Gravitating positiv–negative mass systems

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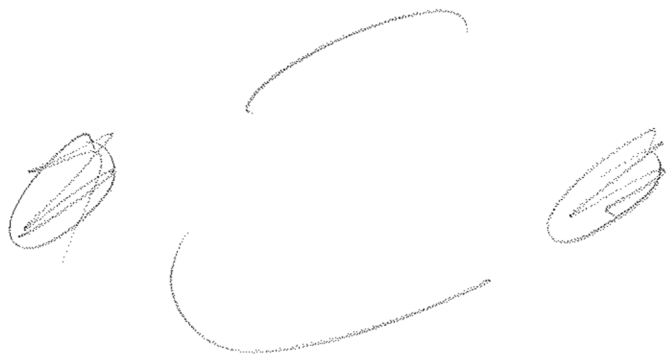
Motivation:

Lichtenstein(1920)

$L\omega^2 = G\frac{M}{(2L)^2}$



In GR such a system radiates. Does there exist a solution in which incoming radiation balances the outgoing radiation?

Complicated; $M_{ADM} = \infty$; the meaning of the helical symmetry $\partial_t + \partial_\phi$ is unclear.

A simpler systen:

Newton: $b = \frac{MG}{L^2}$ for point particles

Plan of the talk:

1. The Bonnor–Swaminarayan solutions

This solution has an interpretation as self-accelerating + − point particles

2. Positiv–negative mass systems in GR

Here I will try to give a formulation of the problem in GR.

1. The Bonnor–Swaminarayan solutions (1964)

The metric: $(\mathrm{t}, \mathrm{x}, \mathrm{y}, \mathrm{z}) \in \mathbf{R}^4$

$$
\begin{aligned}
\mathrm{ds}^2 &= \mathrm{e}^{\lambda}\mathrm{d}\rho^2 + \rho^2\mathrm{e}^{\mu}\mathrm{d}\phi^2 \\
&+(\mathrm{z}^2 - \mathrm{t}^2)^{-1}(\mathrm{z}^2\mathrm{e}^{\lambda} - \mathrm{t}^2\mathrm{e}^{\mu})\mathrm{dz}^2 \\
&-(\mathrm{z}^2 - \mathrm{t}^2)^{-1}(\mathrm{z}^2\mathrm{e}^{\mu} - \mathrm{t}^2\mathrm{e}^{\lambda})\mathrm{dt}^2 \\
&-2(\mathrm{z}^2 - \mathrm{t}^2)^{-1}\mathrm{zt}(\mathrm{e}^{\lambda} - \mathrm{e}^{\mu})\mathrm{dzdt}
\end{aligned}
$$

$$
\lambda(\rho^2, \mathrm{z}^2 - \mathrm{t}^2), \quad \mu(\rho^2, \mathrm{z}^2 - \mathrm{t}^2)
$$

$\lambda = \mu = 0$ is Minkowski space.

This geometry is characterized by two hypersurface orthogonal Killing vectors.

Weyl solutions, Eintein Rosen waves ......

