High order well balanced schemes for hyperbolic systems with source

Giovanni Russo
Department of Mathematics
and Computer Science
University of Catania, Italy

In collaboration with Alexander Khe
Lavrentyev Institute for hydrodynamics
Novosibirsk, Russia

General features of the new approach

Goal: introduce a methodology to derive

- Well balanced property for static and moving equilibria
- High order accuracy (depends on WENO)
- Applicable to a wide class of systems with source
- Applicable to staggered and unstaggered finite volume schemes
- At a numerical level requires the solution of local equilibria
- Conceptually simple

Outline

- Well balanced schemes
- Conservative and equilibrium variables
- Staggered finite volume
- Non staggered FV schemes
- Conservative reconstruction of equilibrium variables
- Numerical tests:
  - Shallow water equations
  - Nozzle flow
- Well balanced ADER schemes
- Numerical reconstruction of equilibrium states
- Work in progress

Well balanced schemes

Consider a system of balance laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u,x)}{\partial x} = g(u,x)$$

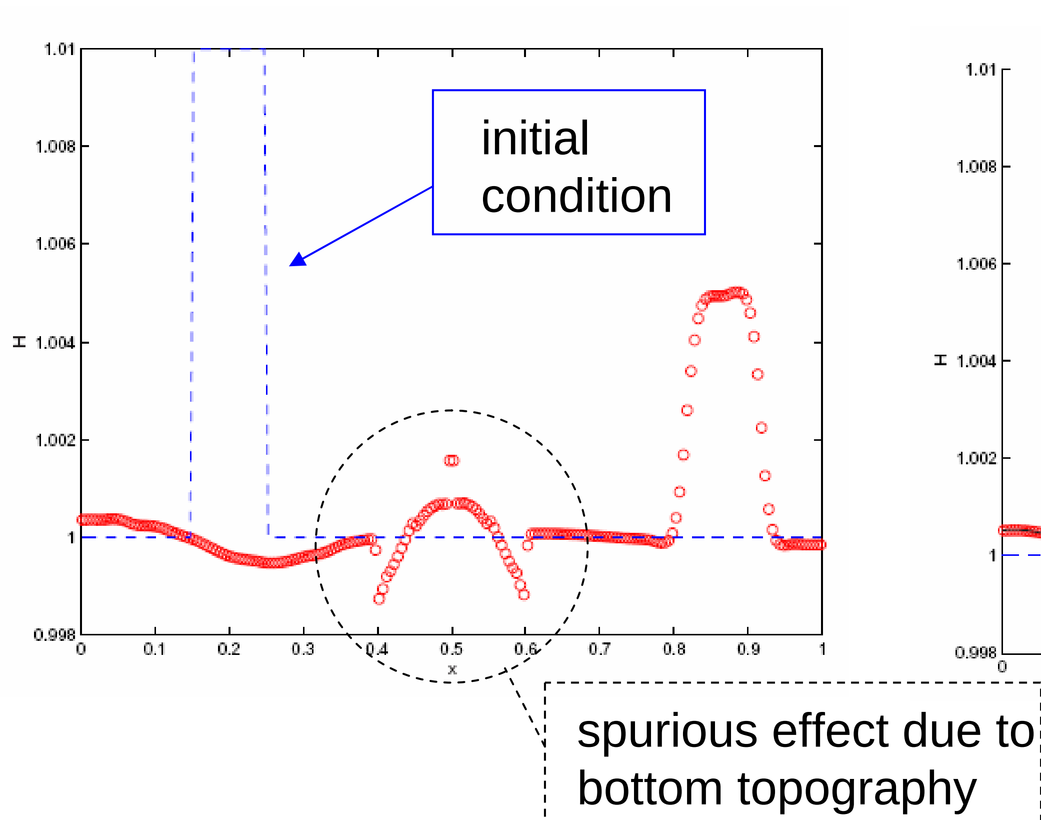
Denote by u^e the stationary solution, satisfying equil. Eqn.

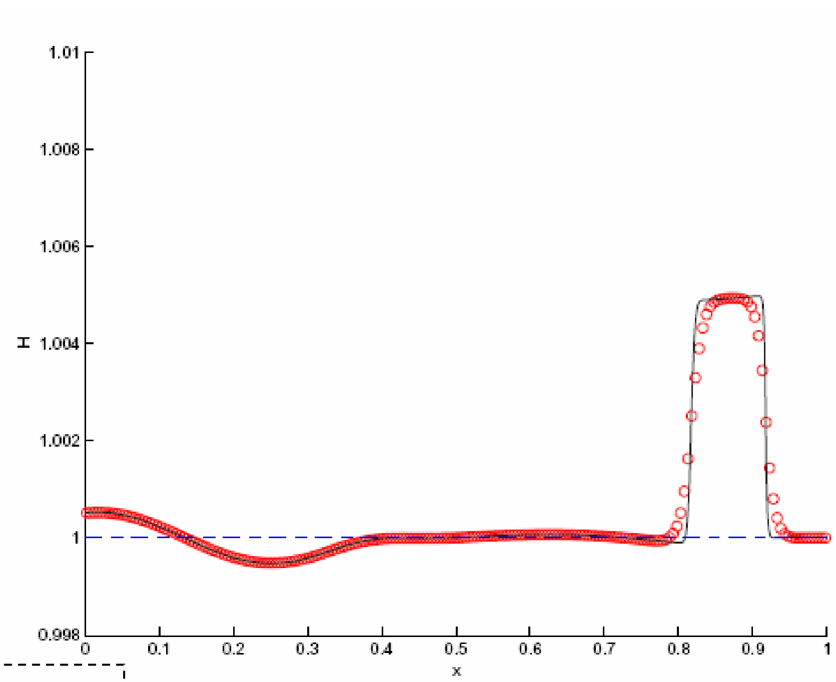
$$\frac{\partial f(u^e,x)}{\partial x} = g(u^e,x)$$

A method is *well balanced* if it satisfies a discrete version of the equilibrium equation.
If a method *is not* well balanced, truncation error of solutions near equilibrium may be larger than $u(x,t) - u^e(x)$

Difference between WB and non WB schemes

Eample with shallow water equations with batimetry:





Solution obtained by a Standard scheme (second order central)

Solution obtained by a Well-balanced scheme (second order central)

Some references (not complete!)

