- Update formulae

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[\widetilde{F}(u_i^n, u_{i+1}^n) - \widetilde{F}(u_{i-1}^n, u_i^n)], \tag{38}$$

with

$$\widetilde{F}(u_L, u_R) = F(\widetilde{W}, \widetilde{Z}) = \begin{cases} \dfrac{w(u_L)z(u_R)}{z(u_R) - w(u_L)} & \text{if } g(.) < 0, \\[2ex] \dfrac{w(u_R)z(u_L)}{z(u_L) - w(u_R)} & \text{if } g(.) > 0. \end{cases} \tag{39}$$

- If the subcharacteristic condition is met, then the Born-Infeld flux $\widetilde{F}(u_L, u_R)$ is monotone, i.e.,

$$\frac{\partial \widetilde{F}}{\partial u_L}(u_L, u_R) \geq 0, \qquad \frac{\partial \widetilde{F}}{\partial u_R}(u_L, u_R) \leq 0. \tag{40}$$

Numerical results

- The flux and data

$$f(u) = u(1-u)(1+u), \tag{41a}$$
$$(u_L, u_R) = (0.5, 1) \tag{41b}$$

give rise to a shock wave propagating at the speed

$$\frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{0 - 0.375}{1 - 0.5} = -0.75. \tag{42}$$

- We compare the Born-Infeld relaxation with two variants of the Jin-Xin relaxation, namely,
  - JX1: uniform parameter

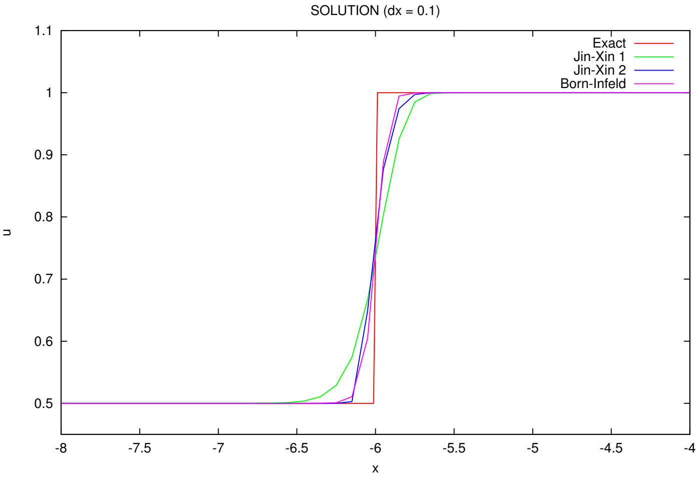
$$a = \max_{i \in \mathbf{Z}} |f'(u_i^n)|; \tag{43}$$

  - JX2: local parameter

$$a_{i+1/2} = \max\{|f'(u_i^n)|, |f'(u_{i+1}^n)|. \tag{44}$$

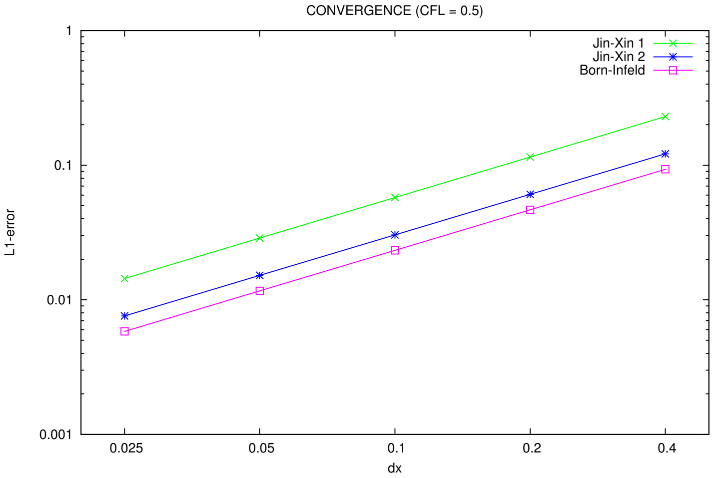
Solution snapshot

$T = 8$; CFL $= 0.5$



Convergence study

JX1 = 0.99992; JX2 = 1.00021; BI = 1.00573



Recapitulative review

- Jin-Xin relaxation
  - designed for any f
  - diagonal variables have no physical meaning, except for kinetic interpretation
  - require a parameter a, subject to subchar. cond.
  - tunable viscosity
  - local max. princ. on u thanks to subchar. cond. ; global max. princ. on u under cond. on a
  - no max. princ. on w and z
  - monotone numerical flux
  - discrete entropy inequality

