Quantum hydrodynamics with large data

Pierangelo Marcati
marcati@univaq.it
(joint work with P. Antonelli)

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QHD system with collision terms

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div} \frac{J \otimes J}{\rho} + \nabla P(\rho) \\ +\rho \nabla V + f(\sqrt{\rho}, J, \nabla\sqrt{\rho}, D^2\sqrt{\rho}) = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ -\Delta V = \rho \end{cases} \tag{1}$$

where the initial data

$$\rho(0) = \rho_0, \qquad J(0) = J_0 \tag{2}$$

satisfy

$$\sqrt{\rho_0} \in H^1(\mathbb{R}^3), \qquad \Lambda_0 := \frac{J_0}{\sqrt{\rho_0}} \in L^2(\mathbb{R}^3).$$

and the *pressure* is given by $P(\rho) = \frac{p-1}{p+1}\rho^{(p+1)/2}$, $1 \leq p < 5$.

Connection with Schrödinger equation (Noncollisional case f=0)

Given a solution ψ to the Schrödinger equation

$$\begin{cases} i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = |\psi|^{p-1}\psi + V\psi \\ -\Delta V = |\psi|^2 \\ \psi(0) = \psi_0. \end{cases} \tag{3}$$

$$\rho = |\psi|^2, J = \hbar \mathrm{Im}(\overline{\psi}\nabla\psi) \tag{4}$$

are (weak) solutions to QHD

$$\begin{cases} \partial_t\rho + \mathrm{div}\, J = 0 \\ \partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla P(\rho) + \rho\nabla V = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ -\Delta V = \rho \\ \rho(0) = \rho_0 := |\psi_0|^2, \quad J(0) = J_0 := \hbar\mathrm{Im}(\overline{\psi_0}\nabla\psi_0). \end{cases} \tag{5}$$

The collision term

References to Nonlinear Schrödinger equation: Cazenave book, 2003,Tao book 2006, papers from Ginibre–Velo in the '70s, 80's, Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T. Ann. Math. 167(3), 1–100 (2007))

The collision term can have for instance the form

$$f = \alpha J + \rho \nabla g(t, x, \sqrt{\rho}, \Lambda, \nabla\sqrt{\rho}, D^2\sqrt{\rho}), \tag{6}$$

where g is a nonlinear operator of $\sqrt{\rho}, \Lambda, \nabla\sqrt{\rho}, D^2\sqrt{\rho}$, under certain Carathéodory-type conditions.
For simplicity from now on we study the case $f(\sqrt{\rho}, J, \nabla\sqrt{\rho}, D^2\sqrt{\rho}) = J$.

Weak Solutions of (1), (1) and Irrotationality

.

$$\sqrt{\rho} \in L^\infty([0,\infty); H^1(\mathbb{R}^3)), \quad \Lambda := J/\sqrt{\rho} \in L^\infty([0,\infty); L^2(\mathbb{R}^3)),$$

- $$\int_0^\infty \int_{\mathbb{R}^3} (\rho \partial_t \eta + J \cdot \nabla \eta) dx dt = -\int_{\mathbb{R}^3} \rho_0 \eta(0) dx; \tag{7}$$
-

$$\int_0^\infty \int_{\mathbb{R}^3} (J \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta + P(\rho) \operatorname{div} \zeta - \rho \nabla V \cdot \zeta - J \cdot \zeta$$
$$+ \hbar^2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla \zeta - \frac{\hbar^2}{4} \rho \Delta \operatorname{div} \zeta) dx dt = -\int_{\mathbb{R}^3} J_0 \cdot \zeta(0) dx \tag{8}$$

$\eta \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^3), \zeta \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^3; \mathbb{R}^3),$

Generalized Irrotationality Condition in distributional sense

$$\nabla \wedge J = 2 \nabla \sqrt{\rho} \wedge \Lambda. \tag{9}$$

Main Theorem

Theorem

Let $\psi_0 \in H^1(\mathbb{R}^3)$ *and* **(Madelung Tranformations)**

$$\rho_0 := |\psi_0|^2, \qquad J_0 := \hbar \mathrm{Im}(\overline{\psi_0} \nabla \psi_0).$$