Bernudez, A., Vazquez, M.E., 1994, Computer Fluids.

L. Gosse, LeRoux, 1996, WB scheme for scalar

Greenberg, LeRoux, 1996, … non conservative products

R. Le Veque, 1998, WB Godunov scheme based on wave propagation

G.R., HYP2000, WB central scheme (staggered)

Perthame and Simeoni, 2001, WB kinetic scheme

S.Jin, 2001, WB FV scheme for systems with geometric source

Kurganov and Levy, 2002, WB central unstaggered

G.R., 2002, central staggered, preserve non static equilibria

Bouchut, 2004, nonlinear stability of finite volume, Birkhauser

...

Noelle, Pankratz, Puppo, Natvig, 2007, High order, FV well balanced

Noelle, Shu, Xing, 2007, high order WB WENO FV scheme

Gallardo, Pares, Castro, JCP, 2007, high order, wb sw topography & dry areas

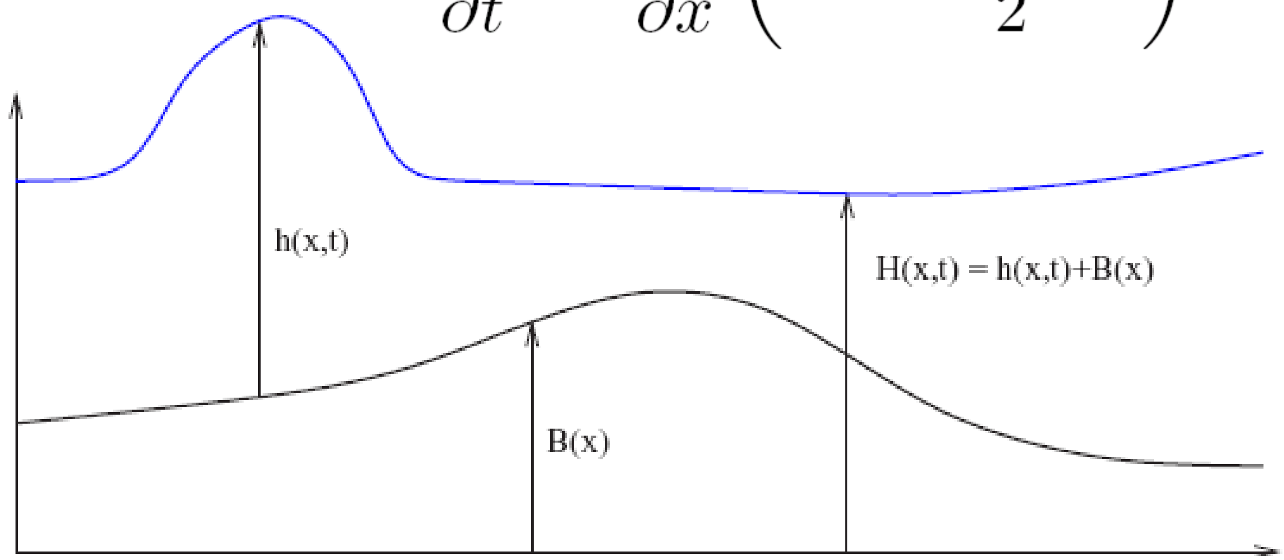
Khe and G.R., 2008, Hyp. conference, Maryland

A second order WB scheme for shallow water

Prototype of hyperbolic system with source term.
Probably the most studied case.
Valid when the wavelength >> water depth.
Pressure is hydrostatic, and horizontal velocity w does not depend on vertical coordinate.

Equations:

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hw) &= 0, \\
\frac{\partial hw}{\partial t} + \frac{\partial}{\partial x}\left(hw^2 + \frac{1}{2}\tilde{g}h^2\right) &= -\tilde{g}hB_x
\end{aligned}
$$



Main ingredients for WB
(staggered schemes)

(G.R. HYP2000, Magdeburg)

- Use $u=(H,q)$ where $H=h+B$, in place of $u=(h,q)$ as independent unknowns ($\Rightarrow$ $f=f(u,x)$)
- Compute space derivatives of f as

$$\frac{\partial f}{\partial x}=A\frac{\partial u}{\partial x}+\left.\frac{\partial f}{\partial x}\right|_u$$

- Suitable approximation of space derivatives

$$B_j=\frac{1}{2}\left(B(x_j+\Delta x/2)+B(x_j-\Delta x/2),\right)$$

$$\frac{B'_j}{\Delta x}=\frac{B(x_j+\Delta x/2)-B(x_j-\Delta x/2)}{\Delta x}.$$

When and why does it work?

- The method preserves static equilibria because if $w = 0$ then $H = h + B$ is constant.
- The method works because the unknown variables are also equilibrium variables.
- It would be nice to use equilibrium variables for the evolution, but then one would loose conservation.
- Key point: conservative mapping between equilibrium and conservative variables.

Conservative and equilibrium variables 1/2

In many cases, equilibrium (for smooth solutions) $\Longleftrightarrow$ some functions of the conservative variables are constant.

For example, for shallow water:

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hw) &= 0, \\ \frac{\partial hw}{\partial t} + \frac{\partial}{\partial x}\left(hw^2 + \frac{1}{2}\tilde{g}h^2\right) &= -\tilde{g}hB_x \end{aligned}$$

At equilibrium $q \equiv hw =$ const, and
from the second equation the following variable is constant:

$\eta = w^2/2 + \tilde{g}(h+B)$ [Energy density – mathematical entropy]

Conservative and equilibrium variables 2/2

Some notation:

Let us denote by u the conservative variable

and by v the equilibrium variable.