- Born-Infeld relaxation
  - designed for $f = u(1-u)g$
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Outline

Porous media with discontinuous coefficients

- Simplistic model

$$\partial_t u + \partial_x (u(1-u)k) = 0, \quad (45a)$$
$$\partial_t k = 0, \quad (45b)$$

with $u \in [0,1]$ and $k > 0$, studied by Seguin and Vovelle (2003).

- Set

$$w(u,k) = (1-u)k, \qquad W = (1-U)K, \quad (46a)$$
$$z(u,k) = -uk, \qquad Z = -UK, \quad (46b)$$

and consider the relaxation model

$$\partial_t W^\lambda + Z^\lambda \, \partial_x W^\lambda = \lambda [w(U(W^\lambda, Z^\lambda), k) - W^\lambda], \quad (47a)$$
$$\partial_t Z^\lambda + W^\lambda \partial_x Z^\lambda = \lambda [z(U(W^\lambda, Z^\lambda), k) - Z^\lambda], \quad (47b)$$
$$\partial_t k = 0, \quad (47c)$$

where $U(W,Z) = Z/(Z-W)$.

Porous media with discontinuous coefficients

- Conservative form

$$\partial_t U^\lambda + \partial_x(U^\lambda(1-U^\lambda)K^\lambda) = 0, \tag{48a}$$

$$\partial_t K^\lambda + (K^\lambda)^2 \partial_x U^\lambda = \lambda[k - K^\lambda], \tag{48b}$$

$$\partial_t k = 0. \tag{48c}$$

- Chapman-Enskog analysis

$$\partial_t U^\lambda + \partial_x(U^\lambda(1-U^\lambda)k) = \lambda^{-1}\partial_x\{U^\lambda(1-U^\lambda)k^2\partial_x U^\lambda\}. \tag{49}$$

Dissipative approximation, no need for a subcharacteristic condition. Actually, we already have

$$\lfloor 0,(1-2u)k\rceil \subset \lfloor -uk,(1-u)k\rceil. \tag{50}$$

Compressible two-phase flow

- Drift-flux model in Eulerian coordinates

$$\partial_t(\rho) + \partial_x(\rho v) = 0, \tag{51a}$$
$$\partial_t(\rho v) + \partial_x(\rho v^2 + P(\mathbf{q})) = 0, \tag{51b}$$
$$\partial_t(\rho Y) + \partial_x(\rho Y v + \rho Y(1-Y)\phi(\mathbf{q})) = 0, \tag{51c}$$

with $\mathbf{q} = (\rho, \rho v, \rho Y)$. Here, $Y \in [0,1]$ is the gas mass-fraction and $\phi(\mathbf{q})$ is the slip velocity, given by a closure law.

- In addition to (51), passive transport

$$\partial_t(\rho\alpha_k) + \partial_x(\rho\alpha_k v + \rho\alpha_k(1-Y)\phi(\mathbf{q})) = 0, \tag{52a}$$
$$\partial_t(\rho\beta_\ell) + \partial_x(\rho\beta_k v - \rho\beta_\ell Y\phi(\mathbf{q})) = 0, \tag{52b}$$

of various partial component-fractions.

Compressible two-phase flow

- Lagrangian velocities

$$w(\mathbf{q}) = \rho(1-Y)\phi(\mathbf{q}), \quad z(\mathbf{q}) = -\rho Y\phi(\mathbf{q}), \quad g(\mathbf{q}) = \rho\phi(\mathbf{q}). \tag{53}$$

- Switch to Lagrangian coordinates first, work out the relaxation model, then go back to Eulerian coordinates.

$$\partial_t(\rho)^\lambda + \partial_x(\rho v)^\lambda = 0, \tag{54a}$$

$$\partial_t(\rho v)^\lambda + \partial_x(\rho v^2 + \Pi)^\lambda = 0, \tag{54b}$$

$$\partial_t(\rho\Pi)^\lambda + \partial_x(\rho\Pi v + a^2 v)^\lambda = \lambda\rho(P(\mathbf{q}^\lambda) - \Pi^\lambda), \tag{54c}$$

$$\partial_t(\rho Y)^\lambda + \partial_x(\rho Y v + Y(1-Y)G)^\lambda = 0, \tag{54d}$$

$$\partial_t(\rho G)^\lambda + \partial_x(\rho G v)^\lambda + (G^\lambda)^2\partial_x Y^\lambda = \lambda\rho(g(\mathbf{q}^\lambda) - G^\lambda). \tag{54e}$$

- Most “real-life” hydrodynamic laws ϕ satisfy the subcharacteristic condition.

