Theory of Fuchsian equations and applications to general relativity

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Joint work with P. LeFloch.
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- Allows to study the full solution space in principle.
- Analytic study: requires global-in-time control.
- Numeric study: requires stable reliable numerical schemes.

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Hope: Forward approach is sometimes easier in practice.

Second-order hyperbolic Fuchsian PDEs

Class of equations

$$u_{tt}(t,x)+\frac{2a(x)+1}{t}u_t(t,x)+\frac{b(x)}{t^2}u(t,x)$$
$$=t^{-2}f(t,x,u,u_x,u_t)+c^2(t,x)u_{xx}(t,x)$$

Here,

1. $u:(0,\delta]\times\mathbb{R}^k\to\mathbb{R}^m$ is the unknown (periodic in space).
2. a, b are smooth periodic functions on $\mathbb{R}^k$.
3. c is the speed of propagation.
4. Left side is called Fuchsian principal part.
5. Right side is the Fuchsian source-term with certain “decay properties” at $t=0$.

Heuristics for canonical leading-order terms

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Canonical two-term expansion

$$u(t,x) = \begin{cases} u_*(x)t^{-\lambda_1} + u_{**}(x)t^{-\lambda_2} + O(t^{-\Re\lambda_2+\alpha}), & a^2(x) \neq b(x), \\ u_*(x)t^{-\lambda_1}\log t + u_{**}(x)t^{-\lambda_1} + O(t^{-\lambda_1+\alpha}), & a^2(x) = b(x), \end{cases}$$

at $t = 0$ for some $\alpha > 0$, where

$$\lambda_1(x) := a(x) + \sqrt{a^2(x) - b(x)}, \quad \lambda_2(x) := a(x) - \sqrt{a(x)^2 - b(x)}.$$

Note: Functions u_*, u_{**} are called asymptotic data.

Examples

1. Euler–Poisson–Darboux equation: For $\kappa \geq 0$,

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Canonical leading-order behavior:

$$u(t,x) = \begin{cases} u_*(x) + u_{**}(x)t^\kappa + \dots & \kappa > 0, \\ u_*(x)\log t + u_{**}(x) + \dots & \kappa = 0. \end{cases}$$

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2. (Main evolution part of the) Gowdy vacuum equations:
$$P_{tt} + P_t/t = P_{xx} + e^{2P}(Q_t^2 - Q_x^2),$$
$$Q_{tt} + Q_t/t = Q_{xx} - 2(P_t Q_t - P_x Q_x).$$
Canonical leading-order behavior:
$$P(t,x) = -k(x)\log t + P_{**}(x) + \dots,$$
$$Q(t,x) = Q_*(x) + Q_{**}(x)t^{2k(x)} + \dots.$$

Singular initial value problem for 2nd-order hyperbolic Fuchsian systems

Singular initial value problem

Look for solutions of the form

$$\text{Solution } u = \underset{\text{(prescribed)}}{\text{Leading-order term}}\ u_0 + \underset{\text{(unknown)}}{\text{Remainder}}\ w,$$

where the remainder must be "higher-order" at $t = 0$ in the sense of weighted Sobolev spaces.

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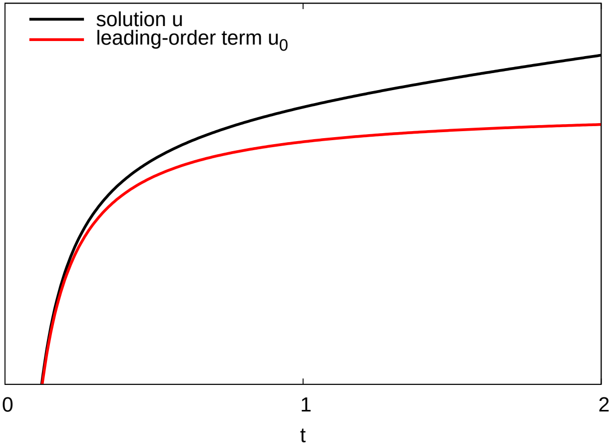
Existence and uniqueness for Fuchsian systems by Kichenassamy and Rendall. However:

- Proof not natural for hyperbolic equations. Make direct use of hyperbolic structure?
- Approximation scheme not of "practical" use for the numerical treatment. Moreover, "practical" error estimates?

New approximation scheme

We use the following new approximation scheme.