We shall assume that there is a 1 to 1 correspondence Between u and v:

$$\boxed{u = U(v,x)} \Leftrightarrow \boxed{v = V(u,x)}$$

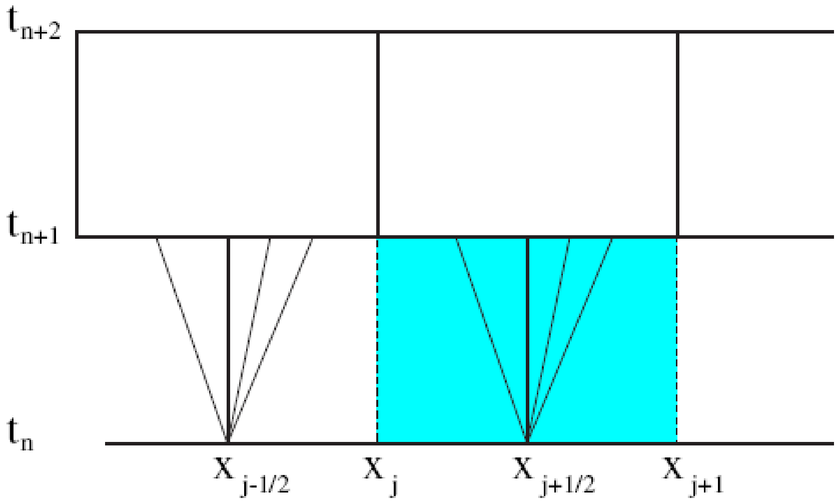
In the case of shallow water, one has

$$u = \begin{pmatrix} h \\ q \end{pmatrix}, v = \begin{pmatrix} q \\ \eta \end{pmatrix} \quad \eta(h,q,x) = q^2/(2h^2) + \tilde{g}(h + B(x))$$

Inversion requires solution of a cubic equation (which we assume we can do. Must be careful for transcritical cases)

Construction of finite volume method: central schemes on a staggered grid

Integrate the balance equation on a staggered cell:



$$\frac{d\bar{u}_{j+1/2}}{dt} = -\frac{1}{\Delta x}\left(f(u(x_{j+1},t)) - f(u(x_j,t))\right) + \langle g \rangle_{j+1/2}(t)$$

where

$$u_{j+1/2} \equiv \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u(x,t)\,dx \qquad \langle g \rangle_{j+1/2}(t) \equiv \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} g(u(x,t),x)\,dx$$

First order scheme

Forward Euler in time

$$\bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^{n} - \frac{\Delta t}{\Delta x}\left(f(u(x_{j+1}), t^n) - f(u(x_j, t^n))\right) + \Delta t < g >_{j+1/2} (t^n)$$

Three qantities have to be defined: given $\{\bar{u}_j^n\}$ compute

- staggered cell values $\{\bar{u}_{j+1/2}^{n}\}$
- pointwise values $u(x_j, t^n)$
- cell average of the source $< g >_{j+1/2} (t^n)$

How to do it in such a way that the scheme is conservative and well balanced?

Basic idea: use **conservative variable** for the **evolution**, and **equilibrium variables** to help the **reconstruction**

First order scheme - 2

Define equilibrium cell average $\bar{v}_j$

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{x+1/2}} U(\bar{v}_j, x)\, dx = \bar{u}_j$$

Note: This definition has been used by Noelle, Shu and Xing

Then define the needed quantities as:

Staggered cell values

$$\bar{u}^n_{j+1/2} = \frac{1}{\Delta x}\left(\int_{x_j}^{x_{j+1/2}} U(\bar{v}_j, x)\, dx + \int_{x_{j+1/2}}^{x_{j+1}} U(\bar{v}_{j+1}, x)\, dx\right)$$

Pointwise values

$$u_j = U(\bar{v}_j, x_j)$$

Staggered source averages

$$< g >_{j+1/2}(t^n) = \frac{1}{\Delta x}\left(\int_{x_j}^{x_{j+1/2}} g(U(\bar{v}_j, x))\, dx + \int_{x_{j+1/2}}^{x_{j+1}} g(U(\bar{v}_{j+1}, x))\, dx\right)$$

Well balanced property

It is easy to show that, if $\{\bar{u}_j^n\}$ are cell averages of equilibrium solution, then

$$\bar{u}^{n+1}_{j+1/2} = \bar{u}^n_{j+1/2} \quad \text{and} \quad \bar{u}^{n+2}_j = \bar{u}^n_j$$

Let $u^e(x)$ be an equilibrium solution, and let v^e be the corresponding equilibrium variable. Then $u^e(x) = U(v^e, x)$

Integrating the equilibrium equation over the space:

$$f(u^e(x_{j+1})) - f(u^e(x_j)) = \int_{x_j}^{x_{j+1}} g(u^e(x))\,dx$$

Making use of this relation in the evolution for the cell average:

$$\bar{u}^{n+1}_{j+1/2} = \bar{u}^n_{j+1/2} - \frac{\Delta t}{\Delta x}(f(u(x_{j+1}), t^n) - f(u(x_j, t^n)) + \Delta t < g >_{j+1/2} (t^n)$$

High order in space: conservative reconstruction

Consider WENO 2-3 in cell j

Conservative variable *u* is reconstructed as

$$u_j(x) = w_L U(P_j^L(x), x) + w_R U(P_j^R(x), x)$$

$$P_j^R(x) = v_j^R + v_j'^R (x - x_j)$$

Obtained by imposing correct cell averages in cells j and $j+1$:

$$< U(P_j^R(x), x) >_j = \bar{u}_j, \quad < U(P_j^R(x), x) >_{j+1} = \bar{u}_{j+1}$$

Set of nonlinear equations for v_j^R and $v_j'^R$

Similarly for $P_j^L(x)$

Remark: conservation property of the mapping is needed to ensure that v is actually constant if the set $\{\bar{u}_j\}$ comes from equilibrium solution, not to ensure that the scheme is conservative!

