Équation de Burgers avec particule ponctuelle

Nicolas Seguin

Laboratoire J.-L. Lions, UPMC Paris 6, France

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En collaboration avec B. Andreianov, F. Lagoutière et T. Takahashi

Outline

- Description of the models of fluid-particle interaction
  - Description: modeling and formal properties
  - Relation with other models
  - Relation with coupling problems
- Mathematical analysis of simple models
  - Definition of solutions
  - The case of one particle with a constant velocity
  - The complete model of interaction
- Numerical approximation of the models of interaction
  - Requirements for the numerical methods
  - Approximation of the fluid
  - A numerical scheme for the complete model of interaction
  - Numerical results

Motivations

Interaction between an inviscid fluid and point-wise particles

- High Reynolds Number: d'Alembert Paradox (1749)...

  In a potential flow, the drag force on an object is zero
  if the fluid is inviscid

- Interaction between small particles and shock waves
- Drafting-kissing-tumbling

But also...

- Tracking of a Dirac measure (math. & num.)
- Complex coupling through the particles
  - Their position and their velocity are unknowns of the problem
  - The interaction depends on the fluid and on the velocity o the particles.

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An inviscid model of fluid-particle interaction

Euler equations for the fluid (ρ: density, $\mathbf{u} \in \mathbb{R}^d$: velocity, p: pressure)
N point-wise particles ($\mathbf{h}_k(t)$: position, m_k: mass, λ_k: drag coefficient)

- Classical Euler equations everywhere except on the particles ($\mathbf{x} \neq \mathbf{h}_k$)
- Point-wise particles: Dirac measures $\delta_0(\mathbf{x} - \mathbf{h}_k)$
- Particles governed by the 2nd Newton law:
  Acceleration = Drag force (function of $(\mathbf{u} - \mathbf{h}_k')$)
- Conservation of the total momentum

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \rho\mathbf{u}\, d\mathbf{x} + \sum_{k=1}^{N} m_k \mathbf{h}_k'\right) = 0$$

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For $t > 0$ and $\mathbf{x} \in \mathbb{R}^d$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$
$$\partial_t (\rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p(\rho)\mathbf{I}) = - \sum_k \mathbf{F}_k \, \delta_0(\mathbf{x} - \mathbf{h}_k(t))$$

For $t > 0$ and $k = 1, ..., N$

$$m_k \, \mathbf{h}_k''(t) = \mathbf{F}_k$$

Coupling by the drag force: $\mathbf{F}_k = \lambda_k \rho D\big(\mathbf{u} - \mathbf{h}_k'\big)$ where $D(v) = v$ or $v|v|$.

A simple model of interaction

Euler equations replaced by the 1D Burgers equation + one particle

[Lagoutière, Seguin, Takahashi]

$$\partial_t u + \partial_x (u^2/2) = \lambda\ D(h'(t) - u)\ \delta_0(x - h(t))$$
$$m\ h''(t) = -\lambda\ D(h'(t) - u(t, h(t)))$$

- $u(t,x)$: velocity of the fluid
- $h(t)$: position of the particle
- λ: drag coefficient (> 0)
- $D(v) = v$ or $v|v|$ ("laminar" or "turbulent" case)

- Fluid-particle coupling by the drag force term
  - Burgers equation + singular source term
  - ODE of relaxation ($h' \to u$)

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Formal properties (also available with Euler equations)

Conservation of the total momentum:

$$\frac{d}{dt}\left(\int_{\mathbb{R}} u\ dx + m\ h'\right) = 0$$

Equation for the total kinetic energy:

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}} u^2\ dx + m(h')^2\right)(t) \leq 0$$

Comparison with other models

Burgers equation coupled with one point-wise particle

$$\partial_t u + \partial_x(u^2/2) = \lambda \; D(h'(t) - u) \; \delta_0(x - h(t))$$
$$m \; h''(t) = -\lambda \; D(h'(t) - u(t, h(t)))$$

Vlasov model - Baranger, Desvillettes, Goudon, Carillo, Domelevo...

Introduce $f(t, x, v)$: density of particles (with velocity v)

$$\partial_t u + \partial_x(u^2/2) = \frac{\lambda}{m} \int_{\mathbb{R}} (v - u) f \; dv$$
$$\partial_t f + v \; \partial_x f + \frac{\lambda}{m} \partial_v((u - v) f) = 0$$

One formally recovers the previous model, setting

$$f(t, x, v) = m \; \delta_0(x - h(t)) \; \delta_0(v - V(t)).$$

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Burgers-Hopf model - Vásquez, Zuazua, Hillairet

Viscous Burgers equation: the velocity u is continuous through the particle

$$\partial_t u + \partial_x(u^2/2) = \varepsilon\ \partial_{xx} u \quad \text{for } x \neq h(t)$$
$$h'(t) = u(t, h(t))$$
$$m\ h''(t) = [\partial_x u](t, h(t))$$

where $[g](x) = \lim_{y \to 0}(g(x+y) - g(x-y))$.

Compatibility with the inviscid model: open problem...

Comparison with other models

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Results on existing models

- Vlasov model:
  - Classical solutions and small times
- Burgers-Hopf model:
  - Well-posedness for strong solutions
  - Several particles: lack of collision

Several studies for Euler and Navier-Stokes extensions.

Definition of solutions

Burgers equation coupled with one point-wise particle

(1) $\partial_t u + \partial_x (u^2/2) = \lambda\ D(h' - u)\ \delta_h$

(2) $m\ h'' = -\lambda\ D(h' - u(t,h))$

- No viscosity in (1) $\Longrightarrow$ consider entropy weak solutions
- Singular source term in (1) $\Longrightarrow$ product of distribution to be defined
- Discontinuous RHS in (2) $\Longrightarrow$ Carathéodory sense or alternative ODE

Alternative formulation: $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, $h \in \mathcal{C}^1(\mathbb{R}_+)$

(1a) $\partial_t u + \partial_x (u^2/2) = 0$ for $x \neq h(t)$

(1b) $(u(t,h(t)^-), u(t,h(t)^+)) \in \mathcal{G}_D(h'(t), \lambda)$

(2') $\partial_t u + m h'' \delta_h + \partial_x (u^2/2) = 0$ (conservation of the total momentum)

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(2') $\partial_t u + mh''\delta_h + \partial_x(u^2/2) = 0$

- Construction of the germ $\mathcal{G}_D(h'(t),\lambda)$
  - Analysis of the wave due to the particle $\{x = h(t)\}$
  - Regularization of the particle
  - Solve the problem inside the regularization
  - Find the compatible traces
- Alternative ODE
  - Jump relations at $\{x = h(t)\}$

Wave due to the particle – Quasilinear form of the PDE

Equivalent form: $\delta_0(x) = \partial_x H(x)$, H Heaviside function

$$\begin{cases} \partial_t u + \partial_x(u^2/2) - \lambda\, D(h' - u)\, \partial_x w = 0 \\ \partial_t w + h'(t)\, \partial_x w = 0 \\ w(0,x) = H(x - h(0)) \qquad \Longrightarrow w(t,x) = H(x - h(t)) \end{cases}$$

Quasilinear formulation:

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} u & -\lambda D(h' - u) \\ 0 & h' \end{pmatrix} \partial_x \begin{pmatrix} u \\ w \end{pmatrix} = 0$$

Hyperbolic system (diagonalizable in $\mathbb{R}$ even if $u = h'$ since $D(0) = 0$).

Wave associated with u: genuinely non linear field
Wave associated with h': linearly degenerate field

What happen when $u = h'$?

