Oscillatory asymptotic behavior for some vacuum Bianchi spacetimes

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Bianchi cosmological models : presentation

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Raisons d'être :

- natural **finite dimensional** class of spacetimes ;
- **BKL conjecture :** *general spacetimes behave like homogeneous spacetimes near their initial singularity*.

Bianchi cosmological models : definitions

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$$M \simeq I \times G \qquad g = -dt^2 + h_t$$

where $I = (t_-, t_+) \subset \mathbb{R}$,
G is 3-dimensional Lie group,
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- A **Bianchi spacetime** amounts to a one-parameter family of left-invariant metrics $(h_t)_{t \in I}$ on a 3-dimensional Lie group G.

Bianchi cosmological models : definitions

We will consider **vacuum type A** Bianchi models.

- **Type A :** G is unimodular.
- **Vacuum :** $\mathrm{Ric}(g) = 0$.

Einstein equation

The space of left-invariant metrics on G is finite-dimensional

$\Longrightarrow$ the Einstein equation $\mathrm{Ric}(g) = 0$ is a system of ODEs.

Einstein equation : coordinate choice

Consider a Bianchi spacetime $(I \times G\ ,\ -dt^2 + h_t)$. There exists a frame field $\left(\frac{\partial}{\partial t}, e_1, e_2, e_3\right)$ such that :

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- (e_1, e_2, e_3) is orthonormal for h_t ;
- the second fundamental form of h_t is diagonal in (e_1, e_2, e_3) ;
- $[e_1, e_2] = n_3(t) e_3$;
  $[e_2, e_3] = n_1(t) e_1$;
  $[e_3, e_1] = n_2(t) e_2$;

Why taking an ortho**normal** frame ?

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Why taking an ortho**normal** frame ?

- One studies the behavior of the **structure constants** n_1, n_2, n_3 instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;
- Key advantage : the various 3-dimensional Lie groups are treated altogether.

Variables

- The three structure constants $n_1(t)$, $n_2(t)$, $n_3(t)$;
- The three diagonal components $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$ of the traceless second fundamental form ;
- The mean curvature of $\theta(t)$.

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Actually, it is convenient to replace

- n_i and σ_i by $N_i = \dfrac{n_i}{\theta}$ and $\Sigma_i = \dfrac{\sigma_i}{\theta}$
- t by τ such that $\dfrac{d\tau}{dt} = -\dfrac{\theta}{3}$.

(Hubble renormalisation ; the equation for θ decouples).

The phase space

With these variables, the phase space $\mathcal{B}$ is a (non-compact) four dimensional submanifold in $\mathbb{R}^6$.

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With these variables, the phase space $\mathcal{B}$ is a (non-compact) four dimensional submanifold in $\mathbb{R}^6$.

$$\mathcal{B} = \left\{(\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3) \in \mathbb{R}^6 \mid \Sigma_1 + \Sigma_2 + \Sigma_3 = 0\,,\ \Omega = 0\right\}$$

where

$$\Omega = 6 - (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \frac{1}{2}(N_1^2 + N_2^2 + N_3^2) - (N_1 N_2 + N_1 N_3 + N_2 N_3).$$

Wainwright-Hsu equations

$$
\begin{aligned}
\Sigma_1' &= (2-q)\Sigma_1 - R_1 \\
\Sigma_2' &= (2-q)\Sigma_2 - R_2 \\
\Sigma_3' &= (2-q)\Sigma_3 - R_3 \\
N_1' &= -(q+2\Sigma_1)N_1 \\
N_2' &= -(q+2\Sigma_2)N_2 \\
N_3' &= -(q+2\Sigma_3)N_3
\end{aligned}
$$

where

$$
\begin{aligned}
q &= \frac{1}{3}\left(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2\right) \\
R_1 &= \frac{1}{3}\left(2N_1^2 - N_2^2 - N_3^3 + 2N_2N_3 - N_1N_3 - N_1N_2\right) \\
R_2 &= \frac{1}{3}\left(2N_2^2 - N_3^2 - N_1^3 + 2N_3N_1 - N_2N_1 - N_2N_3\right) \\
R_3 &= \frac{1}{3}\left(2N_3^2 - N_1^2 - N_2^3 + 2N_1N_2 - N_3N_2 - N_3N_1\right)
\end{aligned}
$$

Wainwright-Hsu equations

We denote by $X_{\mathcal{B}}$ the vector field on $\mathcal{B}$ corresponding to this system of ODEs.