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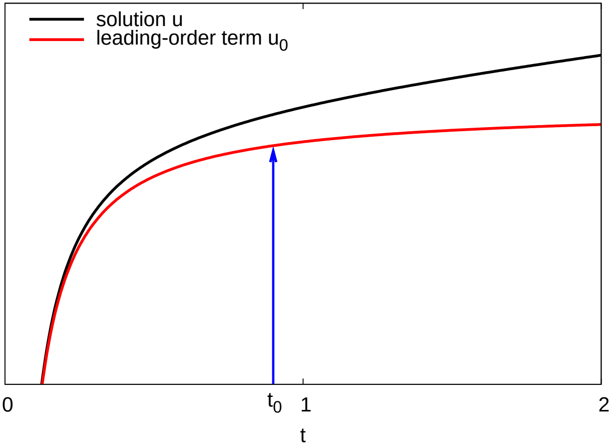

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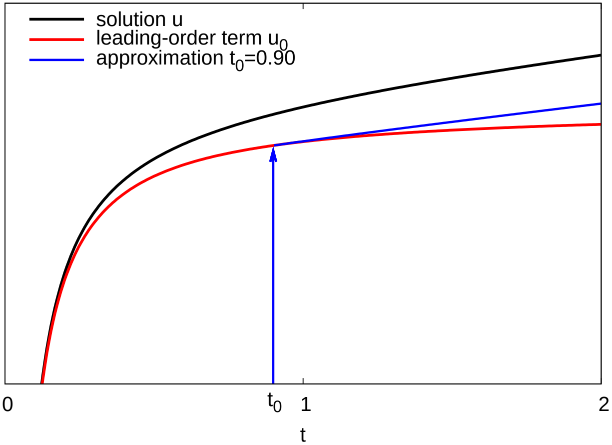

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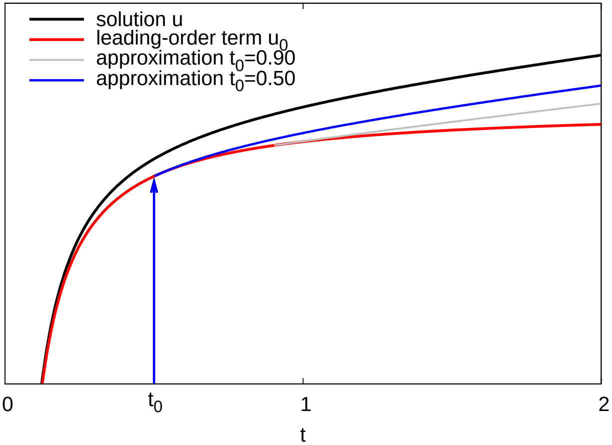

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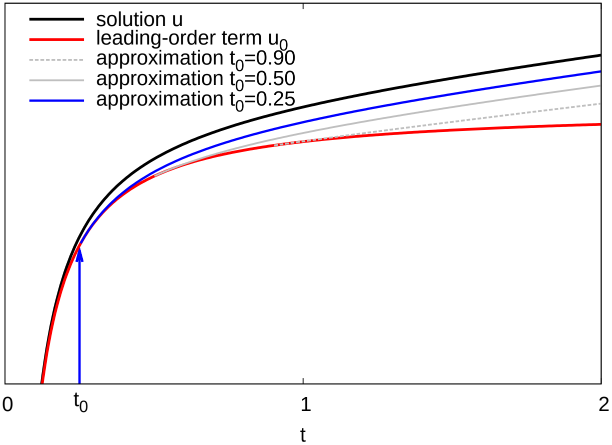

Hope: Convergence in the limit $t_0 \to 0$.

Well-Posedness in an important case

Consider $f(t,x,u,\partial_x u,Du) = f_0(t,x)$ for a given function f_0.

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Result

For any asymptotic data $u_*, u_{**} \in H^2(U)$, there exists a unique solution of the SIVP with remainder $w \in \tilde{X}_{\delta,\alpha,1}$ and the previous sequence of approximate solutions converges, provided:

1. we can choose $\delta, \alpha > 0$ so that the matrix

$$\begin{pmatrix} \lambda_1 - \lambda_2 + \alpha & -\eta/2 & 0 \\ -\eta/2 & \alpha & t\partial_x c - \partial_x(\lambda_1 - \lambda_2)(tc\ln t) \\ 0 & t\partial_x c - \partial_x(\lambda_1 - \lambda_2)(tc\ln t) & \lambda_1 - \lambda_2 + \alpha - 1 - Dc/c \end{pmatrix}$$

   is positive semidef. at each $(t,x) \in (0,\delta) \times U$ for a $\eta > 0$.
2. $f_0 \in X_{\delta,\alpha+\varepsilon,0}$ for some $\varepsilon > 0$.
3. $\alpha < 2(\beta(x)+1) - (\lambda_1(x) - \lambda_2(x))$ for all $x \in U$.