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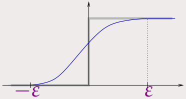
Construction of the germ $\mathcal{G}_D(h', \lambda)$ – Regularization

Regularization of the particle (LeRoux)

H becomes $H_\varepsilon \in \mathcal{C}^1(\mathbb{R})$, a non decreasing function such that

$$\forall |x| \geq \varepsilon \quad H_\varepsilon(x) = H(x)$$



Regularized equivalent form

$$\begin{cases} \partial_t u + \partial_x (u^2/2) - \lambda \ D(h' - u) \ \partial_x w = 0 \\ \partial_t w + h'(t) \ \partial_x w = 0 \\ w(0, x) = H_\varepsilon(x - h(0)) \qquad \Longrightarrow w(t, x) = H_\varepsilon(x - h(t)) \end{cases}$$

We seek for solutions of the form

$$\forall x \in [h(t) - \varepsilon, h(t) + \varepsilon] \qquad U(x - h(t)) = u(t, x)$$

Construction of the germ $\mathcal{G}_D(h', \lambda)$ – Change of frame

The equations "inside" the regularized particle

$$\begin{cases} \partial_t u + \partial_x(u^2/2) - \lambda \; D(h' - u) \; \partial_x w = 0 \\ w(t,x) = H_\varepsilon(x - h(t)) \\ U(x - h(t)) = u(t,x) \end{cases}$$

gives for $\xi = x - h(t) \in [-\varepsilon, \varepsilon]$,

$$-h'(t)U'(\xi) + (U^2/2)'(\xi) - \lambda D(h'(t) - U(\xi))H'_\varepsilon(\xi) = 0 \qquad (\star)$$

in the weak entropy sense:

- Smooth parts: solve $(\star)$ in the classical sense
- Shock wave at ξ_0: $(U(\xi_0^-) + U(\xi_0^+))/2 = h'(t)$ and $U(\xi_0^-) > U(\xi_0^+)$
- Search all the pairs $U(-\varepsilon)$ and $U(+\varepsilon)$ which can be connected

Germ $\mathcal{G}_D(h', \lambda)$ for a linear drag force $D(v) = v$

In the case of a linear drag force: $D(v) = v$
Smooth solutions (t is fixed):

$$
\begin{aligned}
&(U^2/2 - h'(t)U)'(\xi) - \lambda D(h'(t) - U(\xi))H'_\varepsilon(\xi) = 0 \\
\iff &(U(\xi) - h'(t)) \big(U'(\xi) + \lambda H'_\varepsilon(\xi)\big) = 0 \\
\iff &\begin{cases} \text{Either} & U(\xi) = h'(t) \\ \text{Or} & U(\xi) + \lambda H_\varepsilon(\xi) = \text{Cst} \end{cases}
\end{aligned}
$$

Shock wave:

$$
\begin{cases} (U(\xi_0^-) + U(\xi_0^+))/2 = h'(t) \\ U(\xi_0^-) > h'(t) > U(\xi_0^+) \end{cases}
$$

Search all the pairs $U(-\varepsilon)$ and $U(+\varepsilon)$ which can be connected
The germ $\mathcal{G}_D(h', \lambda)$ is the set of all these pairs

Germ $\mathcal{G}_D(h', \lambda)$ *for a linear drag force* $D(v) = v$

Smooth solutions:

$$\begin{cases} \text{Either} & U(\xi) = h'(t) \\ \text{Or} & U(\xi) + \lambda H_\varepsilon(\xi) = \text{Cst} \end{cases}$$

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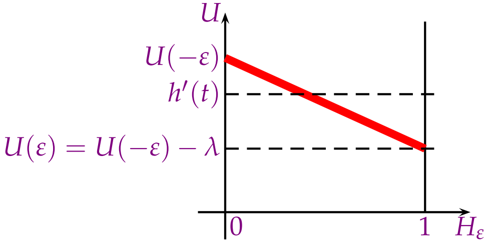

Germ $\mathcal{G}_D(h', \lambda)$ *for a linear drag force* $D(v) = v$

Smooth solutions:

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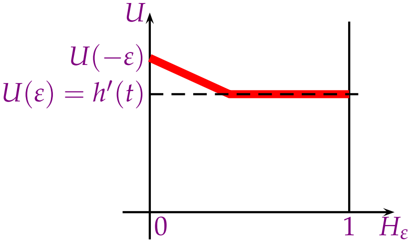

Germ $\mathcal{G}_D(h', \lambda)$ for a linear drag force $D(v) = v$

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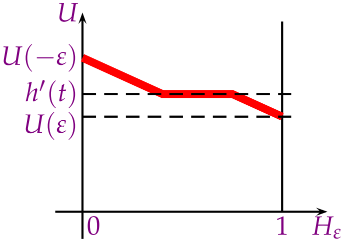

Germ $\mathcal{G}_D(h', \lambda)$ for a linear drag force $D(v) = v$

Smooth solutions:

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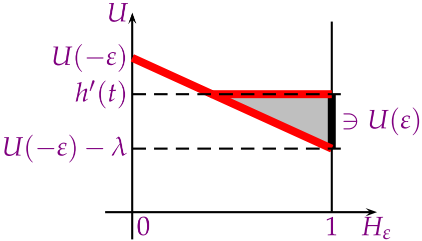

Germ $\mathcal{G}_D(h', \lambda)$ for a linear drag force $D(v) = v$

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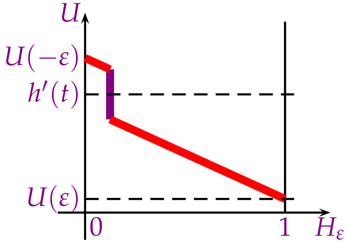

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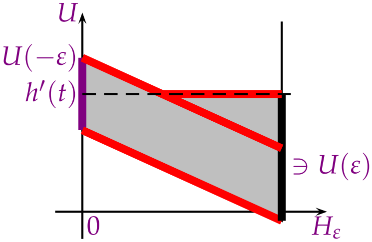
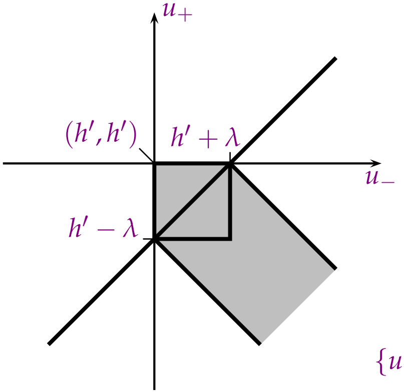


Germ $\mathcal{G}_D(h', \lambda)$ according the drag force

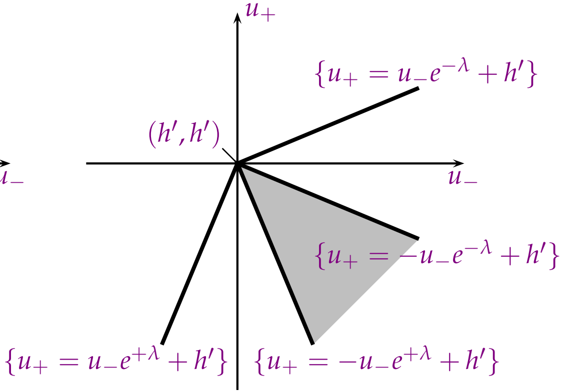
Germ $\mathcal{G}_D(h', \lambda)$: set of all pairs $(u_-, u_+) \in \mathbb{R}^2$ such that

$$(U^2/2 - h'(t)U)'(\xi) - \lambda D(h'(t) - U(\xi))H'_\varepsilon(\xi) = 0$$

$$U(-\varepsilon) = u_- \text{ and } U(+\varepsilon) = u_+$$



$\boxed{D(v) = v}$



$\boxed{D(v) = v|v|}$

Alternative ODE

Instead of

$$m\ h'' = \lambda\ D(u(t,h) - h')$$

we use

$$\partial_t u + mh''\delta_h + \partial_x(u^2/2) = 0$$

Jump relations at $\{x = h(t)\}$:

$$-h'(t)[u] + mh''(t) + [u^2]/2 = 0$$

This leads to the new ODE

$$mh''(t) = (u(t,h(t)^-) - u(t,h(t)^+))\left(\frac{u(t,h(t)^-) + u(t,h(t)^+)}{2} - h'(t)\right)$$

Definition of solutions

Definition

A solution of the Burgers-particle problem
is a pair $(u, h) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{W}^{2,\infty}(\mathbb{R}_+)$ which fulfills:

- The function u is an entropy solution away from the particle:
$$\forall \kappa \in \mathbb{R} \quad \partial_t |u-\kappa| + \partial_x \Phi(u,\kappa) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R} \setminus \{x = h(t)\})$$
- The traces are compatible:
$$\forall t > 0 \quad \left(\gamma^-_{h(t)} u, \gamma^+_{h(t)} u\right) \in \mathcal{G}_D(h'(t), \lambda)$$
- The trajectory of the particle satisfy:
$$m\, h''(t) = \left(\gamma^-_{h(t)} u - \gamma^+_{h(t)} u\right) \left(\frac{\gamma^-_{h(t)} u + \gamma^+_{h(t)} u}{2} - h'(t) \right)$$

$\gamma^\pm_y$: left and right trace operators at $\{x = y\}$ (see Panov, Vasseur...)