There exists a **global weak solution** (ρ, J) *of QHD, with initial data* $(\rho(0), J(0)) = (\rho_0, J_0)$*, such that*

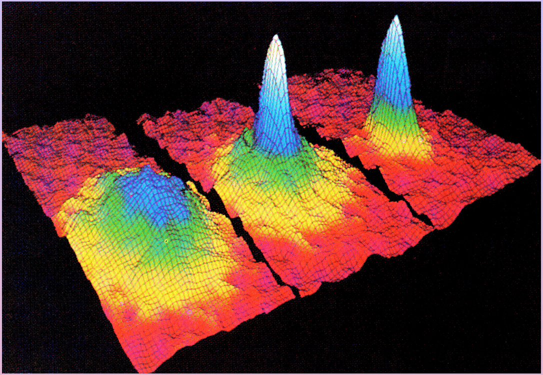
$$\sqrt{\rho} \in L^\infty([0,\infty); H^1_{loc}(\mathbb{R}^3))$$
$$\frac{J}{\sqrt{\rho}} \in L^\infty([0,\infty); L^2_{loc}(\mathbb{R}^3))$$

and such that (7), (8), (9) *hold.*

Some Motivations

- Fluidodynamical description of a quantum system : Madelung, E.: Quantentheorie in hydrodynamischer form. Z. Physik 40, 322 (1927)
- Semiconductor devices: Gardner, SIAM J. Appl. Math. 1994
- Superfluidity and Superconductivity: Landau, Phys. Rev. 1941, Khalatnikov, 1962, Dalfovo, Giorgini, Pitaevskii, Stringari, Rev. Mod. Phys. 1990, Feynman, R.P.: Superfluidity and Superconductivity. Rev. Mod. Phys. 29(2), 205 (1957)
- Two-fluid hydrodynamics for a trapped weakly interacting Bose gas : Zaremba, Nikkuni, Griffin, Stringari, Phys. Review A, 1998.

Source: Dalfovo, Giorgini, Pitaevskii, Stringari, *Rev. Mod. Phys.*, **71**, 3 (1999) (credit to Eric Cornell (JILA U.Colorado))

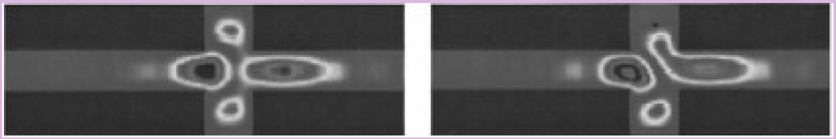


Generalized Irrotationality Condition

Suppose everything is smooth and $J = \rho u$, then (9) implies

$$\rho \nabla \wedge u = 0, \tag{10}$$

which means the current velocity u is irrotational in ρdx.
Possibility of occurrence of vortices in the *nodal region* $\{\rho = 0\}$.



(Source: Barker, *Microelectronic Engineering*, **63**, 223-231, 2002)

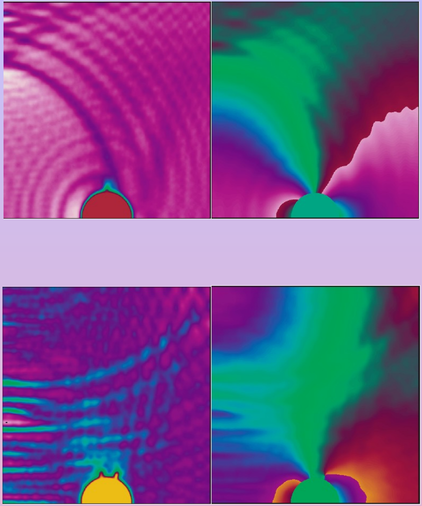


Figure: Source: Josserand, Pomeau, *Nonlinearity*, **14**, 25-62 (2001)

Known results

- Jungel, Mariani, Rial, 2001, M3AS: local existence of smooth solutions via the non standard NLS:

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = |\psi|^{p-1}\psi + V\psi + \tilde{V}\psi, \tag{11}$$

  with $\tilde{V}(\psi) = \frac{\hbar}{2i}\log\frac{\psi}{\bar{\psi}}$. The higher order Sobolev regularity requires the mass density **bounded away from** 0 **and the phase to be small around a constant value**.