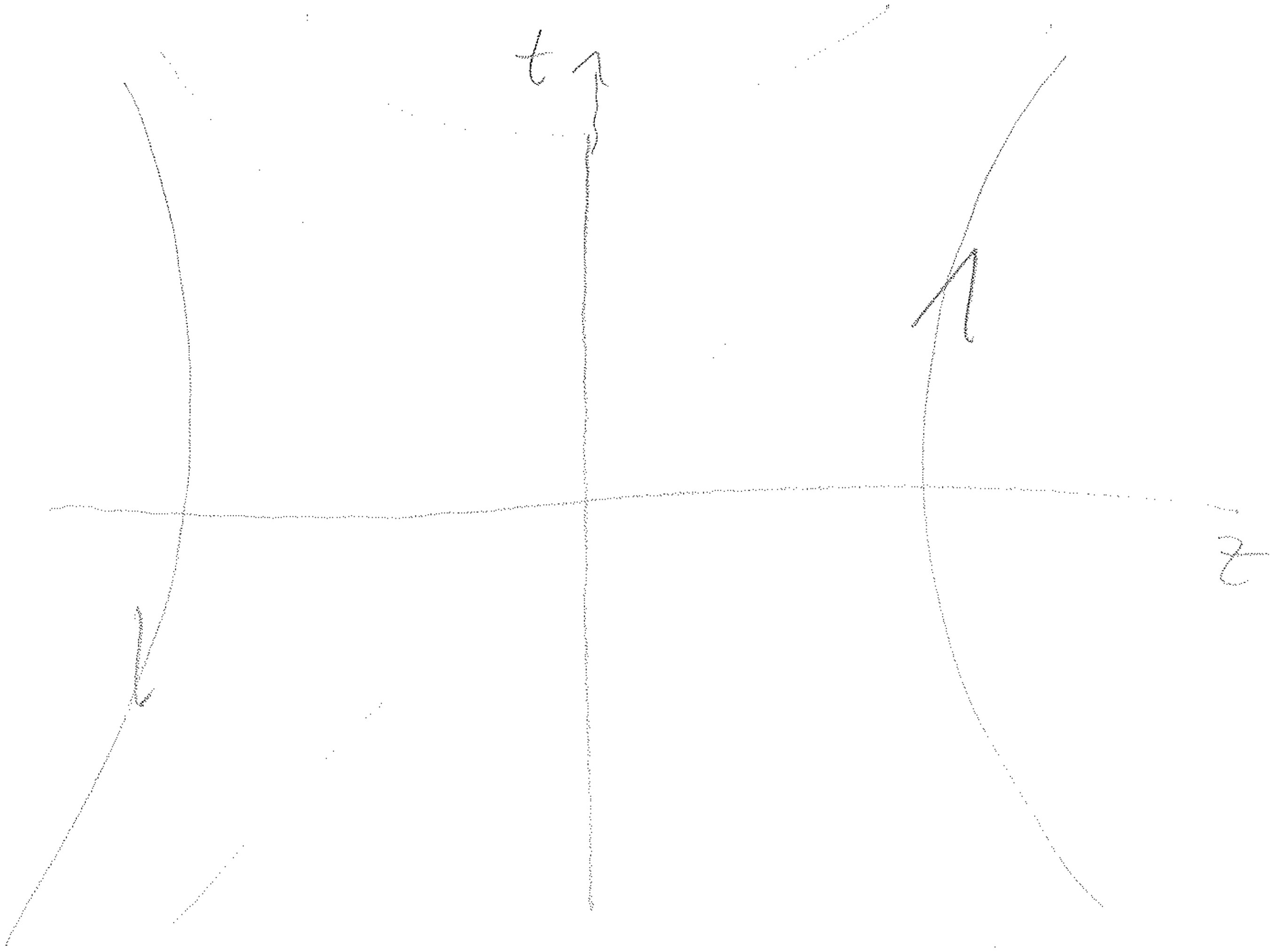
In Minkowski space we have Killing vectors

$$
\partial_{\mathrm{t}}, \mathrm{x}\partial_{\mathrm{y}} - \mathrm{y}\partial_{\mathrm{x}} \quad , \quad \partial_{\mathrm{z}}, \mathrm{x}\partial_{\mathrm{y}} - \mathrm{y}\partial_{\mathrm{x}} \quad , \partial_{\mathrm{x}}, \partial_{\mathrm{y}}
$$

Fixed points and asymptotic bahaviour determines the character of the symmetries.

Boost –rotational symmetry:

$$\mathbf{z}\partial_{\mathbf{t}} + \mathbf{t}\partial_{\mathbf{z}} \ , \quad \mathbf{x}\partial_{\mathbf{y}} - \mathbf{y}\partial_{\mathbf{x}}$$