The case of one particle with a zero velocity

The case $(h \equiv 0)$

$$[\mathcal{P}_0] \qquad \begin{cases} \partial_t u + \partial_x (u^2/2) + \lambda D(u)\ \delta_0 = 0 \\ u_{|t=0} = u_0 \end{cases}$$

See also Isaacson-Temple, LeRoux, Dielh, Gosse, Vasseur, Bürger-García-Karlsen-Towers...

Definition

A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution of $[\mathcal{P}_0]$ if

- $\forall \kappa \in \mathbb{R} \quad \partial_t |u - \kappa| + \partial_x \Phi(u, \kappa) \leq 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R} \setminus \{x = 0\})$
- $\forall t > 0 \quad \big(\gamma_0^- u(t), \gamma_0^+ u(t)\big) \in \mathcal{G}_D(0, \lambda)$

The Riemann problem

Theorem (with $D(v) = v$ or $v|v|$)

The Riemann problem

$$\begin{cases} \partial_t u + \partial_x (u^2/2) + \lambda D(u)\ \delta_0 = 0 \\ u(0,x) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases} \end{cases}$$

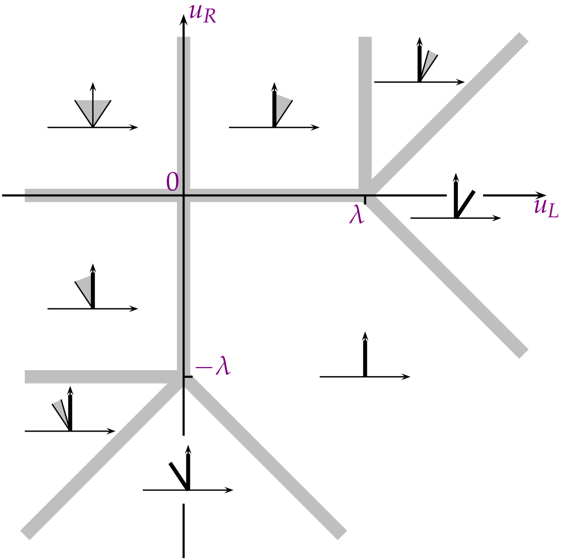
admits one and only one self-similar entropy solution for all $(u_L, u_R) \in \mathbb{R}$.

Construct the set $\mathcal{U}_-(u_L) = \{W(0^-; u_L, \bar{u}), \bar{u} \in \mathbb{R}\}$
Construct the set $\mathcal{U}_+(u_R) = \{W(0^+; \bar{u}, u_R), \bar{u} \in \mathbb{R}\}$
Existence and uniqueness of $(u_-, u_+) \in \mathbb{R}^2$ such that

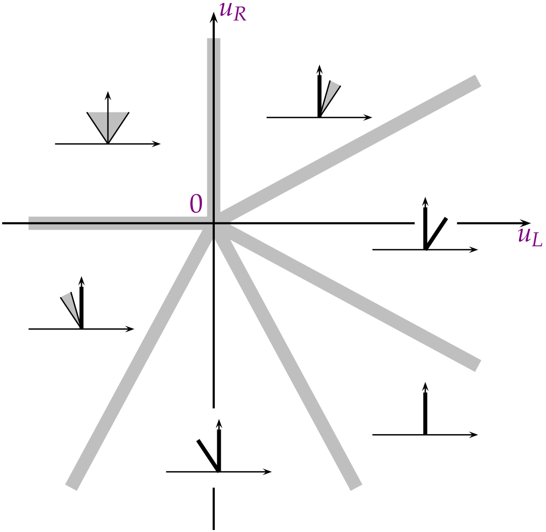
$u_- \in \mathcal{U}_-(u_L)$, $u_+ \in \mathcal{U}_+(u_R)$ and $(u_-, u_+) \in \mathcal{G}_D(0, \lambda)$. □

N.B. Uniqueness fails if $\lambda < 0$!

The solution of the Riemann problem when $D(v) = v$



The solution of the Riemann problem when $D(v) = v|v|$



The Cauchy problem for a particle with a zero velocity

Alternative definition: Adapted Kruzhkov entropies

Let c be defined by $c(x) = \begin{cases} c_L & \text{if } x < 0, \\ c_R & \text{if } x > 0. \end{cases}$.

A function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution of $[\mathcal{P}_0]$ iff

$$\partial_t |u - c| + \partial_x \Phi(u, c) \leq 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R} \setminus \{x = 0\})$$

for all $(c_L, c_R) \in \mathcal{G}_D(0, \lambda)$.

Baiti-Jenssen, Audusse-Perthame, Adimurthi-Mishra-Veerappa Gowda, Bürger-Karlsen-Towers, Andreianov-Karlsen-Risebro...

- If $u_{|t=0}(x) = c(x)$, then c is a stationary entropy solution of $[\mathcal{P}_0]$, for all $(c_L, c_R) \in \mathcal{G}_D(0, \lambda)$.
- An entropy solution is "dissipative" w.r.t. any stationary solution.

The Cauchy problem for a particle with a zero velocity

Proposition (dissipative germ)

For all $(c_L, c_R) \in \mathcal{G}_D(0, \lambda)$ and $(d_L, d_R) \in \mathcal{G}_D(0, \lambda)$, we have

$$\Phi(c_R, d_R) - \Phi(c_L, d_L) \leq 0.$$

This property enables to compare two solutions, in the spirit of Kruzhkov's uniqueness proof.

Theorem

Let the drag force be $D(v) = v$ *or* $v|v|$*.*
The Cauchy problem $[\mathcal{P}_0]$ *admits one and only one entropy solution in* L^∞*.*

The existence will follow, using the convergence of numerical schemes.

N.B. Also available for $h'(t) = V$, $V \in \mathbb{R}$.

The full model

Theorem ($D(v) = v$)

The Riemann problem for the Burgers-particle model

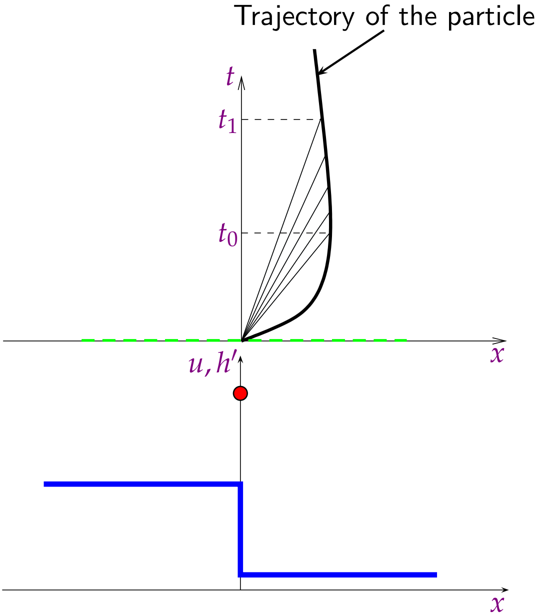
$$\begin{cases} \partial_t u + \partial_x (u^2/2) = \lambda \ (h'(t) - u) \ \delta_0(x - h(t)) \\ m \ h''(t) = \left(\gamma^-_{h(t)} u - \gamma^+_{h(t)} u \right) \left((\gamma^-_{h(t)} u + \gamma^+_{h(t)} u)/2 - h'(t) \right) \\ u_{|t=0} = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases} \\ h(0) = 0, \quad h'(0) = V \end{cases}$$

admits at least one solution for all $u_L, u_R, V \in \mathbb{R}$.