The vaccum type A Bianchi spacetimes can be seen as the orbits of $X_{\mathcal{B}}$.

The stratification of $\mathcal{B}$ given by the various isomorphism classes of Lie groups is $X_{\mathcal{B}}$-invariant.

Bianchi classification

Name	N_1	N_2	N_3	G
I	0	0	0	$\mathbb{R}^3$
II	$+$	0	0	Heis_3
VI_0	$+$	$-$	0	$O(1,1) \ltimes \mathbb{R}^2$
VII_0	$+$	$+$	0	$O(2) \ltimes \mathbb{R}^2$
VIII	$+$	$+$	$-$	$SL(2,\mathbb{R})$
IX	$+$	$+$	$+$	$SO(3,\mathbb{R})$

Type I models ($G = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- The subset of $\mathcal{B}$ corresponding to type I Bianchi spacetimes is a euclidean circle : the *Kasner circle* $\mathcal{K}$.

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- Every point of $\mathcal{K}$ is a fixed point for the flow.

Type I models ($G = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- For every $p \in \mathcal{K}$, the derivative $DX_{\mathcal{B}}(p)$ has :
  - two distinct negative eignevalues,
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- For every $p \in \mathcal{K}$, the derivative $DX_{\mathcal{B}}(p)$ has :
  - two distinct negative eignevalues,
  - a zero eigenvalue,
  - a positive eigenvalue.
- Except if p is one of the three special points T_1, T_1, T_3, in which case $DX_{\mathcal{B}}(p)$ has :
  - a negative eigenvalue,
  - a triple-zero eigenvalue.

Type II models ($G = \text{Hein}_3$, one of the N_i's is non-zero)

- The subset $\mathcal{B}_{\text{II}}$ of $\mathcal{B}$ corresponding to type II models is the union of three ellipsoids which intersect along the Kasner circle.

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- Every type II orbit converges to a point of $\mathcal{K}$ in the past, and converges to another point of $\mathcal{K}$ in the future.

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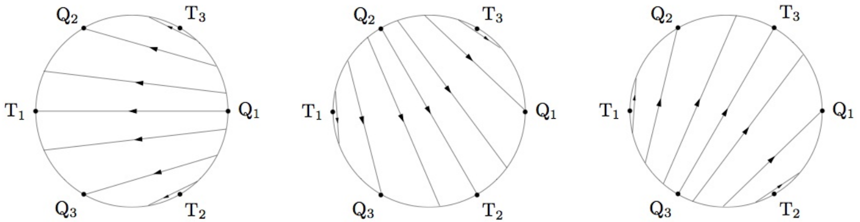
- Every type II orbit converges to a point of $\mathcal{K}$ in the past, and converges to another point of $\mathcal{K}$ in the future.

- The orbits on one ellipsoid "take off" from one third of $\mathcal{K}$, and "land on" the two other thirds.