Time integration: central Runge-Kutta

Evolution equation

$$\frac{d\bar{u}_{j+1/2}}{dt} = -\frac{1}{\Delta x}(f(u_{j+1}) - f(u_j)) + < g >_{j+1/2} (t)$$

CRK approach: **numerical solution** on the staggered cell

$$\begin{aligned} \bar{u}^{n+1}_{j+1/2} &= \bar{u}^{n}_{j+1/2} + \Delta t \sum_{i=1}^{\nu} b_i K_i, \\ K_i &= \frac{1}{\Delta x}(f(u_j^{(i)}) - f(u_{j+1}^{(i)})) + < g >^{(i)}_{j+1/2} \end{aligned}$$

Stage values computed by the evolution equation in non conservative form on the edge of the staggered cell (i.e. at the center of the cells)

Computation of the stage values 1/2

Observe that
$$\frac{\partial f}{\partial x} = A\frac{\partial u}{\partial x} = A\left(\frac{\partial U}{\partial v}v_x + \frac{\partial U}{\partial x}\right)$$

where $A = \nabla_u f$

Consider the evolution equation

$$\frac{\partial u}{\partial t} + A\left(\frac{\partial U}{\partial v}v_x + \frac{\partial U}{\partial x}\right) = g(u,x)$$ At equilibrium

This relation in fact can be used to define the equilibrium variable v

Evolution equation for the stage values:

$$\frac{\partial u}{\partial t} + A\frac{\partial U}{\partial v}\frac{\partial v}{\partial x} = 0$$

Computation of the stage values 2/2

$$u_j^{(i)} = u_j^{(1)} - \Delta t \sum_{\ell=1}^{i-1} a_{i\ell} A(u_j^{(\ell)}) \left. \frac{\partial U}{\partial v} \right|_{\ell} D_x v^{(\ell)}(x_j)$$

$a_{i\ell}$ Runge-Kutta coefficients

The term $D_x v^{(\ell)}(x_j)$ can be obtained using WENO on the derivatives of the reconstruction $v_j(x)$

The WENO reconstruction $v_j(x)$ is obtained from pointwise reconstruction of *v* obtained from the stage value

Practical considerations

Integrals on each interval appearing in the non-linear Equations for the reconstructions, e.g. for WENO 2-3

$$< U(P_j^R(x), x) >_j = \bar{u}_j, \quad < U(P_j^R(x), x) >_{j+1} = \bar{u}_{j+1}$$

are replaced by Gaussian quadrature formulas.
[We used 4 point Gauss-Legendre formulas]
[which in practice guarantee WB property within round off Error in our tests]

WB error depends on the tolerance for the solution of the nonlinear equations

Other reconstruction techniques are possible (see last part)

Unstaggered grids

skip

Evolution equation

$$\frac{d\bar{u}_j}{dt} = -\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + < g >_j (t)$$

Numerical flux function

$$f_{j+1/2} = F(u^-_{j+1/2}, u^+_{j+1/2})$$

Boarder values

$$u^-_{j+1/2} = u_j(x_{j+1/2})$$

$$u^+_{j+1/2} = u_{j+1}(x_{j+1/2})$$

With (WENO 2-3)

$$u_j(x) = w_L U(P^L_j(x), x) + w_R U(P^R_j(x), x)$$

Source average

$$< g >_j = \int_{x_{j-1/2}}^{x_{j+1/2}} g(U(v_j(x), x)\, dx$$

Forward Euler

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}\left(F(u^-_{j+1/2}, u^+_{j+1/2}) - F(u^-_{j-1/2}, u^+_{j-1/2})\right) + < g >_j^n \Delta t$$

Let us show WB property

Assume equilibrium initial data $u^0(x) = u^e(x)$

Then $v_j(x) = v_{j+1}(x) = \bar{v}$ Therefore, by consistency:

$$F(u^-_{j+1/2}, u^+_{j+1/2}) = F(u^e(x_{j+1/2}), u^e(x_{j+1/2})) = f(U(\bar{v}, x_{j+1/2})$$

Source cell average

$$< g >_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} g(U(\bar{v}, x), x)\, dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} g(u^e(x))\, dx$$

Above relations imply $f(u_{j+1/2}) - f(u_{j-1/2}) + < g >_j^0 \Delta x = 0$

And therefore $\boxed{\bar{u}_j^{n+1} = \bar{u}_j^n}$

High order schemes

Can be obtained by applying RK schemes in time

Numerical solution $\bar{u}_j^{n+1} = \bar{u}_j^n + \Delta t \sum_{i=1}^{\nu} b_i K_j^i$

$$K_j^i = \frac{1}{\Delta x}\left(F((u^{(i)}_{j-1/2})^-, (u^{(i)}_{j-1/2})^+) - F((u^{(i)}_{j+1/2})^-, (u^{(i)}_{j+1/2})^+)\right) + < g(u^{(i)}) >_j$$

Stage values $\bar{u}_j^{(i)} = \bar{u}_j^n - \Delta t \sum_{\ell=1}^{i-1} a_{i\ell} K_j^\ell$

Values at cell edges:

$$(u^{(i)}_{j+1/2})^- = u_j^{(i)}(x_{j+1/2})$$

$$(u^{(i)}_{j+1/2})^+ = u_{j+1}^{(i)}(x_{j+1/2})$$

[conservative reconstruction]

Using same argument as Euler scheme, it can be proved it is WB, i.e. $K_j^\ell = 0$ because it is proportional to derivatives of v

Numerical tests: staggered FV schemes

Schemes compared

- First order: piecewise constant, forward Euler
- Second order: piecewise linear, MM limiter, CRK2 (modified Euler)
- Third order: compact WENO, CRK3
- Fourth order: Central WENO (3 parabolas), CRK4