Proof by construction (solutions are not self-similar).

When $t \to +\infty$ or $\lambda \to +\infty$, the solution tends to a solution of the classical Burgers equation and $h'(t)$ to a predictable constant.

Examples of solution of the Riemann problem



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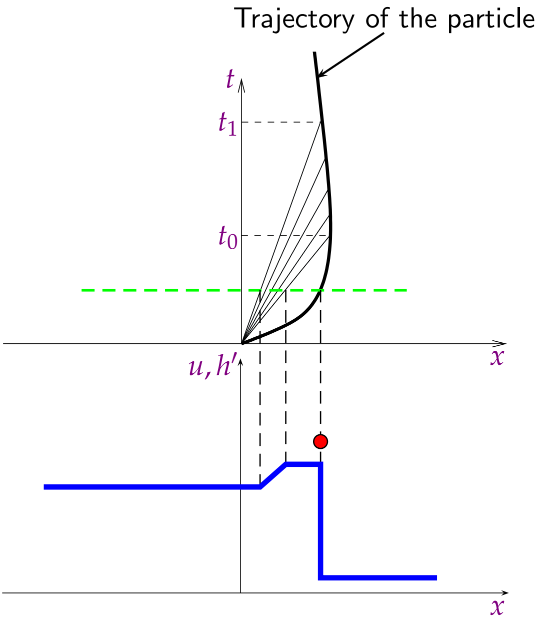

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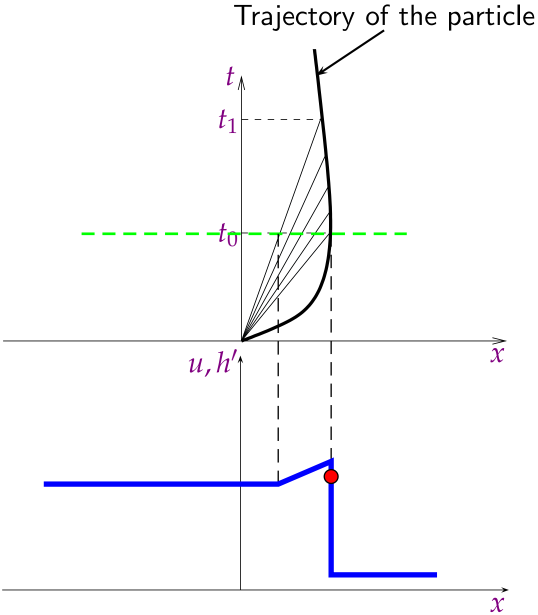

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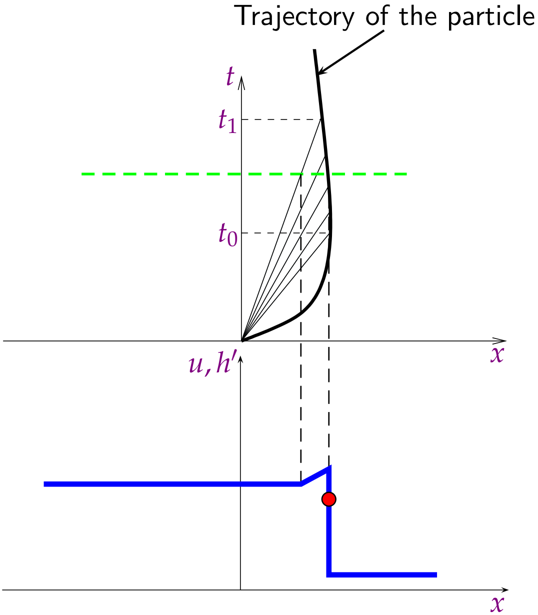

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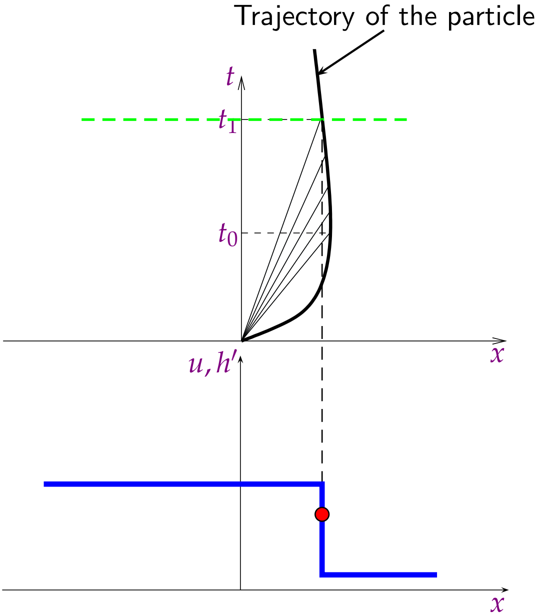

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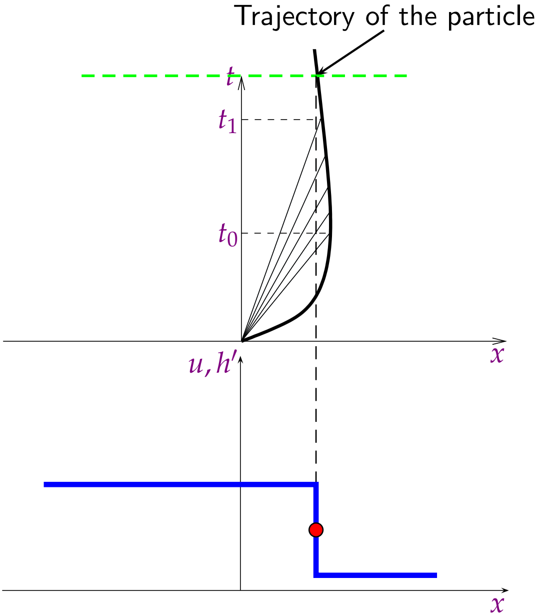

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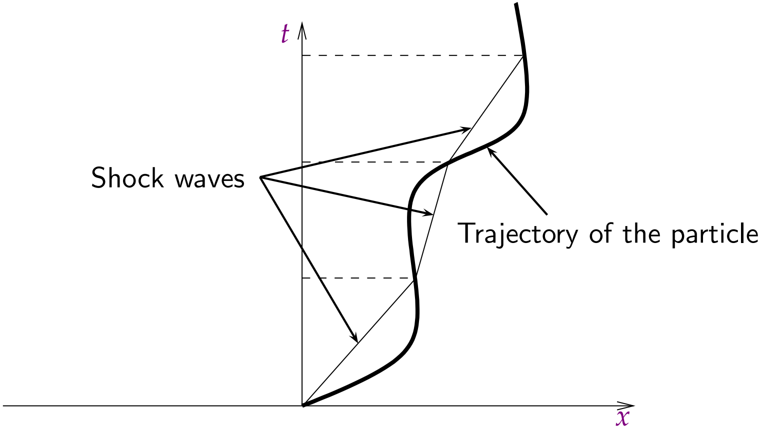

Well-posedness of the Burgers-particle model

Riemann problem

- The solution is not self-similar
- Existence by construction
- Uniqueness... Probably equivalent to study the Cauchy problem

Cauchy problem: still open

- Existence *via* the methods devoted to hyperbolic systems?
- Uniqueness by comparison, continuity...

Numerical methods

Requirements for the numerical methods

- Based on classical Finite Volume schemes for the fluid part
- Accurate tracking of the particle(s) (no diffusion)
- Avoid as much as possible remeshing techniques
- Avoid as much as possible complex Riemann solvers

What we are able to do...