Type II models (G = Hein$_3$, one of the N_i's is non-zero)



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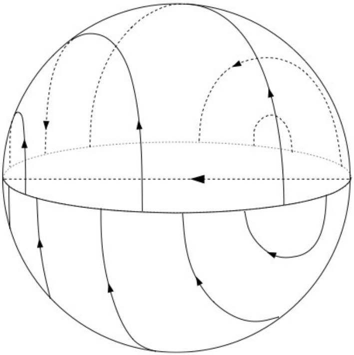

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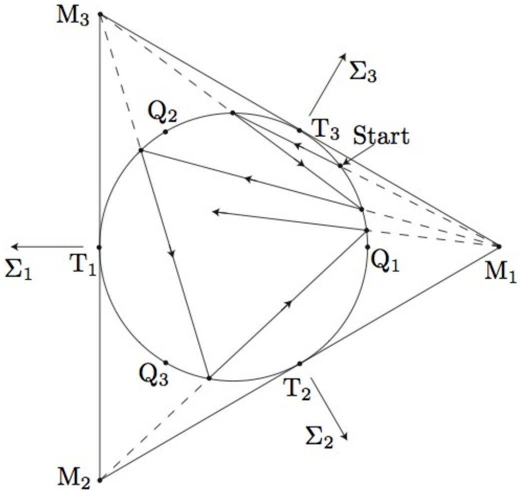
- We restrict to the subset $\mathcal{B}^+$ of $\mathcal{B}$ where the N_i's are non-negative.
- For every $p \in \mathcal{K}$, there is one (and only one) type II orbit "taking off" from p. In the future, this orbit "land on" at some point $f(p) \in \mathcal{K}$.
- This defines a map $f : \mathcal{K} \longrightarrow \mathcal{K}$: the *Kasner map*.

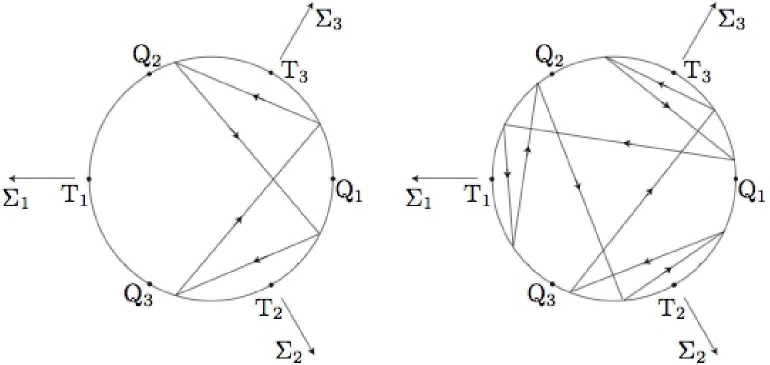
The Kasner map



The Kasner map

The Kasner map is a prototype of a chaotic dynamical system.

The Kasner map



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- **Vague conjecture.** The dynamics of of type IX orbits "reflects" the dynamics of the Kasner map.

- **Example of more precise conjecture.** Almost every type IX orbit accumulates on the whole Kasner circle.

Ringström's theorem

Let $\mathcal{A} := \mathcal{K} \cup \mathcal{B}_{\mathsf{II}}$ be the union of all type I and type II orbits.

Theorem (Ringström 2000). $\mathcal{A}$ is attracting all type IX orbits (except for the Taub type orbits).

Ringström's theorem

- Ringström's result does not imply that the dynamics of type IX orbits "reflects" the dynamics of the Kasner map.

- For example, it could be possible that every type IX orbit is attracted by the period 3 orbit of f.

The dynamics of type IX reflects at least part of the dynamics of the Kasner map

For $q \in \mathcal{K}$, we can define the stable manifold $W^s(q)$ as follows

$$W^s(q) = \left\{ r \in \mathcal{B} \mid \exists\ t_0 < t_1 < \dots \text{ s.t. } \mathrm{dist}(X_{\mathcal{B}}^{t_n}(r), f^n(q)) \to 0 \right\}$$

The dynamics of type IX reflects at least part of the dynamics of the Kasner map

Theorem (Béguin 2010) Consider $q \in \mathcal{K}$ such that the closure of the orbit of q under f does not contain any periodic orbit of f. Then $W^s(q)$ is non-empty.