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Note: It is possible to generalize the result to non-linear sources.

Results

Approximation scheme for numerics of the singular initial value problem (F.B., P. LeFloch, arXiv:1006.2525)

We have found an approximation scheme for numerical construction of solutions of the singular initial value problem with "practical" error estimates.

Numerics of the singular initial value problem

General numerical approach, using EPD-equation as a model:

$$u_{tt} - (\kappa - 1)u_t/t = u_{xx}.$$

1. Introduce time variable $\tau := \ln t$. Then equation becomes

   $$\partial_\tau^2 u - \kappa \partial_\tau u - e^{2\tau}\partial_x^2 u = 0.$$

   Singularity is "shifted to $\tau = -\infty$".
2. Write the equation for the remainder w, after having fixed asymptotic data u_*, u_{**}.
3. Direct discretization of the second-order equation (following the ideas of Kreiss et al.).
4. Solve sequence (w_n) of solutions of RIVPs with initial times $\tau_n \to -\infty$ with data

   $$w_n(\tau_n) = 0, \quad \partial_\tau w_n(\tau_n) = 0.$$

Application: Gowdy vacuum spacetimes

Theorem (Kichenassamy, Rendall; F.B., P.LeFloch)

Let $k, P_{**}, Q_{*}, Q_{**} \in C^\infty(U)$ and $0 < k < 1$. Choose as leading-order term

$$P(t,x) = -k(x)\log t + P_{**}(x) + \dots,$$
$$Q(t,x) = Q_*(x) + Q_{**}(x)t^{2k(x)} + \dots.$$

Then there exists a unique solution of the Gowdy equations

$$P_{tt} + P_t/t = P_{xx} + e^{2P}(Q_t^2 - Q_x^2),$$
$$Q_{tt} + Q_t/t = Q_{xx} - 2(P_tQ_t - P_xQ_x),$$

which obeys this leading-order term (up to some minor subtleties).

Application: Gowdy vacuum spacetimes

Remarks:

- Interpretation of the condition $0 < k < 1$: prevent the solutions from forming "spikes".
- We are allowed to choose $k \geq 1$ if $Q_* = const$. Note that the curvature stays bounded at points at $t = 0$ where $k = 1$.

Application: Gowdy vacuum spacetimes

To solve Einstein's field equations, we need to solve additionally

$$\Lambda_{tt} - \Lambda_{xx} = P_x^2 - P_t^2 + e^{2P}(Q_x^2 - Q_t^2).$$

and

$$\Lambda_x = 2t\left(P_x P_t + e^{2P} Q_x Q_t\right), \quad \Lambda_t = t\left(P_x^2 + te^{2P} Q_x^2 + P_t^2 + e^{2P} Q_t^2\right).$$

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We can show:

- Implies a 2nd-order hyperbolic Fuchsian equation for Λ. Canonical leading-order term:

$$\Lambda(t,x) = \Lambda_*(x)\ln t + \Lambda_{**}(x) + \dots$$

- Constraints imply:

$$\Lambda_* = k^2,$$
$$\Lambda_{**} = \Lambda_0 + 2\int_0^x k(-\partial_{\tilde{x}} P_{**} + 2e^{2P_{**}} Q_{**} \partial_{\tilde{x}} Q_*)\, d\tilde{x},$$

$\Rightarrow$ Solution of full Einstein's field equations for all $t > 0$.

Gowdy solutions with Cauchy horizons

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But: For Gowdy solutions, we know conditions on asymptotic data which guarantee $t = 0$-surface to be a Cauchy horizon.

Hence: It should be possible to compute such solutions numerically with the Fuchsian forward approach!