$$-w^2 d\tau^2 + dw^2 + d\rho^2 + \rho^2 d\varphi^2$$

The vacuum field equations $R_{ab} = 0$ imply

$$\mu_{\rho\rho} + \frac{\mu_\rho}{\rho} + \mu_{ww} + \frac{\mu_w}{w} = 0$$

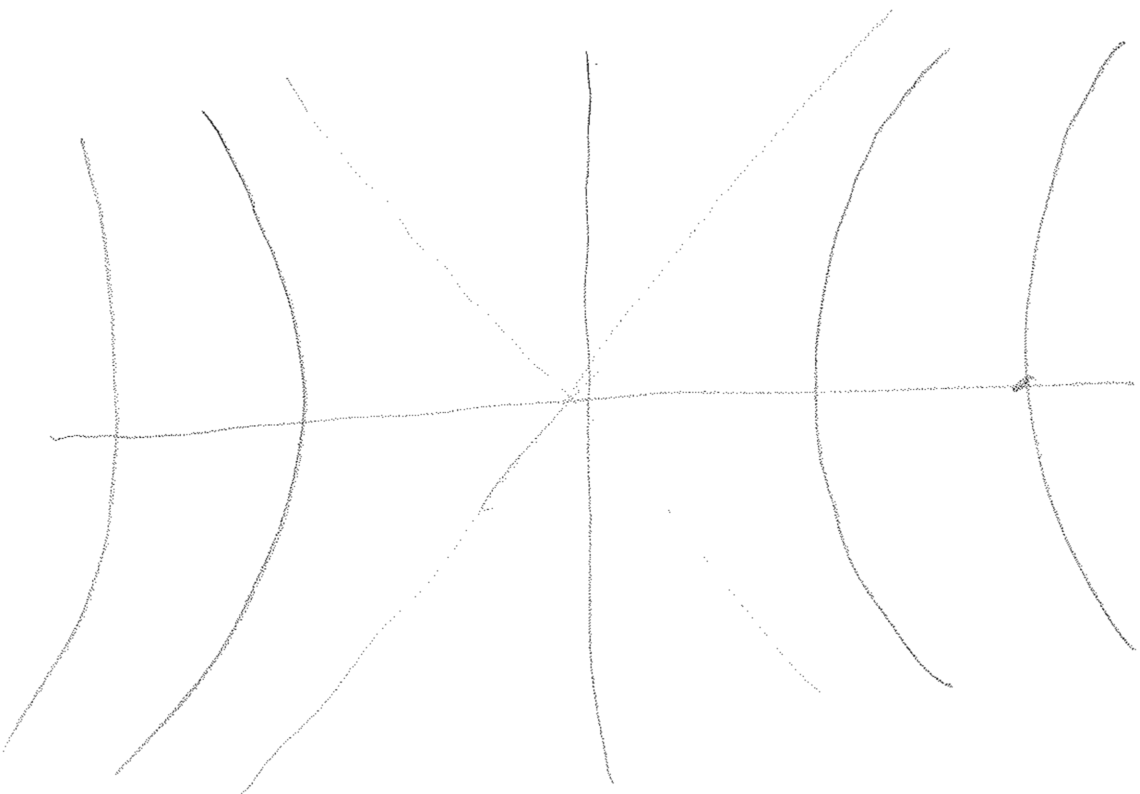
$(\rho^2 = x^2 + y^2, w^2 = z^2 - t^2)$

λ is determined by integration once μ is known.

A boost– rotation invariant solution of the flat space scalar wave equation determines a boost rotationaly symmetric space time.

This class of solutions is analysed in detail by Bicak,Schmidt (1989).

Bonnor,Swaminarayan (1964) decribed a particularly nice solution which is characterized by "4 point particles" moving on boost orbits.



$$\mu = -\frac{2a_1}{R_1} - \frac{2a_2}{R_2} + \frac{2a_1}{h_1} + \frac{2a_2}{h_2}$$

$$R_i = [(R - h_i)^2 + 2\rho^2 h_i]^{\frac{1}{2}}$$

$$R = \frac{1}{2}(\rho^2 + z^2 - t^2)$$

$$\lambda = \ldots$$

The solution has only singularities at the particle positions and is asymptotically flat for

$$a_1 = \frac{(h_1 - h_2)^2}{(2h_2)^{-1}}\ , a_2 = -\frac{(h_1 - h_2)^2}{(2h_1)^{-1}}$$

Bondy (1957) found the solution in the right static sector and gave arguments that the singularities correspond to a positive– negative pair of particles running to the right.

2.Positiv–negative mass systems in GR

The idea is to try to replace the singularities of the B–S–solution by bodies of finite extent.

For the Newtonian case this is posssible. One can show the existence of two elastic bodies with different sign of their masses which move with constant acceleration.

(Elastic bodies because the boundary value problem for time indpendent situations is better understood than the case of fluids.)

Elasticity in GR:

Hergloz, Rayner, Carter, Quintana, Kijowski, Magli ...