Numerical tests

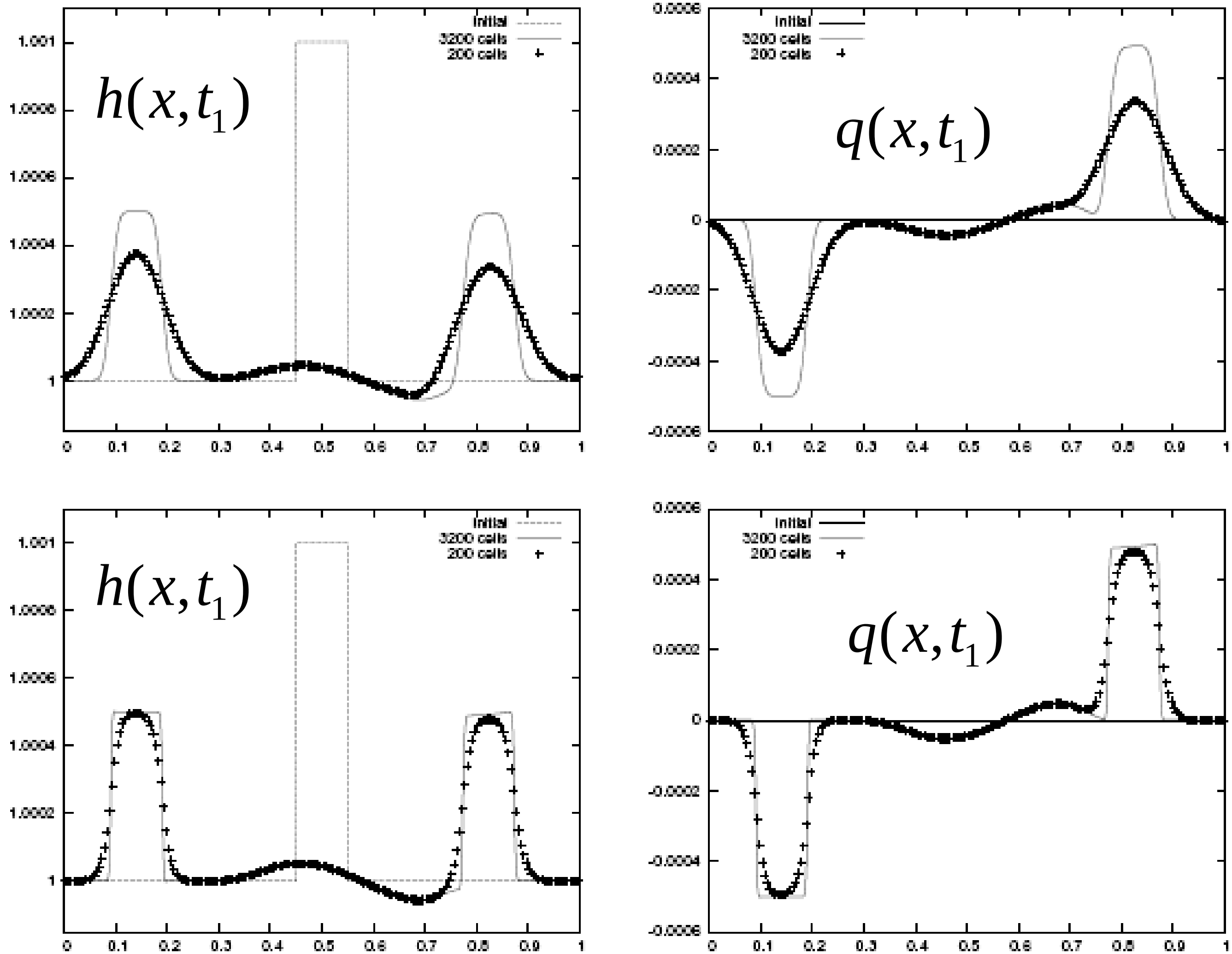
- Models considered:
  - Saint Venant equations of shallow water
  - Nozzle flow for Euler equations
- Test performed to check
  - WB property for
    - Static equilibria
    - Moving equilibria
  - Order of accuracy
  - Shock capturing property