- Burgers equation with a particle with a zero velocity
  - Simple numerical schemes (no Riemann solver)
  - Analysis of convergence
- The full Burgers-particle model
  - Use of the Riemann solver for a particle with a constant velocity
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Burgers equation with a particle with a zero velocity

Numerical methods for

$$\partial_t u + \partial_x(u^2/2) + \lambda\ D(u)\ \delta_0 = 0$$

Mesh: interfaces $x_{i+1/2} = i\Delta x$, cells $K_i = [x_{i-1/2}, x_{i+1/2}[$

- Away from the particle: classical consistent monotone scheme

$$\forall i \neq 0,1 \qquad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right)$$

- Near the particle

$$u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x}\left(g^-_{\lambda,D}(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n)\right)$$

$$u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x}\left(g(u_1^n, u_2^n) - g^+_{\lambda,D}(u_0^n, u_1^n)\right)$$

$$g^+_{\lambda,D}(u^-, u^+) - g^-_{\lambda,D}(u^-, u^+) \approx \lambda\ D(u)$$

Burgers equation with a particle with a zero velocity

Well-balanced schemes (Greenberg-LeRoux, Gosse-LeRoux...)

Preserve some stationary solutions: $\forall (c_L, c_R) \in \mathcal{G}_D^0(0,\lambda) \subset \mathcal{G}_D(0,\lambda)$

If $u_i^n = \begin{cases} c_L & \forall i \leq 0 \\ c_R & \forall i > 0 \end{cases}$, then $u_i^{n+1} = u_i^n$ for all $i \in \mathbb{Z}$.

Insert such a solution in the numerical scheme for $i = 0, 1$:

$$\begin{aligned} c_L = c_L - \frac{\Delta t}{\Delta x}(g^-_{\lambda,D}(c_L, c_R) - f(c_L)) \quad &\Longrightarrow \quad g^-_{\lambda,D}(c_L, c_R) = f(c_L) \\ c_R = c_R - \frac{\Delta t}{\Delta x}(f(c_R) - g^+_{\lambda,D}(c_L, c_R)) \quad &\Longrightarrow \quad g^+_{\lambda,D}(c_L, c_R) = f(c_R) \end{aligned}$$

OK if $\begin{matrix} g^-_{\lambda,D}(c_L, c_R) = g(c_L, \phi^-_{\lambda,D}(c_R)) \\ g^+_{\lambda,D}(c_L, c_R) = g(\phi^+_{\lambda,D}(c_L), c_R) \end{matrix}$ and $\begin{matrix} \phi^-_{\lambda,D}(c_R) = c_L \\ \phi^+_{\lambda,D}(c_L) = c_R \end{matrix}$.

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$$c_R = c_R - \frac{\Delta t}{\Delta x}\left(f(c_R) - g^+_{\lambda,D}(c_L, c_R)\right) \implies g^+_{\lambda,D}(c_L, c_R) = f(c_R)$$

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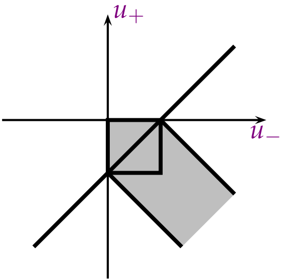
Burgers equation with a particle with a zero velocity

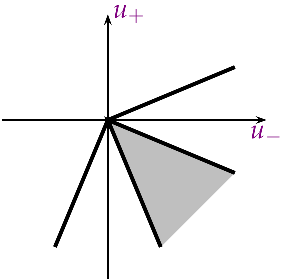
Requirement for the reconstruction functions $\phi^{\pm}_{\lambda,D}$

$\phi^{\pm}_{\lambda,D} : \mathbb{R} \mapsto \mathbb{R}$ invertible functions such that

$$\forall (c_L, c_R) \in \mathcal{G}^0_D(0,\lambda) \subset \mathcal{G}_D(0,\lambda) \qquad \begin{cases} \phi^-_{\lambda,D}(c_R) = c_L \\ \phi^+_{\lambda,D}(c_L) = c_R \end{cases}$$



$\boxed{D(v) = v}$



$\boxed{D(v) = v|v|}$

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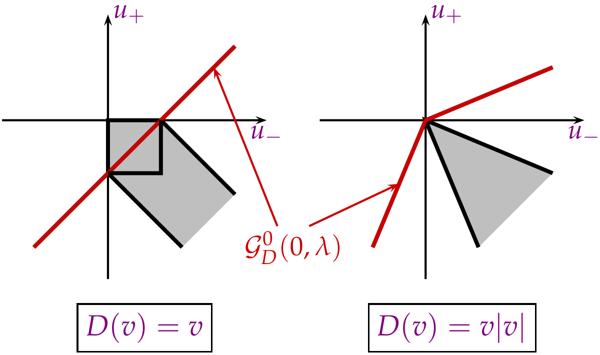
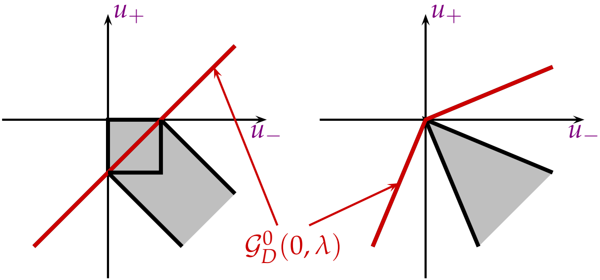


Burgers equation with a particle with a zero velocity

Reconstruction functions $\phi^{\pm}_{\lambda,D}$

$$
\begin{array}{l|l}
D(v) = v & D(v) = v|v| \\
\phi^-(s) = s + \lambda & \phi^-(s) = s\ e^{\mathrm{sgn}(s)\lambda} \\
\phi^+(s) = s - \lambda & \phi^+(s) = s\ e^{-\mathrm{sgn}(s)\lambda}
\end{array}
$$



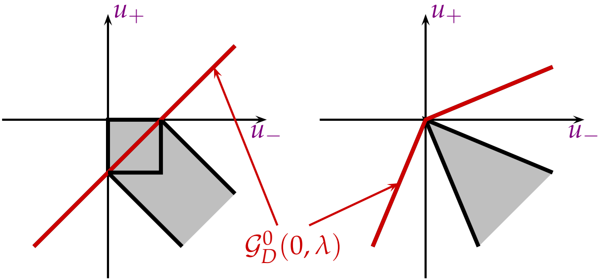
$\boxed{D(v) = v}$ $\boxed{D(v) = v|v|}$

Burgers equation with a particle with a zero velocity

Well-balanced scheme

$$u_0^{n+1} = u_0^n - \frac{\Delta t}{\Delta x}\left(g^-_{\lambda,D}(u_0^n, u_1^n) - g(u_{-1}^n, u_0^n)\right) \qquad g^-_{\lambda,D}(u,v) = g(u, \phi^-_{\lambda,D}(v))$$
$$u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x}\left(g(u_1^n, u_2^n) - g^+_{\lambda,D}(u_0^n, u_1^n)\right) \qquad g^+_{\lambda,D}(u,v) = g(\phi^+_{\lambda,D}(u), v)$$

Exactly preserves any initial data obtained from $\mathcal{G}_D^0(0,\lambda)$.



$\boxed{D(v) = v}$ $\qquad$ $\boxed{D(v) = v|v|}$

Burgers equation with a particle with a zero velocity

Analysis of the well-balanced scheme

Let u_Δ be the piecewise constant function obtained from $(u_i^n)_{i,n}$
Under a classical CFL condition:

- u_Δ is uniformly bounded in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$
- u_Δ is uniformly bounded in $BV_{\mathrm{loc}}([0,T] \times (\mathbb{R} \setminus \{0\}))$ [BGKT 08]
- u_Δ satisfies discrete (adapted) entropy inequalities
- u_Δ converges in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ to the entropy weak solution

Corollary. Existence of the entropy weak solution.