- Li, M., 2004, CMP: Global existence. Higher order Sobolev regularity fot small perturbation around steady states, proof via nonstandard energy estimates **under a Quantum Subsonic Condition**.

From QHD to nonlinear Schrödinger ?

Problem: (ρ, J), weak solutions of (5), s.t. $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ and $\Lambda := \frac{J}{\sqrt{\rho}} \in L^2(\mathbb{R}^3)$ find $\psi \in H^1(\mathbb{R}^3)$ solution of (3).

Answer: in general NO

Obstructions :

- **The WKB formulation** : $\psi = \sqrt{\rho}e^{iS/\hbar}$ doesn't work, since in the vacuum $\{\rho = 0\}$ the phase S cannot be defined. The velocity $u = \nabla S$ is not defined.
- GMT, Federer, Ziemer, 1972: $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ implies the nodal region $\{\rho = 0\}$ may contain a nontrivial singular set (Quantum Dots).
- **Pauli Problem**: one cannot determine a wave function just by knowing its mass and momentum densities.

Difficulty of using WKB - Searching the phase 1

- Degenerate elliptic equation:

$$\mathrm{div}(\rho\nabla S) = \mathrm{div}\, J. \tag{12}$$

Problem: degenerate elliptic equation ($\rho \geq 0$); ρ doesn't satisfy Muckenhoupt's conditions.

- Quantum Hamilton-Jacobi eq: ($u = \nabla S$)!

$$\partial_t S + \frac{1}{2} u \cdot \nabla S = -h(\rho) - V + \frac{\hbar^2}{2}\frac{\Delta\rho}{\sqrt{\rho}}. \tag{13}$$

- Transport equation (Ambrosio's theory):

$$\rho\partial_t S + J \cdot \nabla S = -\rho h(\rho) - \rho V + \frac{\hbar^2}{2}\sqrt{\rho}\Delta\sqrt{\rho} \tag{14}$$

Problem: $\mathrm{div}_{(t,x)}(\rho, J) = 0$, but J is not in BV_{loc}.

The vacuum problem - Searching the phase - 2

Nelson's stochastic mechanics (description of quantum phenomena in term of diffusions: Nelson, Carlen, Guerra, Morato).
Carlen, 1984: equivalence between Nelson's mechanics and quantum mechanics (Schrödinger equation). Definition of a stochastic process, with probability density ρdx, through a drift determined by the velocity

$$v(t,x) := \begin{cases} \hbar \mathrm{Im} \frac{\nabla\psi(t,x)}{\psi(t,x)} & \psi(t,x) \neq 0 \\ 0 & \psi(t,x) = 0. \end{cases} \quad (15)$$

Problem: too much smoothness needed for ψ.

Searching the phase - 3

Aim: given ψ, find a function θ such that $\psi = \sqrt{\rho}e^{i\theta}$ (similar to the problem of *lifting* in Sobolev spaces, see Bourgain, Brezis, Mironescu)
From above discussion, this is difficult.

Idea: find the *unitary factor* $e^{i\theta}$ and work with it (similar to the *polar factorization* of L^p spaces, see Brenier).

Lemma (trivial)

Let B_R be the ball centered at the origin with radius R. Let us define the set

$$S_R := \{\phi \in L^2(B_R) \ \ s.t. \ \|\phi\|_{L^\infty(B_R)} \leq 1\}$$

and consider the functional

$$\Phi[\phi] := \mathrm{Re} \int_{B_R} \overline{\psi(x)}\phi(x)dx$$

Then the maximization problem $\sup_{\phi \in S_R} \Phi[\phi]$ *has a solution* $\hat{\phi} \in S_R$ *such that* $\psi = \sqrt{\rho}\hat{\phi}$ *a.e. in* B_R *(and thus* $|\hat{\phi}| = 1$ *for* $\sqrt{\rho}dx$*-a.e.* $x \in B_R$*.*