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Actually, $W^s(q)$ is a three-dimensional injectively immersed manifold which depends continuously on q (when q ranges in a closed f-invariant subset of $\mathcal{K}$ without any periodic orbit).

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Actually, $W^s(q)$ is a three-dimensional injectively immersed manifold which depends continuously on q (when q ranges in a closed f-invariant subset of $\mathcal{K}$ without any periodic orbit).

Proposition. The set of the points q satisfying the hypothesis of the theorem above is dense in $\mathcal{K}$, but has zero Lebesgue measure.

The dynamics of type IX reflects at least part of the dynamics of the Kasner map

Theorem (Georgi, Häterich, Liebscher, Webster, 2010).
Consider a point $q \in \mathcal{K}$ which is periodic point for f.
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Theorem (Georgi, Häterich, Liebscher, Webster, 2010).
Consider a point $q \in \mathcal{K}$ which is periodic point for f.
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Theorem (Reiterer, Trubowitz, 2010). There is a full Lebesgue measure subsets of points q in $\mathcal{K}$ such that $W^s(q)$ is non-empty.

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Theorem (Reiterer, Trubowitz, 2010). There is a full Lebesgue measure subsets of points q in $\mathcal{K}$ such that $W^s(q)$ is non-empty.

Caution. This does not imply that almost every Bianchi spacetime is in $W^s(q)$ for some q.

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Interpretation. Close to the initial singularity :

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).

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- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).
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Interpretation. Close to the initial singularity :

- For all Bianchi spacetimes, the spacelike slice $G \times \{t\}$ is curved in only one direction (Ringström).
- For "many" Bianchi spacetimes, this direction oscillates in a complicated periodic or aperiodic way.
- The way this direction oscillates is sensitive to initial conditions.

Link with the asymptotic silence conjecture

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Formally : for every past inextendible timelike curve γ, the diameter of the set $J^+(\gamma) \cap (\{t\} \times G)$ goes to 0 as $t \to t_-$.

Theorem. For q as in one of the three preceding theorems, the orbits in $W^s(q)$ correspond to asymptotically silent spacetimes.

Bianchi VIII cosmological models

All this is also valid for Bianchi VIII spacetimes.

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- The key is to understand what happens to type IX orbits when they pass close to the Kasner circle.
- For this purpose, we need to find some "good" coordinates.

Hartman Grobman theorem.

Consider a vector field X and a point p such that $X(p) = 0$.

Theorem. Assume that $DX(p)$ does not have any purely imaginary eigenvalue.

Then, there is a C^0 local coordinate system on a neighborhood of p, such that X is linear in these coordinates.

Sternberg's theorem

Theorem. Assume that $DX(p)$ does not have any purely imaginary eigenvalue. Assume moreover that the eigenvalues of $DX(p)$ are independent other $\mathbb{Q}$.

Then, there is a C^∞ local coordinate system on a neighbourhood of p, such that X is linear in these coordinates.

Takens' theorem

Generalization of Sternberg's theorem to the case where $DX(p)$ has some purely imaginary eigenvalues.

There is a C^A local coordinate system on a neighbourhood of p, such that "X depends linearly on the coordinates corresponding to non purely imaginary eigenvalues".

Linearization of the Wainwright-Hsu vector field near of point of $\mathcal{K}$

Let $X_{\mathcal{B}}$ be the Wainwright-Hsu vector field and p be a point of the Kasner circle.

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Proposition. If the three non-zero eigenvalues of $DX_{\mathcal{B}}(p)$ are independant over $\mathbb{Q}$, then there is a C^∞ local coordinate system (x, x', y, z) on a neighbourhood of p, such that

$$X(x, x', y, z) = \lambda^s(y)x\frac{\partial}{\partial x} + \lambda^{s\prime}(y)x'\frac{\partial}{\partial x'} + \lambda^u(y)z\frac{\partial}{\partial z}$$

with $\lambda^s(y) < \lambda^{s\prime}(y) < 0 < \lambda^u(y)$.