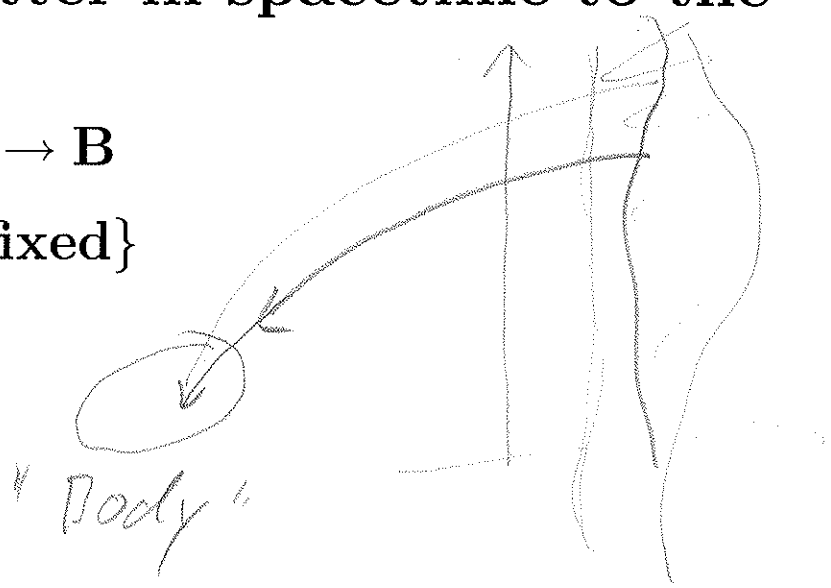
The PDE aspect was always neglected. The dynamical field is a map from the support of matter in spacetime to the "body":

$\mathbf{f^A(x^\alpha) : sup(T^{\alpha\beta}) \to B}$

$\mathbf{\{x^\alpha | f^A(x^\alpha) = X^b \text{ fixed}\}}$

timelike curve

$\mathbf{f^A{}_{|\alpha} u^\alpha = 0}$



The theory is defined by the action

$$\mathbf{S(g_{\alpha\beta}, f^A{}_{|\mu}) = \int R\sqrt{-g}d^4x + \int \rho\sqrt{-g}d^4x}$$

$\rho = n\epsilon$ **is the elastic energy in the rest sytem**

$$\epsilon = \mathbf{epsilon(H^{AB}, f^C)}\ , \mathbf{H^{AB} = f^A{}_{|\mu} f^B{}_{|\nu} g^{\mu\nu}}$$

$$\mathrm{n} = \det(\mathrm{f^A}_{|\mu})$$

The theory is diffeomorphism invariant.

Cauchy stress tensor:

$$\sigma_{\mu\nu} = -2\mathrm{n}\frac{\partial\epsilon}{\partial \mathrm{H^{AB}}}\mathrm{f^A}_{|\mu}\mathrm{f^B}_{|\nu}$$

Energy momentum tensor

$$\mathrm{T}_{\mu\nu} = 2\frac{\partial\rho}{\partial \mathrm{g}^{\mu\nu}} - \rho \mathrm{g}_{\mu\nu}$$

Elasticity equations

$$\mathrm{T}^{\mu\nu}{}_{;\nu} \iff \delta\int \rho\sqrt{-\mathrm{g}}\,\mathrm{d}^4\mathrm{x}$$

Quasilinear second order PDEs for $\mathrm{f^A}$

$$\mathrm{f^A}_{|\mu}\left[\mathrm{M}^{\nu\lambda}{}_{\mathrm{AB}}\partial_\nu\partial_\lambda \mathrm{F^B} + \mathrm{l.o.}\right] = 0$$

Theorem:(Beig, Schmidt 2003)

For reasonable conditions on ϵ the equations are symmetric hyperbolic.

Local Cauchy problem solvable in a given spacetime.

Suppose we want to find the solution first just in the static boost sector. This means to find a solution satisfying:

1.) There is a Killing vector ξ which behaves globally like a boost in Minkowski space.

2.) The spacetime is asymptotically flat in the sense of admitting future and past infinities as well as a "regular" spacelike infinity.

4.) There are two elastic bodies whose matterflow is aligned with the Killing vector.