Trivial because $\hat{\phi}$ is actually the Radon-Nikodym derivative of ψ w.r.t. $\sqrt{\rho}$.
The variational approach could be relevant if we ask for some smoothness of the maximizer (e.g. H^1)

Lemma

*(**Stability lemma**) Let $\psi \in H^1(\mathbb{R}^3)$, $\phi \in L^\infty(\mathbb{R}^3)$ s.t. $\psi = \sqrt{\rho}\phi$ a.e. in $\mathbb{R}^3$, then*

$$\nabla\sqrt{\rho} = \mathrm{Re}(\overline{\phi}\nabla\psi) \in H^1(\mathbb{R}^3); \Lambda := \hbar \mathrm{Im}(\overline{\phi}\nabla\psi) \in L^2(\mathbb{R}^3) \qquad (16)$$

*moreover (**trivial null form structure**)*

$$\hbar^2 \mathrm{Re}(\partial_j\overline{\psi}\partial_k\psi) = \hbar^2\partial_j\sqrt{\rho}\partial_k\sqrt{\rho} + \Lambda^{(j)}\Lambda^{(k)}. \qquad (17)$$

If $\{\psi_n\} \subset H^1(\mathbb{R}^3)$, $\psi_n \to \psi$ in $H^1(\mathbb{R}^3)$, then

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \Lambda_n \to \Lambda \qquad in\ L^2(\mathbb{R}^3) \qquad (18)$$

.

Corollary

(Stability of Irrotationality) Let ψ and $\{\psi_n\}$ in $H^1(\mathbb{R}^3)$ as before, then $2i\nabla\sqrt{\rho} \wedge \Lambda$ is $H^1(\mathbb{R}^3)$ stable.

Basic Idea - Restricted to f = -J

Discretize time: take $\tau > 0$ and split $[0, \infty)$ in many subintervals $[k\tau, (k+1)\tau)$, $k \geq 0$.

- Step A: **solve** the nonlinear Schrödinger-Poisson system (3)
- Step B: **update** the wave funtion in order to take into account collisions (phase shift)

From the QHD point of view :

- Step A: **solve** the non-collisional QHD system
- Step B: **solve** the collision via an ODE:

$$\begin{cases} \frac{d}{dt}\rho = 0 \\ \frac{d}{dt}J + J = 0. \end{cases}$$

Remark: The update mechanisms act only on the polar factor not on ρ

Formal Iteration Procedure – Fractional Step Method

Step $k = 0$.

- Take $\psi_0 \in H^1(\mathbb{R}^3)$ and solve (3) in $[0, \tau) \times \mathbb{R}^3$.
- Factorize $\psi^\tau(\tau-) := \sqrt{\rho_\tau}\phi_\tau$.
- Define $\psi^\tau(\tau+) := \sqrt{\rho_\tau}\phi_\tau^{(1-\tau)}$.

From step $k-1$ to step k, $k \geq 1$:

- Take $\psi^\tau(k\tau+)$ (defined at previous step) and solve (3) in $[k\tau, (k+1)\tau) \times \mathbb{R}^3$ with initial datum $\psi^\tau(k\tau+)$
- Factorize $\psi^\tau((k+1)\tau-) = \sqrt{\rho_{(k+1)\tau}}\phi_{(k+1)\tau}$
- Define $\psi^\tau((k+1)\tau+) := \sqrt{\rho_{(k+1)\tau}}\phi_{(k+1)\tau}^{(1-\tau)}$

Formal since you actually need Approximate Polar Factorization Lemma.

Lemma (Approximate Polar Factorization)