Gowdy solutions with Cauchy horizons

A particular choice of asymptotic data:
Solution with an incomplete Cauchy horizon (smooth, but not analytic!):

$$k(x) = \begin{cases} 1, & x \in [\pi, 2\pi], \\ 1 - e^{-1/x} e^{-1/(\pi - x)}, & x \in (0, \pi), \end{cases}$$

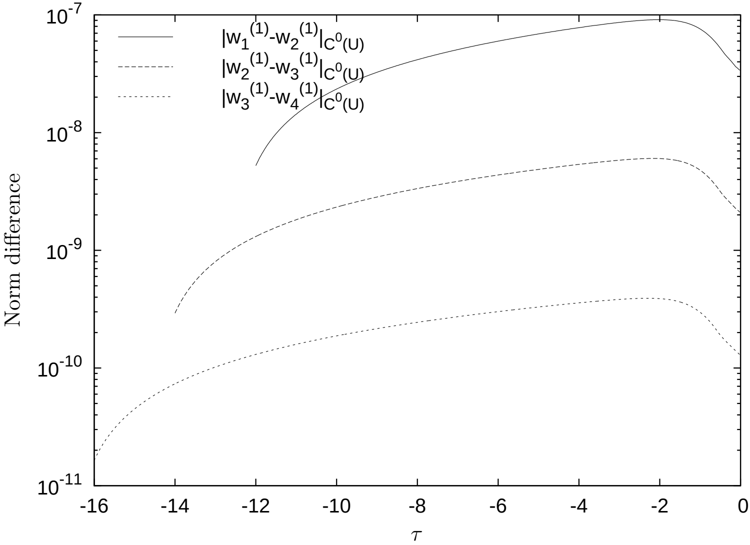
$$P_{**}(x) = 1/2, \quad Q_*(x) = 0,$$

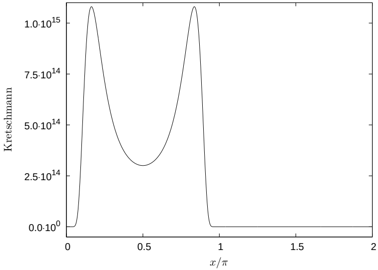
$$Q_{**}(x) = \begin{cases} 0, & x \in [\pi, 2\pi], \\ e^{-1/x} e^{-1/(\pi - x)}, & x \in (0, \pi), \end{cases}$$

$$\Lambda_*(x) = k^2(x), \quad \Lambda_{**}(x) = 2.$$

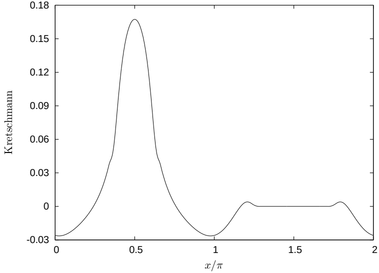
Num. solutions of Gowdy equations for previous data

Convergence of approximate solutions:

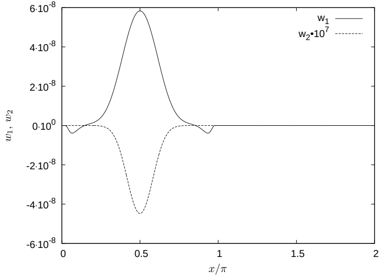




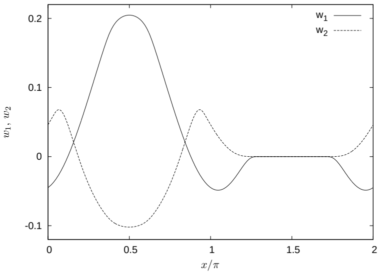
(a) Kretschmann at $\tau = -10$.



(b) Kretschmann at $\tau = 0$.



(c) Remainders of P, Q at $\tau = -10$.



(d) Remainders of P, Q at $\tau = 0$.

Summary and outlook

We have applied the theory so far to the $\mathbb{T}^3$-Gowdy vacuum equations:

- Simpler proof of well-posedness than Kichenassamy/Rendall.
- Computed numerical solutions in various test cases in order to understand the numerical approximation scheme.
- Numerical solutions for Gowdy solutions with (incomplete) Cauchy horizons.

We plan to do in the near future:

- Analyze the geometry of Gowdy solutions with (incomplete) Cauchy horizons.
- Reconsider the problem of Gowdy solutions with spatial topologies $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$ (Stahl).
- Behavior of solutions of the Einstein-Euler equations under Gowdy symmetry. “Interaction of shocks and cosmological singularity”?