Analysis of $\partial_t u + \partial_x(u^2/2) + \lambda\, D(u)\, \delta_0(x - Vt) = 0$

- Existence and uniqueness of the entropy weak solution
- Convergence of well-balanced schemes (in the frame of the particle)

The full Burgers-particle model

The numerical method

- The mesh is fixed
- The particle is always located at an interface (random sampling)
- The fluid equation is solved using the Riemann problem for the case of a particle with a constant velocity
- The trajectory of the particle is computed by the explicit Euler method

The full Burgers-particle model

The numerical method

From $\{(u_i^n)_{i\in\mathbb{Z}}, h^n, v^n\}$ to $\{(u_i^{n+1})_{i\in\mathbb{Z}}, h^{n+1}, v^{n+1}\}$:

- The particle is located at an interface of the mesh: $h^n \in \{x_{i+1/2}, i\}$
- Solve the linearized (in h') fluid equation for $(t,x) \in [t^n, t^{n+1}] \times \mathbb{R}$

$$\begin{cases} \partial_t u + \partial_x(u^2/2) - \lambda\, D(v^n - u)\, \delta_0(x - v^n t) = 0 \\ u(t^n, x) = u_i^n \quad \text{if } x \in K_i \end{cases}$$

  (exact resolution of self-similar Riemann problems)
  Note $\tilde{u}(x)$ the solution of this problem at time t^{n+1}

- Compute the trajectory of the particle using an explicit Euler method

$$\begin{cases} \tilde{h} = h^n + v^n \Delta t \\ v^{n+1} = v^n + \frac{\Delta t}{m}(\tilde{u}(\tilde{h}^-) - \tilde{u}(\tilde{h}^+))\left(\frac{\tilde{u}(\tilde{h}^-)+\tilde{u}(\tilde{h}^+)}{2} - v^n\right) \end{cases}$$

- Random sampling to define $(u_i^{n+1})_{i\in\mathbb{Z}}$ and h^{n+1} from $\tilde{u}$ and $\tilde{h}$

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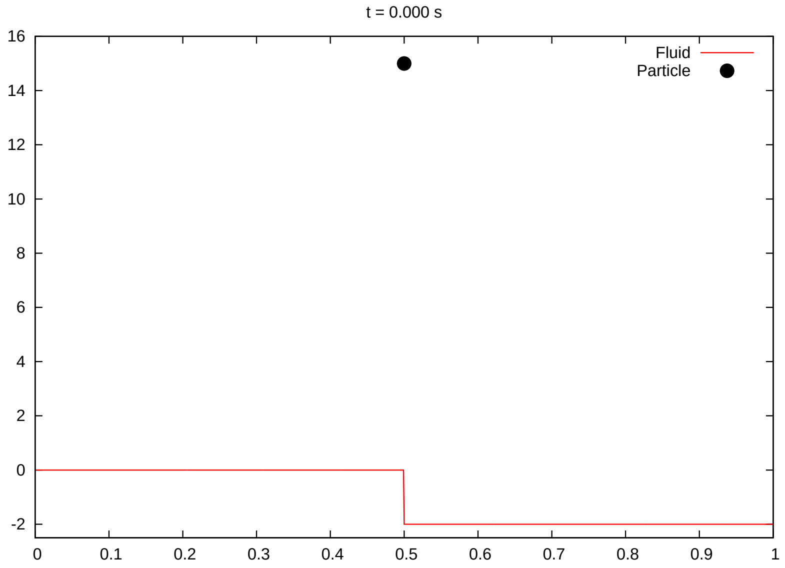
Numerical results: Riemann problem – $\lambda = 10, m = 0.1$



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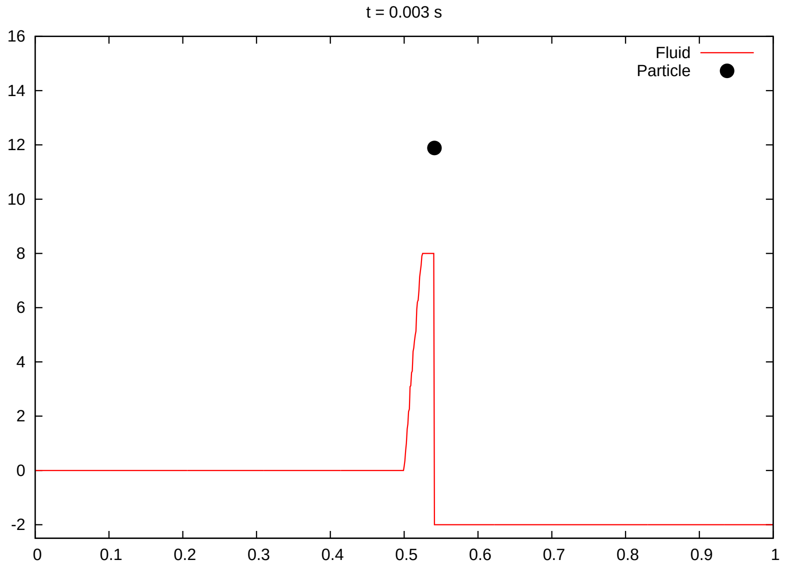

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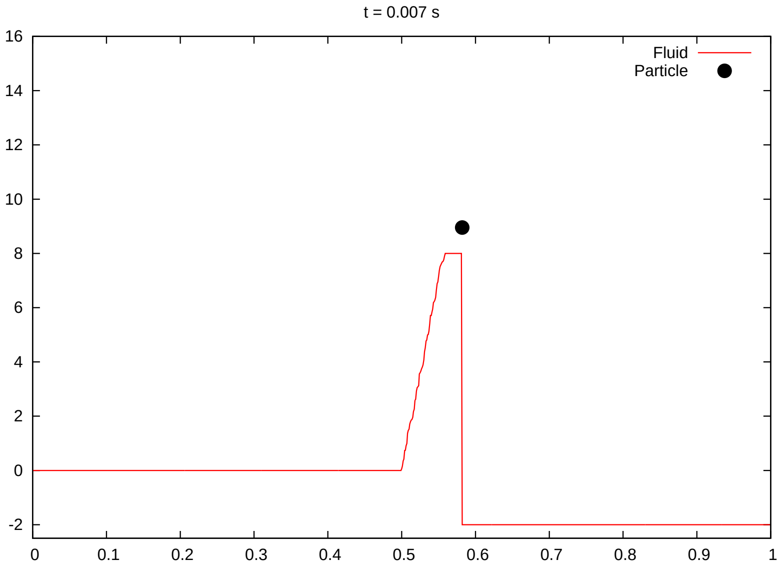

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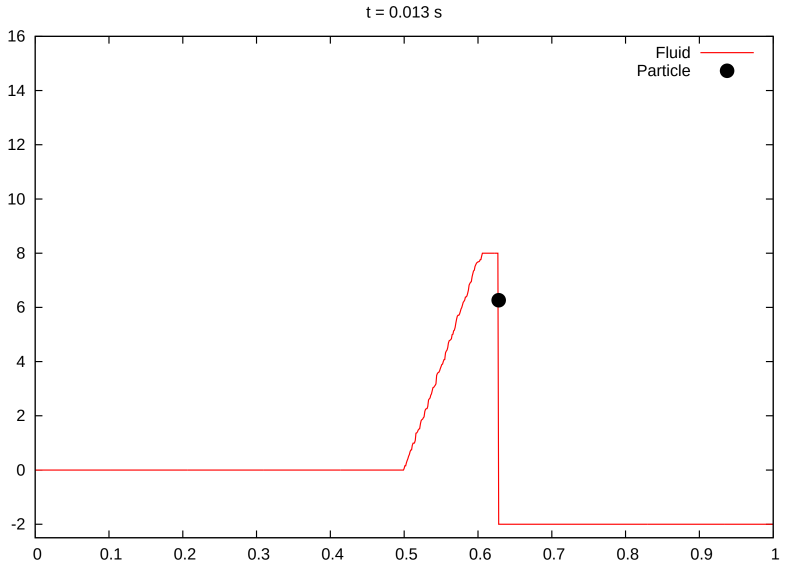