Let $\psi \in H^1(\mathbb{R}^3)$, *and let* $\tau, \varepsilon > 0$. *Then there exists* $\tilde{\psi} \in H^1(\mathbb{R}^3)$ *such that*

$$\tilde{\rho} = \rho, \qquad \tilde{\Lambda} = (1-\tau)\Lambda + r_\varepsilon,$$

where $\sqrt{\rho} := |\psi|, \sqrt{\tilde{\rho}} := |\tilde{\psi}|, \Lambda := \hbar \mathrm{Im}(\bar{\phi}\nabla\psi), \tilde{\Lambda} := \hbar \mathrm{Im}(\bar{\tilde{\phi}}\nabla\tilde{\psi})$, $\phi, \tilde{\phi}$ *are polar factors for* $\psi, \tilde{\psi}$ *respectively, and*

$$\|r_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq \varepsilon.$$

Furthermore we have

$$\nabla\tilde{\psi} = \nabla\psi - i\frac{\tau}{\hbar}\phi^\star\Lambda + r_{\varepsilon,\tau}, \tag{19}$$

where $\|\phi^\star\|_{L^\infty(\mathbb{R}^3)} \leq 1$ *and* $\|r_{\varepsilon,\tau}\|_{L^2(\mathbb{R}^3)} \leq C(\tau\|\nabla\psi\|_{L^2(\mathbb{R}^3)} + \varepsilon)$.

Define $(\rho^\tau, J^\tau) := (|\psi^\tau|^2, \hbar \mathrm{Im}(\overline{\psi^\tau} \nabla \psi^\tau))$.

Lemma (Consistency)

(ρ^τ, J^τ) are approximate solutions of (1), in the sense that

$$\int_0^\infty \int_{\mathbb{R}^3} \rho^\tau \partial_t \eta + J^\tau \cdot \nabla \eta dxdt + \int_{\mathbb{R}^3} \rho_0 \eta(0) dx = o(1), \tag{20}$$

$$\int_0^\infty \int_{\mathbb{R}^3} J^\tau \cdot \partial_t \zeta + \Lambda^\tau \otimes \Lambda^\tau : \nabla \zeta + P(\rho^\tau) \operatorname{div} \zeta - \rho^\tau \nabla V^\tau \cdot \zeta - J^\tau \cdot \zeta$$
$$+ \hbar^2 \nabla \sqrt{\rho^\tau} \otimes \nabla \sqrt{\rho^\tau} : \nabla \zeta - \frac{\hbar^2}{4} \rho^\tau \Delta \operatorname{div} \zeta dxdt + \int_{\mathbb{R}^3} J_0 \cdot \zeta(0) dx = o(1) \tag{21}$$

as $\tau \to 0$, for any $\eta \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^3), \zeta \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^3; \mathbb{R}^3)$.

Energy estimates

Define the energy

$$E^{\tau}(t) := \int_{\mathbb{R}^3} \frac{\hbar^2}{2}|\nabla\sqrt{\rho^{\tau}}|^2 + \frac{1}{2}|\Lambda^{\tau}|^2 + f(\rho^{\tau}) + \frac{1}{2}|\nabla V^{\tau}|^2 dx \tag{22}$$

Lemma (Dissipation of energy for approximate solutions)

Let $t \in [N\tau, (N+1)\tau)$, and let $\rho^{\tau}, \Lambda^{\tau}$ be defined as above. Then

$$E^{\tau}(t) \leq -\frac{\tau}{2}\sum_{k=1}^{N} \|\Lambda^{\tau}(k\tau-)\|_{L^2(\mathbb{R}^3)} + (1+\tau)E_0. \tag{23}$$

Strichartz estimates for Schrödinger in $\mathbb{R}^3$

Admissible pair (q, r)

$$\frac{1}{q} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \tag{24}$$

$$2 \leq q \leq \infty,\ 2 \leq r \leq 6 \tag{25}$$

Let $U(\cdot)$ be the free Schrödinger group then for any admissible pair

$(q, r), (\tilde{q}, \tilde{r})$ we have (Keel, Tao 1998)

$$\|U(\cdot)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)} \tag{26}$$

$$\|\int_{s<t} U(t-s)F(s)ds\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbb{R}^3)} \tag{27}$$

$$\|\int_{\mathbb{R}} U(t-s)F(s)ds\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbb{R}^3)} \tag{28}$$