Shallow water – static equilibium

$N = 200, 3200$

First order

Second order

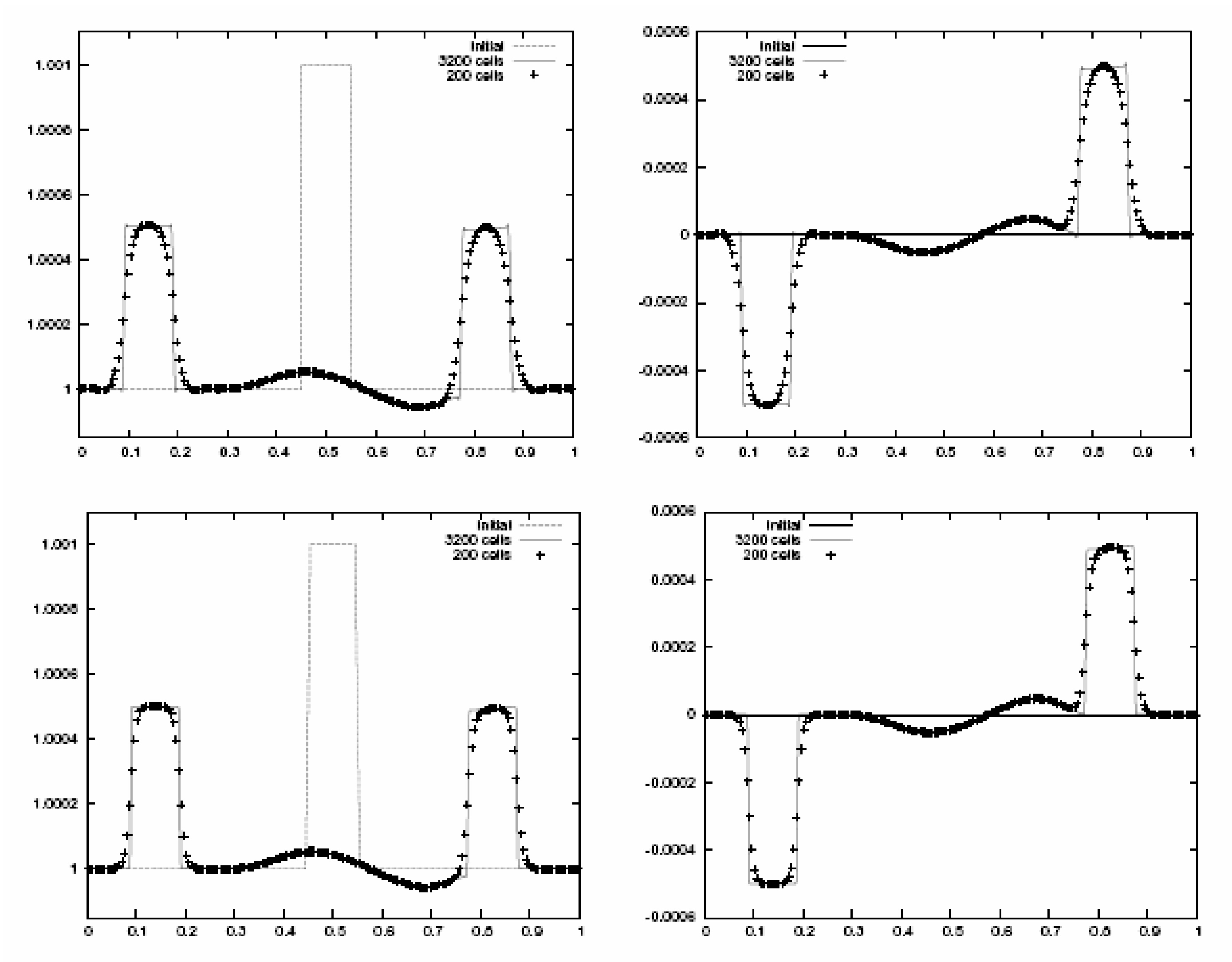


Shallow water – static equilibium

$$N = 200, 3200$$

Third order

Fourth order

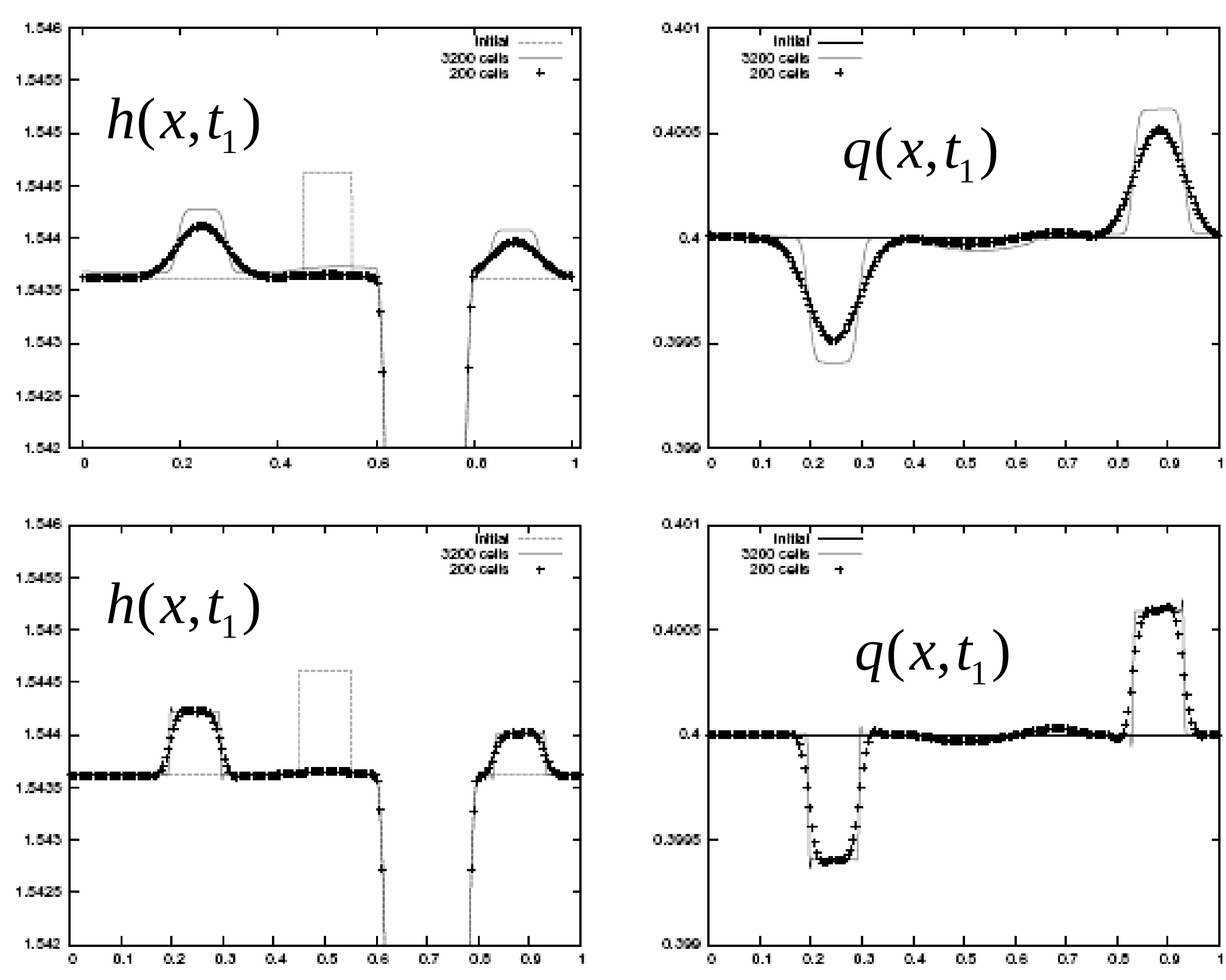


Shallow water – moving equilibium

$N = 200, 3200$

First order

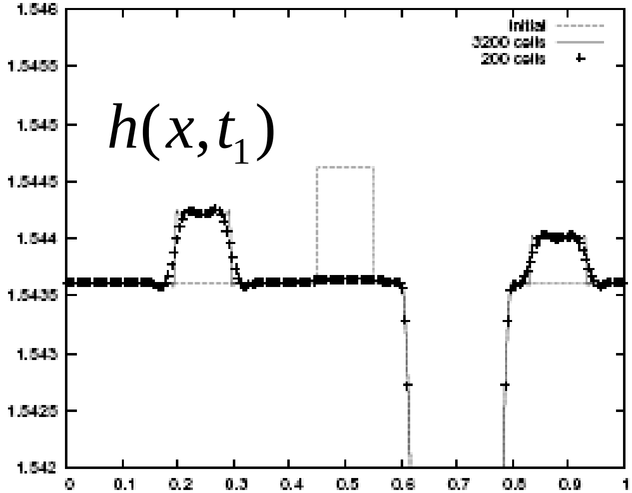
Second order

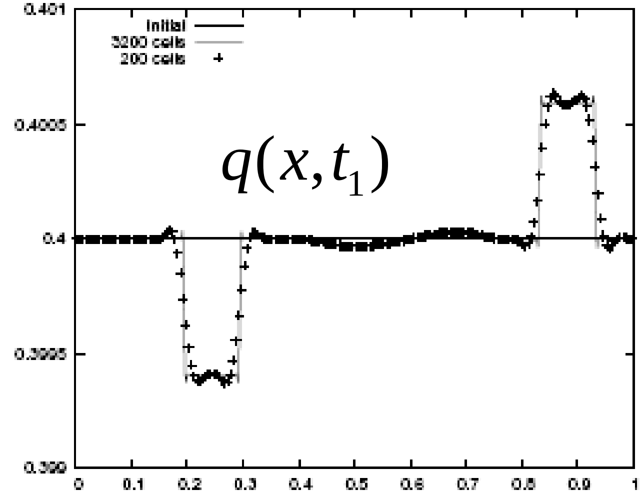


Shallow water – moving equilibium

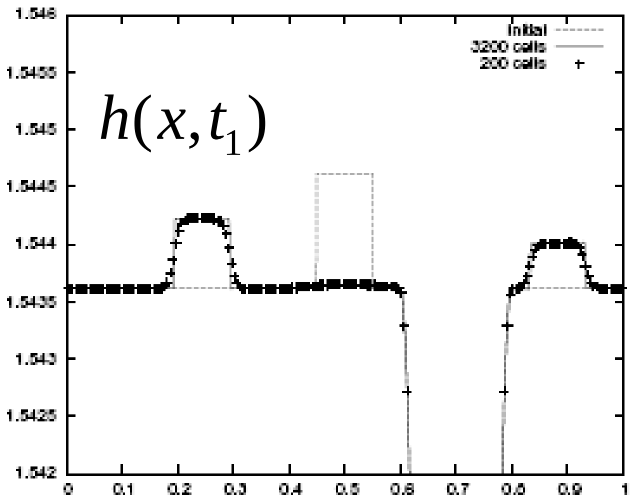
$$N = 200, 3200$$

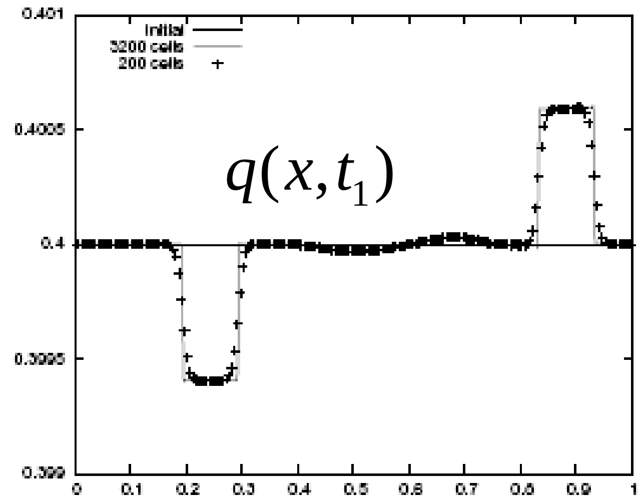
Third order





Fourth order





Equilibrium preservation

Shallow water, Moving equilibrium, 1st order scheme
L1 errors in h and q, dt/dx = 0.2
Cells: 100: 2.7693E-010 1.9099E-010
Cells: 200: 5.4388E-010 3.8929E-010
Cells: 400: 1.0930E-009 7.8746E-010

Shallow water, Moving equilibrium, 4th order scheme
L1 errors in h and q, dt/dx = 0.2
Cells: 100: 2.6651E-013 4.2220E-014
Cells: 200: 5.6666E-013 3.8355E-014