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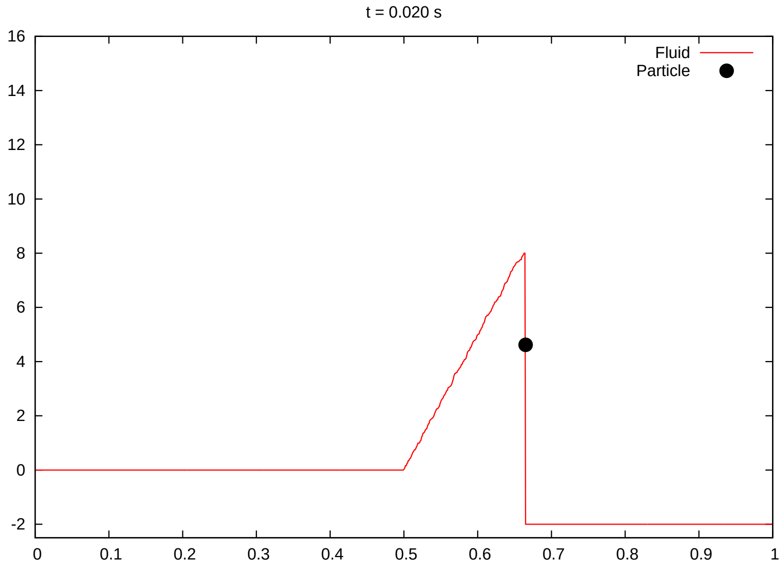

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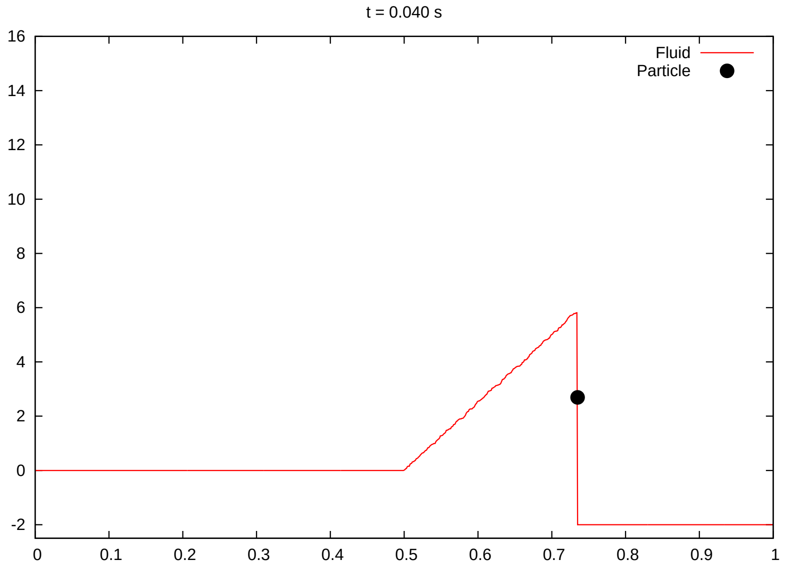

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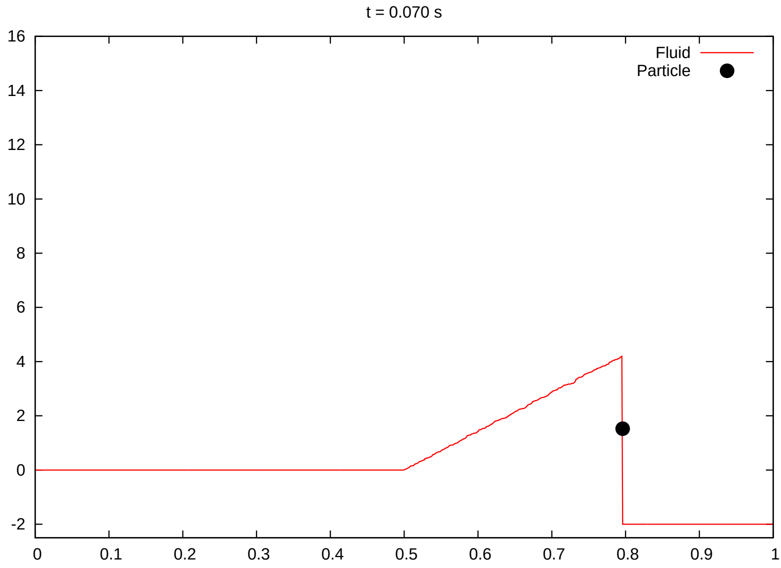

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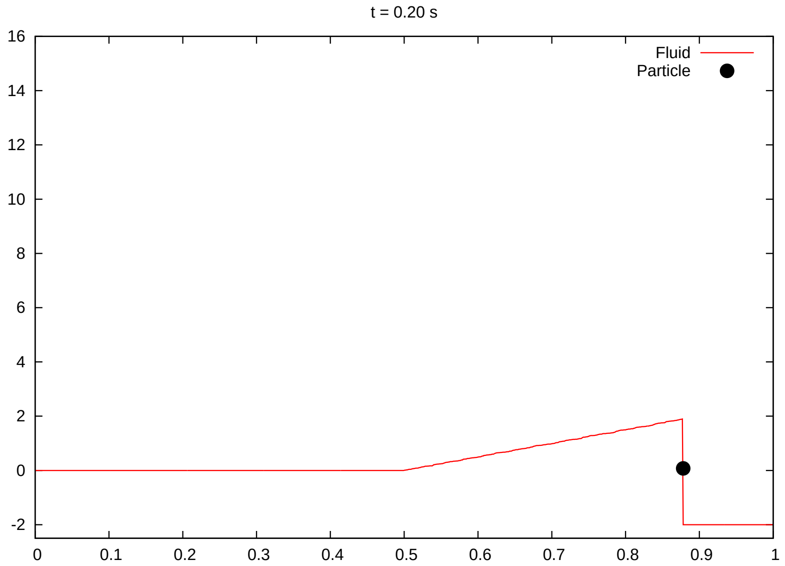

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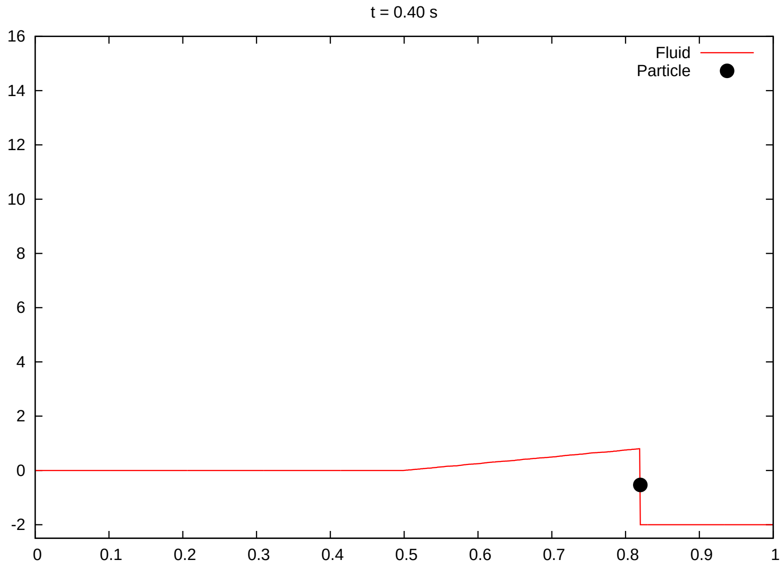