Following the iteration procedure we need to estimate in Strichartz norms this unpleasent expression :

$$\begin{aligned}\nabla\psi^\tau(t) =&\sigma^\tau_t U(t-N\tau)\sigma^\tau_{N\tau}U(\tau)\cdots\sigma^\tau_\tau U(\tau)\nabla\psi_0\\ &- i\frac{t-N\tau}{\hbar}\phi_t\Lambda(t) - i\frac{\tau}{\hbar}\sigma^\tau_t U(t-N\tau)\phi_{N\tau}\Lambda^\tau(N\tau-)\\ &+\ldots - i\frac{\tau}{\hbar}\sigma^\tau_t U(t-N\tau)\cdots U(\tau)\phi_\tau\Lambda^\tau(\tau-)\\ &- i\sigma^\tau_t\int_{N\tau}^t U(t-s)F(s)ds\\ &- i\sigma^\tau_t U(t-N\tau)\sigma^\tau_{N\tau}\int_{(N-1)\tau}^{N\tau}U(N\tau-s)F(s)ds\\ &+\ldots - i\sigma^\tau_t U(t-N\tau)\sigma^\tau_{N\tau}U(\tau)\cdots\sigma^\tau_\tau\int_0^\tau U(\tau-s)F(s)ds,\end{aligned}$$

where $\sigma^\tau_{k\tau} := \overline{\phi_{k\tau}}\phi^{(1-\tau)}_{k\tau}$ and $F = \nabla(|\psi^\tau|^{p-1}\psi^\tau + V^\tau\psi^\tau)$.
It is **impossible** to **estimate** the commutator $[U(\tau), \sigma^\tau_{k\tau}]$ (low regularity!!).

Thanks to Approximate Polar Factorization Lemma, we have the following expression

$$\begin{aligned}\nabla\psi^\tau(t) =& U(t)\nabla\psi_0 - i\frac{\tau}{\hbar}\sum_{k=1}^{N} U(t-k\tau)\left(\phi_k \Lambda^\tau(k\tau-)\right) \\ &+ \tau\sum_{k=1}^{N} U(t-k\tau)(\tilde{\phi}_k\nabla\psi_k) - i\int_0^t U(t-s)F(s)ds \\ &+ \sum_{k=1}^{N} U(t-k\tau)R_k,\end{aligned}$$

with $\|\tilde{\phi}_k\|_{L^\infty} \leq 1$, $\|R_k\|_{L^2} \lesssim 2^{-k}\tau E_0$.

Strichartz estimates for $\nabla\psi^\tau$

Lemma (Strichartz estimates for $\nabla\psi^\tau$)

Let $0 < T < \infty$, ψ^τ *be as before, then one has*

$$\|\nabla\psi^\tau\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^3)} \leq C(E_0^{\frac{1}{2}}, \|\rho_0\|_{L^1(\mathbb{R}^3)}, T) \tag{29}$$

for each admissible pair of exponents (q, r).

$$
\begin{aligned}
\|\nabla\psi^\tau\|_{L^q_t L^r_x([0,T_1]\times\mathbb{R}^3)} \leq \\
& \|U(t)\nabla\psi_0\|_{L^q_t L^r_x([0,T_1]\times\mathbb{R}^3)} \\
& + \frac{\tau}{\hbar}\sum_{k=1}^{N}\|U(t-k\tau)\left(\phi_k\Lambda^\tau(k\tau-)\right)\|_{L^q_t L^r_x([0,T_1]\times\mathbb{R}^3)} \\
& + \sum_{k=1}^{N}\|U(t-k\tau)r_k^\tau\|_{L^q_t L^r_x([0,T_1]\times\mathbb{R}^3)} \\
& + \left\|\int_0^t U(t-s)F(s)\mathrm{d}s\right\|_{L^q_t L^r_x([0,T_1]\times\mathbb{R}^3)} \\
=: A + B + C + D.
\end{aligned}
$$

The estimate of A is straightforward, since

$$\|U(t)\nabla\psi_0\|_{L_t^q L_x^r([0,T_1]\times\mathbb{R}^3)} \lesssim \|\nabla\psi_0\|_{L^2(\mathbb{R}^3)}.$$

The estimate of B follows from

$$\frac{\tau}{\hbar}\sum_{k=1}^{N}\|U(t-k\tau)\left(\phi_k\Lambda^\tau(k\tau-)\right)\|_{L_t^q L_x^r([0,T_1]\times\mathbb{R}^3)} \lesssim \tau\sum_{k=1}^{N}\|\Lambda^\tau(k\tau-)\|_{L^2(\mathbb{R}^3)} \lesssim T_1 E_0^{\frac{1}{2}}. \quad (30)$$