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The field equation decompose in the well known way: $(\xi = \partial_\tau\ )$

$$-\mathbf{V}(\mathbf{x}^{\mathbf{i}})^2 \mathbf{d}\tau^2 + \mathbf{g}_{\mathbf{ik}}(\mathbf{x}^{\mathbf{l}})\mathbf{d}\mathbf{x}^{\mathbf{i}}\mathbf{d}\mathbf{x}^{\mathbf{k}}$$

$$\triangle_{\mathbf{g}} \mathbf{V} = 4\pi \mathbf{G}\mathbf{V}(\rho + \sigma)$$

$$\mathbf{R}_{\mathbf{ik}} - \frac{1}{\mathbf{V}}\mathbf{V}_{,\mathbf{i};\mathbf{k}} = 4\pi\mathbf{G}((\rho - \sigma)\mathbf{g}_{\mathbf{ik}} + 2\sigma_{\mathbf{ik}})$$

The equation of motion for the elastic matter written on the quotient of the boost Killing vector are $(\mathrm{U} = \ln \mathrm{V})$

$$e^{-U}\nabla_i(e^U \sigma^i{}_k) + n\epsilon\nabla_k U = 0$$

Andersson,Beig and Schmidt (2007) have shown the existence of a solution for small G describing a small, static elastic body without symmetries. The solution is determined by the boundary conditions of asymptotic flatness.

In our case the boundary conditions are: (w, x, y) are coordinates on the quotient.

1.)$V \sim wb$ at $w = 0$ and at infinity

2.) $g_{AB,w} = 0$

$g_{ww} = 1$ at $w = 0$

$g_{Aw} = 0$

$g_{ik} \to \delta_{ik}$ at infinity

In harmonic coordinates the differential operator in the Ricci tensor becomes elliptic.

The condition $V \sim wb$ at $w = 0$ is not a good boundary condition.

The equations are singular at the fixed point of the boost Killing vector, $w = 0$.

So, the first challenge is to show existens for this singular elliptic boundary value problem.

Suppose we have a solution as described above. How do we find the global 4–dimensional spacetime?

The metric in the static boost sector is

$(\mathrm{x^i} = (\mathrm{w}, \mathrm{x^A}))$

$$-\mathrm{V}(\mathrm{x^i})^2 \mathrm{d}\tau^2 + \mathrm{g_{ik}}(\mathrm{x^l}) \mathrm{dx^i dx^k}$$

Introducing coordinates

$$\mathrm{z} = \mathrm{w} \cosh \tau \ , \quad \mathrm{t} = \mathrm{w} \sinh \tau$$

which implies

$$\mathrm{dw}^2 = \frac{1}{\mathrm{z}^2 - \mathrm{t}^2} (\mathrm{zdz} - \mathrm{tdt})^2$$

$$\mathrm{w}^2 \mathrm{d}\tau^2 = \frac{1}{\mathrm{z}^2 - \mathrm{t}^2} (\mathrm{zdt} - \mathrm{tdz})^2$$

The boundary conditions at $\mathrm{w} = 0$ imply that the metric is regular for all $(\mathrm{t}, \mathrm{z}, \mathrm{x^A})$.

It is in fact analytic if all quantities depend only on w^2 which implies that $\mathrm{z} \to -\mathrm{z}$ is an isometry.

How can we prove the existence of solutions? In all the time independent problems we analysed, we found a solution near a known solution using the implicit function theorem.

The only solution we know are two stress–free elastic bodies with $b = G = 0$. No gravity, no acceleration.

Try to find nearby solutions. We need $b \sim G$ because gravity has to balance the acceleration. Hence small b implies small G. Hence the bodies are further and further away from the fixed point of the boost.

Comparison with rotation suggests to use

$$\partial_t + b(t\partial_z + z\partial_t)$$

as the matterflow and the boost Killing vector. Its fixed point is at $z = -\frac{1}{b}$ and moves off to infinity for $b \to \infty$.

Linearized solution:

$$V = 1 + bz + G\delta V$$

$$\Delta_0 \delta V = 4\pi\rho_0$$

If we take the Poisson integral for δV the bondary condition is not satisfied; however, $\delta V(-\frac{1}{b})$ is of order bG, hence second order.

Using δV one can solve the linearized elasticity equations provided $G \sim b$. Here the Newtonian equilibration condition appears.

Finally one solves linear elliptic equtions for the first order metric. ($\delta V_{,ww}$ is regular at the boundary).

Unfortunately, I do not know how to proceed to higher orders in the spirit of the implicit function theorem.