General scalar conservation law

- The homogeneous Born-Infeld system

$$\partial_t W + Z\partial_x W = 0, \tag{55a}$$

$$\partial_t Z + W\partial_x Z = 0 \tag{55b}$$

has an entropy-entropy flux pair

$$\partial_t U(W,Z) + \partial_x F(W,Z) = 0 \tag{56}$$

if and only if

$$U(W,Z) = \frac{A(W)+B(Z)}{Z-W} \quad \text{and} \quad F(W,Z) = \frac{ZA(W)+WB(Z)}{Z-W}. \tag{57}$$

- Goursat equation

$$U_W - U_Z + (Z-W)U_{WZ} = 0. \tag{58}$$

General scalar conservation law

- The natural idea is to relax the conservation law $\partial_t u + \partial_x f(u) = 0$ by the generalized system

$$\partial_t W^\lambda + Z^\lambda \partial_x W^\lambda = \lambda W_F[f(U(W^\lambda, Z^\lambda)) - F(W^\lambda, Z^\lambda)], \tag{59a}$$

$$\partial_t Z^\lambda + W^\lambda \partial_x Z^\lambda = \lambda Z_F[f(U(W^\lambda, Z^\lambda)) - F(W^\lambda, Z^\lambda)]. \tag{59b}$$

- The difficulty lies in obtaining close-form expressions for $W(U,F)$ and $Z(U,F)$. Moreover, the equilibrium values

$$w(u) = W(u, f(u)), \qquad z(u) = Z(u, f(u)) \tag{60}$$

do not always have an obvious physical meaning but have to remain bounded.

- But an abstract framework can be worked out, in which *all* of the results obtained for $f = u(1-u)g$ can be extended. Most notably, the subcharacteristic condition and the monotonicity of the numerical flux.

Examples

- $A(W) = 0$ and $B(Z) = Z$ [linear BI] lead to $U = Z/(Z-W)$ and $F = WZ/(Z-W)$, the inverse of which is

$$W(U,F) = \frac{F}{U} \quad \text{and} \quad Z(U,F) = -\frac{F}{1-U}. \tag{61}$$

  If $f(u) = u(1-u)g(u)$, where g keeps a constant sign, then $w(u)$ and $z(u)$ remain well-defined.

- $A(W) = 0$ and $B(Z) = Z^2$ [quadratic BI] lead to $U = Z^2/(Z-W)$ and $F = WZ^2/(Z-W)$, the inverse of which is

$$W(U,F) = \frac{F}{U} \quad \text{and} \quad Z(U,F) = \frac{U + \sqrt{U^2 - 4F}}{2}. \tag{62}$$

  If $f(u) = -uh(u)$, where $h > 0$, then $w(u)$ and $z(u)$ remain well-defined.

Examples

- $A(W) = 0$ and $B(Z) = Z^3$ [cubic BI] lead to $U = Z^3/(Z - W)$ and $F = WZ^3/(Z - W)$, the inverse of which is

$$W(U,F) = \frac{F}{U} \quad \text{and} \quad Z(U,F) = \text{negative root of } Z^3 - UZ + F. \tag{63}$$