The term C can be estimated in a similar way, namely

$$\sum_{k=1}^{N}\|U(t-k\tau)r_k^\tau\|_{L_t^q L_x^r} \lesssim \sum_{k=1}^{N}\|r_k^\tau\|_{L^2(\mathbb{R}^3)} \lesssim T_1\|\psi_0\|_{H^1(\mathbb{R}^3)}. \quad (31)$$

To estimate D decompose $F = F_1 + F_2 + F_3$, where $F_1 = \nabla(|\psi^\tau|^{p-1}\psi^\tau)$, $F_2 = \nabla V^\tau \psi^\tau$ and $F_3 = V^\tau \nabla \psi^\tau$.
There exists $\alpha > 0$, depending on p, such that

$$\| |\psi^\tau|^{p-1} \nabla \psi^\tau \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}([0,T_1]\times\mathbb{R}^3)} \lesssim T_1^\alpha \|\psi^\tau\|_{\dot{S}^1([0,T_1]\times\mathbb{R}^3)}. \qquad (32)$$

Choose $(q_2', r_2') = (1, 2)$, hence by Hölder and Hardy-Littlewood-Sobolev

$$\begin{aligned} \|V^\tau \nabla \psi^\tau\|_{L_t^1 L_x^2([0,T_1]\times\mathbb{R}^3)} &\leq T_1^{\frac{1}{2}} \|V^\tau\|_{L_t^\infty L_x^2} \|\nabla \psi^\tau\|_{L_t^2 L_x^6} \\ &\lesssim T_1^{\frac{1}{2}} \|\psi^\tau\|_{L_t^\infty L_x^2}^2 \|\nabla \psi^\tau\|_{L_t^2 L_x^6}. \end{aligned}$$

Choose $(q_3', r_3') = (\frac{2}{2-3\varepsilon}, \frac{2}{1+2\varepsilon})$ and use again Hardy-Littlewood-Sobolev and Hölder

$$\|\nabla V^\tau \psi^\tau\|_{L_t^{\frac{2}{2-3\varepsilon}} L_x^{\frac{2}{1+2\varepsilon}}([0,T_1]\times\mathbb{R}^3)} \leq T_1^{\frac{1}{2}} \|\nabla V^\tau\|_{L_t^{\frac{2}{1-3\varepsilon}} L_x^{\frac{1}{\varepsilon}}} \|\psi^\tau\|_{L_t^\infty L_x^2}$$

$$\lesssim T_1^{\frac{1}{2}} \|\nabla |\psi^\tau|^2\|_{L_t^{\frac{2}{1-3\varepsilon}} L_x^{\frac{3}{2+3\varepsilon}}} \|\tilde{\psi}^\tau\|_{L_t^\infty L_x^2} \lesssim T_1^{\frac{1}{2}} \|\psi^\tau\|_{L_t^\infty L_x^2}^2 \|\nabla \psi^\tau\|_{L_t^{\frac{2}{1-3\varepsilon}} L_x^{\frac{6}{1+6\varepsilon}}}$$

we summarize the previous estimates

$$\|\nabla\psi^\tau\|_{\dot{S}^0([0,T_1]\times\mathbb{R}^3)}$$
$$\lesssim (1+T)E_0^{\frac{1}{2}} + T_1^\alpha \|\nabla\psi^\tau\|^p_{\dot{S}^0([0,T_1]\times\mathbb{R}^3)} + T_1^{\frac{1}{2}}\|\psi_0\|^2_{L^2(\mathbb{R}^3)}\|\nabla\psi^\tau\|_{\dot{S}^0([0,T_1]\times\mathbb{R}^3)}.$$