Characterization of linearizable points

Proposition. For $p \in \mathcal{K}$, the following conditions are equivalent :

1. the non-zero eigenvalues of $DX(p)$ are independent over $\mathbb{Q}$;
2. the orbit of p under the Kasner map is not pre-periodic.

Dulac map near a point $p \in \mathcal{K}$

- Consider $p \in \mathcal{K}$ such that there is a C^∞ local coordinate system (x, x', y, z) on a neighbourhood of p, such that

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- We consider the neighborhood of p in $\mathcal{B}^+$

$$V = \{0 \leq x \leq 1\ ,\ 0 \leq x' \leq 1\ ,\ -1 \leq y \leq 1\ ,\ 0 \leq z \leq 1\}.$$

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- Given $q \in M$, if $z(q) > 0$, the orbit of q will exit V by crossing N at some point $\Phi(q)$.

- This defines a map

$$\Phi : M \cap \{z > 0\} \longrightarrow N.$$

Dulac map near a point $p \in \mathcal{K}$

- $\Phi(1, x', y, z) = \left(z^{-\lambda_s(y)/\lambda_u(y)}\ ,\ x'.z^{-\lambda_s{}'(y)/\lambda_u(y)}\ ,\ y\ ,\ 1\right).$

Dulac map near a point $p \in \mathcal{K}$

- $\Phi(1, x', y, z) = \left(z^{-\lambda_s(y)/\lambda_u(y)}\ ,\ x'.z^{-\lambda_s{}'(y)/\lambda_u(y)}\ ,\ y\ ,\ 1\right).$
- **Important observation.** The negative eigenvalues dominate the positive one :

$$-\lambda_s(y)/\lambda_u(y) > 1 \qquad -\lambda_s{}'(y)/\lambda_u(y) > 1.$$

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- **Important observation.** The negative eigenvalues dominate the positive one :

$$-\lambda_s(y)/\lambda_u(y) > 1 \qquad - \lambda_s{}'(y)/\lambda_u(y) > 1.$$

- **Consequence.** Φ extends to a C^1-map on $M \cap \{z = 0\}$.

  If $z(q) = 0$, $\qquad d\Phi(q).\frac{\partial}{\partial x'} = d\Phi(c).\frac{\partial}{\partial z} = 0$

$$d\Phi(q).\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

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- The distance from the orbit to $\mathcal{A}$ is contracted when the orbit passes close to the Kasner circle. This contraction is "super-linear".

- The drift in the direction tangent to $\mathcal{A}$ is neglectible as compared to this contraction.

- No matter what happens far from the Kasner circle! This will never compensate the "super-linear contraction".

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Assume that there is a splitting $T_q M = E^s_q \oplus E^u_q$ for every $q \in C$ and some constant $\mu < 1$ and $\nu > 1$ such that, for every $q \in C$,

- $d\psi(q)(E^s_q) \subset E^s_{\psi(q)}$ and $\|d\psi(q).v\| \leq \mu.\|v\|$ for every $v \in E^s_q$;
- $d\psi(q)(E^u_q) = E^u_{\psi(q)}$ and $\|d\psi(q).v\| \geq \nu.\|v\|$ for every $v \in E^u_q$.

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- $d\psi(q)(E^u_q) = E^u_{\psi(q)}$ and $\|d\psi(q).v\| \geq \nu.\|v\|$ for every $v \in E^u_q$.

Then, for every $q \in C$, the set

$$W^s(q) := \{r \in M \mid \mathrm{dist}(\psi^n(r), \psi^n(q)) \to 0 \text{ when } n \to \infty\}$$

contains a $\dim(E^s_q)$-dimensional disc C^1-embedded in M.

How does this applies to our situation ?

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- **Proposition.** The first return map of the orbits of the Wainwright-Hsu vector field on $M_{p_1} \cup \cdots \cup M_{p_n}$ satisfies the hypotheses of the stable manifold theorem.