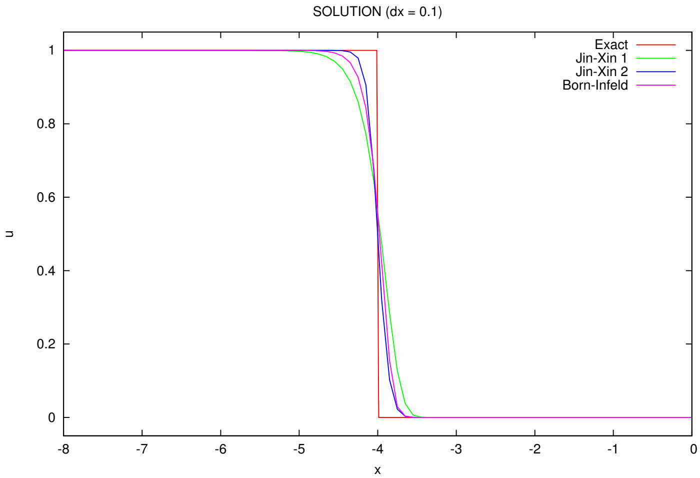
  If $f(u) = uh(u)$, where $h > 0$, then $w(u)$ and $z(u)$ remain well-defined.
- The function

$$h(u) = \frac{1}{(1+u)^{\alpha}}, \qquad 0 \leq \alpha \leq 1, \tag{64}$$

  satisfies the subcharacteristic condition for the quadratic and cubic Born-Infeld relaxations.

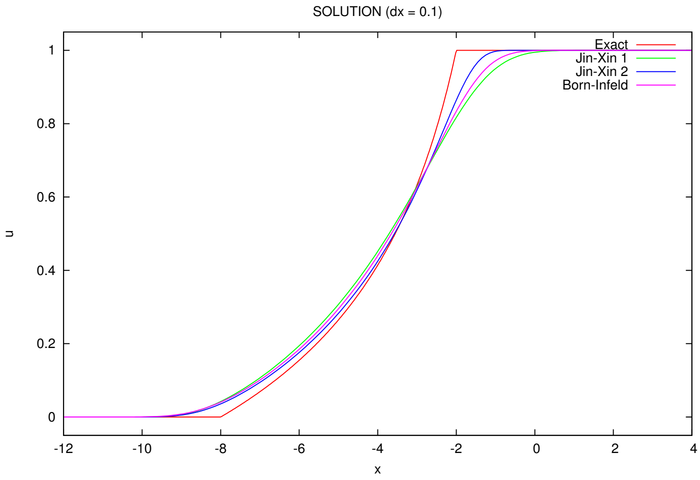
Solution snapshot

Quadratic Born-Infeld for $f(u) = -\dfrac{u}{1+u}$



Solution snapshot

Cubic Born-Infeld for $f(u) = \dfrac{u}{1+u}$



Rich systems

- Prototype of a 3×3 linearly degenerate rich system

$$\partial_t W_1 + (W_2 + W_3)\partial_x W_1 = 0, \tag{65a}$$

$$\partial_t W_2 + (W_3 + W_1)\partial_x W_2 = 0, \tag{65b}$$

$$\partial_t W_3 + (W_1 + W_2)\partial_x W_3 = 0. \tag{65c}$$

- The entropy-entropy flux pairs are of the form (Serre, 1992)

$$\partial_t(\mathscr{P}_j(\boldsymbol{W})q(W_j)) + \partial_x(\mathscr{P}_j(\boldsymbol{W})q(W_j)v_j(\boldsymbol{W})) = 0, \tag{66}$$

where $q(.)$ is any function,

$$\mathscr{P}_j(\boldsymbol{W}) = -\frac{1}{(W_j - W_i)(W_j - W_k)} \quad \text{and} \quad v_j(\boldsymbol{W}) = W_i + W_k. \tag{67}$$

- Another entropy-entropy flux pair is

$$\partial_t(W_1 + W_2 + W_3) + \partial_x(W_2 W_3 + W_3 W_1 + W_1 W_2) = 0. \tag{68}$$

An attempt at pressureless gas

- Taking $q(W_2) = K_0$ and $2W_2K_0$, we get

$$\partial_t(K_0\mathscr{P}_2(\boldsymbol{W})) + \partial_x(K_0\mathscr{P}_2(\boldsymbol{W})\cdot(W_3+W_1)) = 0, \quad (69a)$$

$$\partial_t(K_0\mathscr{P}_2(\boldsymbol{W})\cdot 2W_2) + \partial_x(K_0\mathscr{P}_2(\boldsymbol{W})\cdot 2W_2\cdot(W_3+W_1)) = 0. \quad (69b)$$

- If $W_3 + W_1 = 2W_2$, then we formally recover

$$\partial_t(\rho) + \partial_x(\rho u) = 0, \quad (70a)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = 0. \quad (70b)$$

Non-strictly hyperbolic, with the resonant eigenvalue u.