Lemma

There exist $T_1(E_0, \|\psi_0\|_{L^2(\mathbb{R}^3)}, T) > 0$, $C_1(E_0, \|\psi_0\|_{L^2(\mathbb{R}^3)}, T) > 0$, *s.t.*

$$\|\nabla\psi^\tau\|_{\dot{S}^0([0,\tilde{T}]\times\mathbb{R}^3)} \leq C_1(E_0, \|\psi_0\|_{L^2(\mathbb{R}^3)}, T) \tag{33}$$

for all $0 < \tilde{T} \leq T_1(E_0, \|\psi_0\|_{L^2(\mathbb{R}^3)})$, *hence for all* $T > 0$,

$$\|\nabla\psi^\tau\|_{\dot{S}^0([0,T]\times\mathbb{R}^3)}$$
$$\leq C_1(E_0, \|\psi_0\|_{L^2}, T)\left(\left[\frac{T}{T_1}\right] + 1\right) = C(\|\psi_0\|_{L^2}, E_0, T).$$

.

Smoothing estimates for Schrödinger

From Constantin, Saut (1988) we have the following smoothing estimates

$$\|U(\cdot)u_0\|_{L^2([0,T];H^{1/2}_{loc}(\mathbb{R}^3))} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)} \tag{34}$$

$$\|\int_0^t U(t-s)F(s)ds\|_{L^2([0,T]:H^{1/2}_{loc}(\mathbb{R}^3))} \lesssim \|F\|_{L^1_tL^2_x([0,T]\times\mathbb{R}^3)}. \tag{35}$$

Use expression for $\nabla\psi^\tau$ and Strichartz estimates: we find

$$\|\nabla\psi^\tau\|_{L^2([0,T];H^{1/2}_{loc}(\mathbb{R}^3))} \leq C(\|\nabla\psi_0\|_{L^2(\mathbb{R}^3)}, T, E_0) \tag{36}$$

thus $\{\nabla\psi^\tau\}$ relatively compact in $L^2([0,T];L^2(\mathbb{R}^3))$.

Compactness

We use Rakotoson-Temam (Appl. Math. Letters, 2001) (Aubin-Lions type Lemma).

- $\{\psi^\tau\}$ uniform bounded in $L^2([0,T]; H^{3/2}_{loc}(\mathbb{R}^3))$ and $H^1_{loc}(\mathbb{R}^3)$ compactly embedded in $H^{3/2}_{loc}(\mathbb{R}^3)$.
- Equi-integrability property from the Energy estimates

Proposition

$\{\psi^\tau\} \subset L^2([0,T]; H^1_{loc}(\mathbb{R}^3))$ **is relatively compact**, *there exists* $\psi \in L^2([0,T]; H^1_{loc}(\mathbb{R}^3))$, *s.t. (up to subsequences)*

$$\psi^\tau \to \psi \in L^2([0,T]; H^1_{loc}(\mathbb{R}^3)) \tag{37}$$

$$\sqrt{\rho^\tau} \to \sqrt{\rho} \in L^2([0,T]; H^1(\mathbb{R}^3)) \tag{38}$$

$$\Lambda^\tau \to \Lambda \in L^2([0,T]; L^2(\mathbb{R}^3)), \tag{39}$$

where $\sqrt{\rho}, \Lambda$ *are the hydrodynamic quantities defined via* ψ *through the Madelung transform. Then* (ρ, J) *is a* **weak solution to the QHD** *but* ψ *is* **not** *a solution of any Schrödinger equation.*

2-D QHD

Theorem

Let ρ_0, J_0 *be such that there exists* $\psi_0 \in H^1(\mathbb{R}^2)$ *satisfying*

$$\rho_0 = |\psi_0|^2, \qquad J_0 = \hbar \mathrm{Im}(\overline{\psi_0} \nabla \psi_0).$$

Furthermore, let us assume that

$$\int_{\mathbb{R}^2} \rho_0 \log \rho_0 \mathrm{d}x < \infty, \tag{40}$$

and that

$$V(0, x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \rho_0(y) \mathrm{d}y \tag{41}$$

satisfies $V(0, \cdot) \in L^r(\mathbb{R}^2)$, *for some* $2 < r < \infty$. *Then, for* $0 < T < \infty$ *there exists a finite energy weak solution of QHD in* $[0, T) \times \mathbb{R}^2$.