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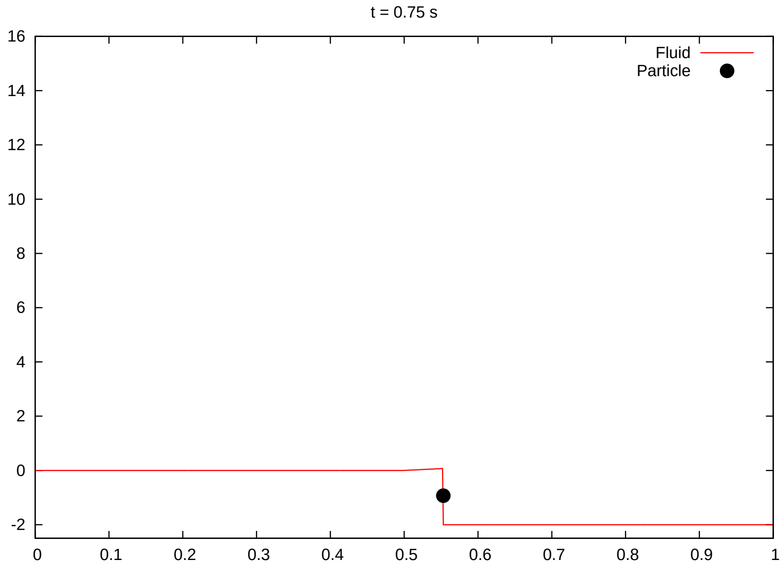
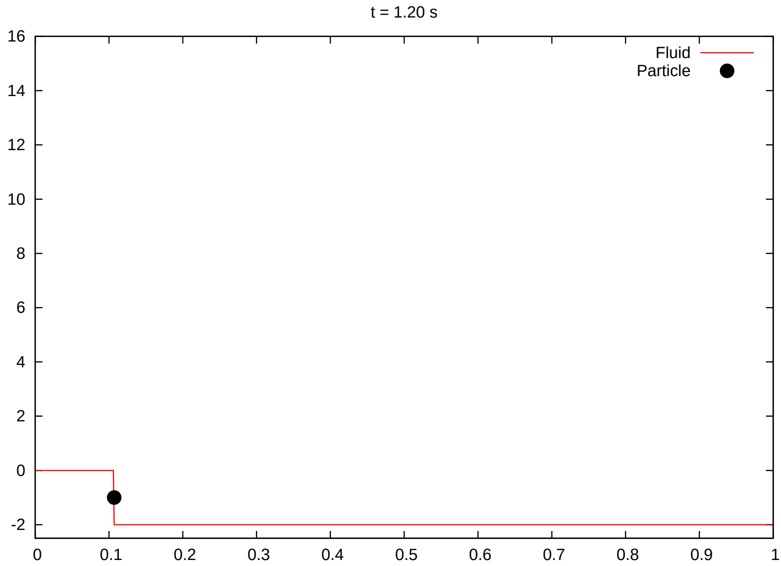
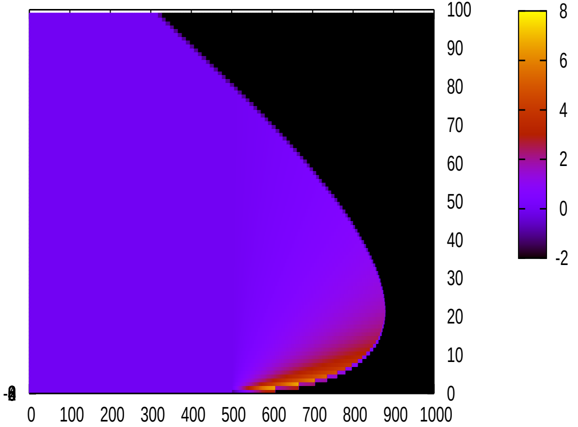


Numerical results: Riemann problem – $\lambda = 10, m = 0.1$



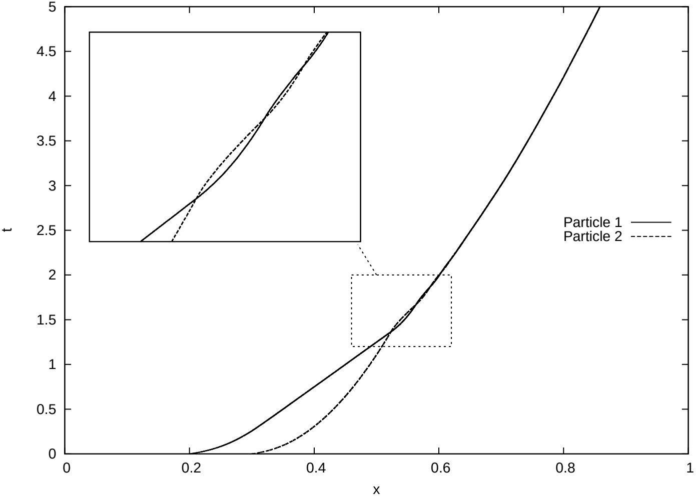
Numerical results: Riemann problem

The solution in the (x, t) plane



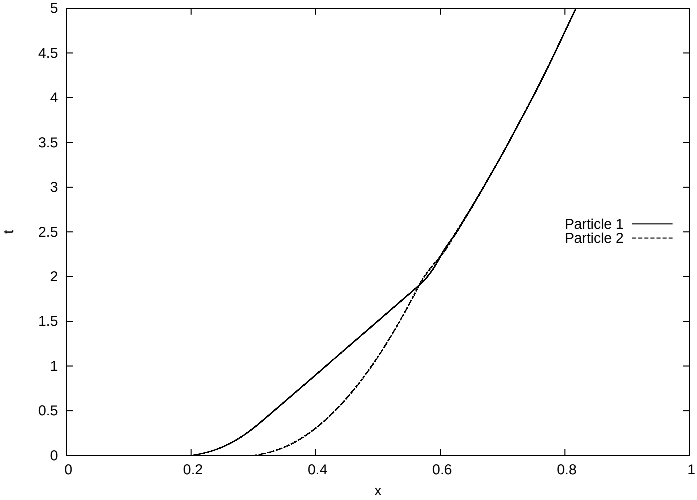
Numerical results: 2 particles (splitting method)

Trajectory of the particles: $m_1 = 2.5 \times 10^{-2}$, $m_2 = 2 \times 10^{-2}$



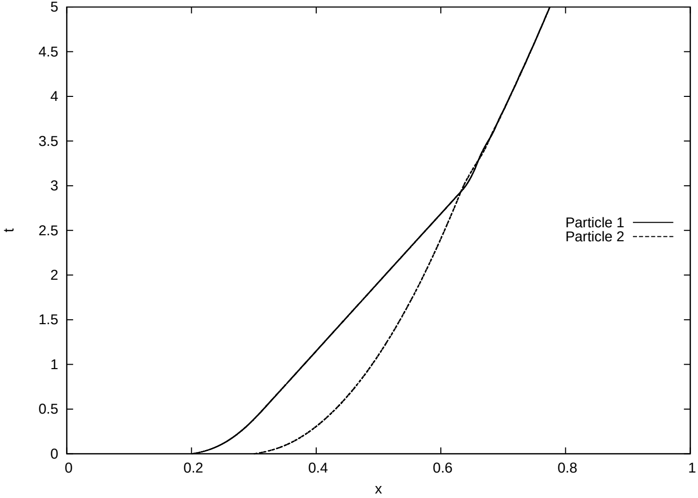
Numerical results: 2 particles (splitting method)

Trajectory of the particles: $m_1 = 2 \times 10^{-2}$, $m_2 = 2 \times 10^{-2}$



Numerical results: 2 particles (splitting method)

Trajectory of the particles: $m_1 = 1.5 \times 10^{-2}$, $m_2 = 2 \times 10^{-2}$



Conclusion

Done...

- Models for the coupling of an inviscid fluid with point-wise particles
- Main difficulties well understood
  - Influence of the particle on the fluid
  - Evolution of the particle
- Several numerical strategies
  - Particle with a constant velocity: simple numerical schemes
  - Burgers-particle model: use the linearized Riemann solver (w.r.t. h')
  - Extension to several particles
- Deep connections with interfacial coupling, with a free boundary
  - Compatibility of the traces of the solution
  - Exchange of informations between both sides (well-balanced schemes)

Conclusion

To do...

- Complete the analysis of the Burgers-particle model
  - Well-posedness of the Cauchy problem
  - Extension to several particles
- Find numerical methods without Riemann solver
  - Use the numerical schemes for a particle with a constant velocity
  - Analysis of convergence
- Extension to 1D-Euler equations for the fluid
- Extension to the multidimensional case
- Adapt these works to the interfacial coupling