- The idea is therefore to supplement (69) with a third equation

$$\partial_t(W_1+W_2+W_3) + \partial_x(W_2W_3 + W_3W_1 + W_1W_2) = \lambda(2W_2 - (W_3+W_1)). \quad (71)$$

Pressureless gas. . . far from vacuum

- Conservative form

$$\partial_t(\rho)^\lambda + \partial_x(\rho V)^\lambda = 0, \tag{72a}$$

$$\partial_t(\rho u)^\lambda + \partial_x(\rho u V)^\lambda = 0, \tag{72b}$$

$$\partial_t(V+u/2)^\lambda + \partial_x(Vu - u^2/4 - K_0/\rho)^\lambda = \lambda(u-V)^\lambda. \tag{72c}$$

- Diagonal variables

$$W_1 = \frac{1}{2}\left(V + \sqrt{(V-u)^2 + 4K_0/\rho}\right) \tag{73a}$$

$$W_2 = \frac{1}{2}\ u \tag{73b}$$

$$W_3 = \frac{1}{2}\left(V - \sqrt{(V-u)^2 + 4K_0/\rho}\right) \tag{73c}$$

- The eigenvalues $v_j = W_i + W_k$ coincide, at equilibrium, with

$$v_1 = u - \sqrt{K_0/\rho}, \qquad v_2 = u, \qquad v_3 = u + \sqrt{K_0/\rho}. \tag{74}$$

The original Born-Infeld equations

- The BI system (6×6)

$$\partial_t D + \nabla \times \frac{-B + D \times P}{h} = 0, \qquad \nabla \cdot D = 0, \tag{75a}$$
$$\partial_t B + \nabla \times \frac{D + B \times P}{h} = 0, \qquad \nabla \cdot B = 0, \tag{75b}$$

with

$$h = \sqrt{1 + |D|^2 + |B|^2 + |D \times B|^2}, \qquad P = D \times B, \tag{76}$$

were intended (1934) to be a nonlinear correction to the Maxwell equations.

- Designed on purpose to be hyperbolic and linearly degenerate, so as to avoid shock wave. No further microscopical theory needed.
- Additional conservation laws on h (energy density) and P (Poynting vector), but h is not a uniformly convex entropy.

The augmented Born-Infeld equations

- In his works on wave-particle transition, Brenier (2004) proposed to consider the conservation laws in h and P as part of a larger system

$$\partial_t D + \nabla \times \frac{-B + D \times P}{h} = 0, \qquad \nabla \cdot D = 0, \tag{77a}$$

$$\partial_t B + \nabla \times \frac{D + B \times P}{h} = 0, \qquad \nabla \cdot B = 0, \tag{77b}$$

$$\partial_t h + \nabla \cdot P = 0, \tag{77c}$$

$$\partial_t P + \nabla \cdot \frac{P \otimes P - D \otimes D - B \otimes B}{h} = \nabla \frac{1}{h}. \tag{77d}$$

- The ABI system (10×10) is hyperbolic, linearly degenerate, and coincides with the BI system on the submanifold defined by (76).
- The uniformly convex entropy $\eta = (1 + |D|^2 + |B|^2 + |P|^2)/2h$ satisfies

$$\partial_t \eta + \nabla \cdot \frac{(\eta h - 1)P + D \times B - (D \otimes D + B \otimes B)P}{h^2} = 0. \tag{78}$$

Plane-wave solution to the ABI system

- Choose $x = x_1$ and look for fields that depend only on (t, x). After simplification, the (h, P)-block becomes

$$\partial_t h + \partial_x P_1 = 0, \tag{79a}$$

$$\partial_t P_1 + \partial_x \frac{P_1^2 - 1}{h} = 0. \tag{79b}$$

- The eigenvalues

$$v^- = \frac{P_1 - 1}{h} \qquad \text{and} \qquad v^+ = \frac{P_1 + 1}{h} \tag{80}$$

are linearly degenerate and are governed by $\partial_t v^\mp + v^\pm \partial_x v^\mp = 0$.

- Setting formally $U = (P_1 + 1)/2$ and $G = -2/h$, we recover

$$\partial_t U + \partial_x U(1 - U)G = 0, \tag{81a}$$

$$\partial_t G + G^2 \partial_x U = 0. \tag{81b}$$