Cells: 400: 1.4794E-011 6.9670E-012

Accuracy test (smooth solution)

Cells	1st order				2nd order			
	h		q		h		q	
	L_1 error	Order	L_1 error	Order	L_1 error	Order	L_1 error	Order
200	2.23E-03		4.67E-02		3.07E-04		2.15E-03	
400	1.16E-03	0.95	2.49E-02	0.91	5.19E-05	2.56	3.41E-04	2.66
800	5.93E-04	0.97	1.29E-02	0.95	9.32E-06	2.48	5.31E-05	2.68
1600	3.00E-04	0.98	6.58E-03	0.97	1.66E-06	2.49	8.79E-06	2.59

Cells	3rd order				4th order			
	h		q		h		q	
	L_1 error	Order	L_1 error	Order	L_1 error	Order	L_1 error	Order
200	1.93E-04		1.90E-03		6.05E-06		7.63E-05	
400	1.83E-05	3.40	2.03E-04	3.23	2.98E-07	4.34	3.71E-06	4.36
800	1.74E-06	3.40	1.84E-05	3.46	1.63E-08	4.19	2.04E-07	4.19
1600	1.75E-07	3.31	1.80E-06	3.36	9.68E-10	4.08	1.19E-08	4.10

Skip non staggered

Non-staggered schemes for shallow water

Numerical Flux — HLL Riemann Solver

CFL number = 0.9

1st order scheme

2nd order scheme:

- Piecewise linear reconstruction with MinMod limiter
- Modified Euler

4th order scheme:

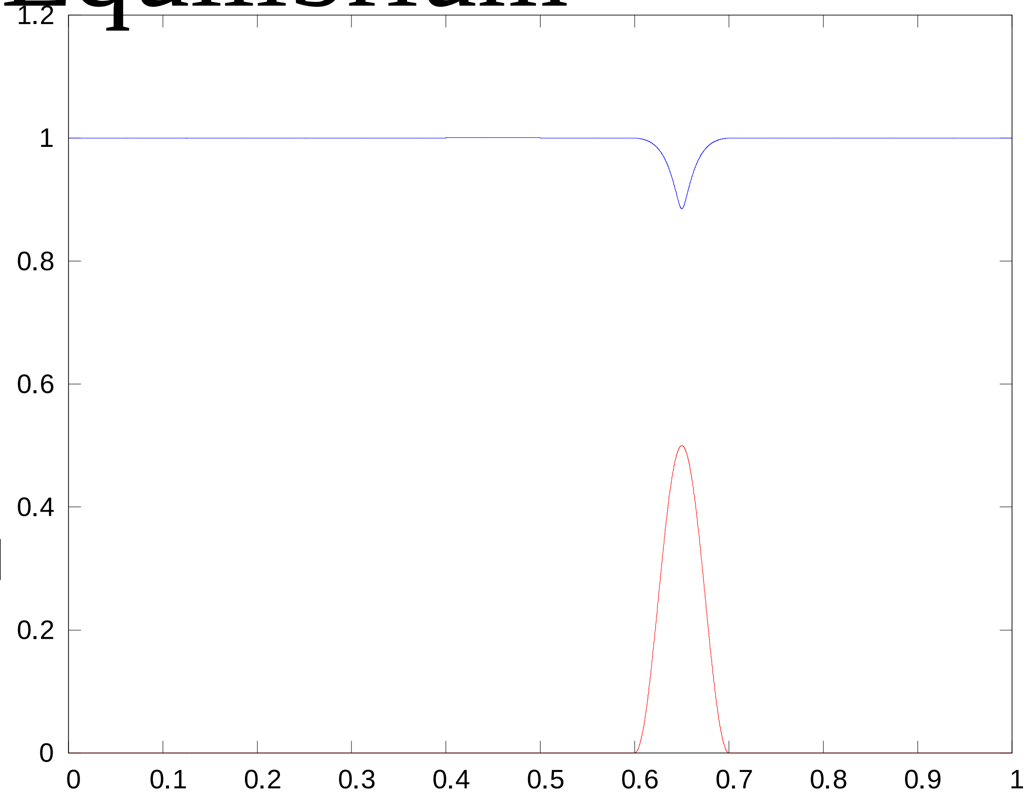
- WENO parabolic reconstruction
- Runge—Kutta 4

Number of cells: 100 and 3200

Moving Equilibrium

- Initial Conditions

Final time = 0.38



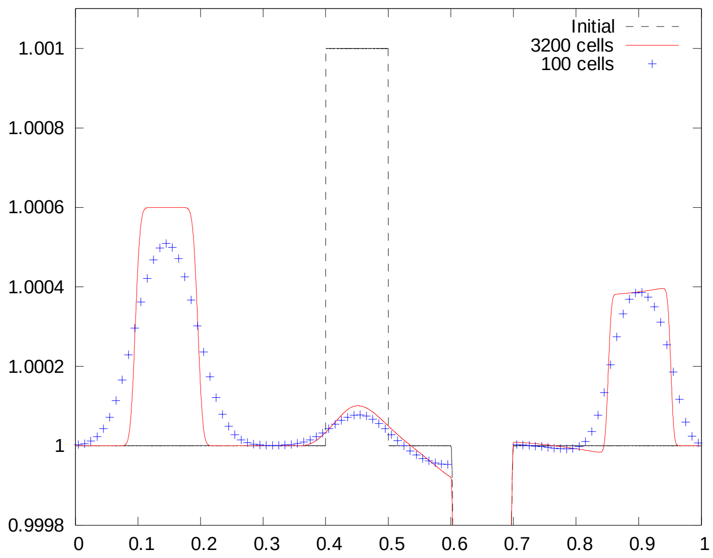
$$h(x) = \begin{cases} H(x, q_0, \varepsilon_0) + 0.001, & x \in [0.45, 0.55] \\ H(x, q_0, \varepsilon_0), & \text{otherwise} \end{cases}$$

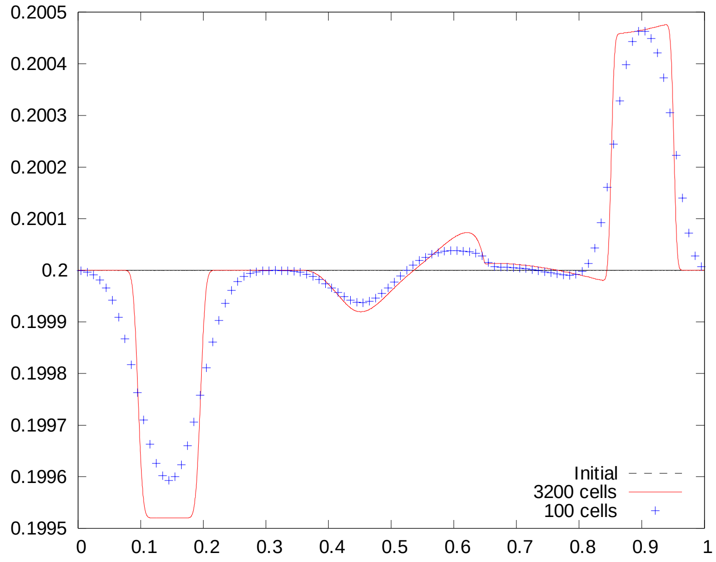
$$u = q_0 / h_0(x), \quad x \in [0,1]$$

$$b(x) = \begin{cases} 0.25\left(1 + \cos 20\pi(x - 0.65)\right), & x \in [0.6, 0.7] \\ 0.0, & \text{otherwise} \end{cases}$$

1^{st} order scheme

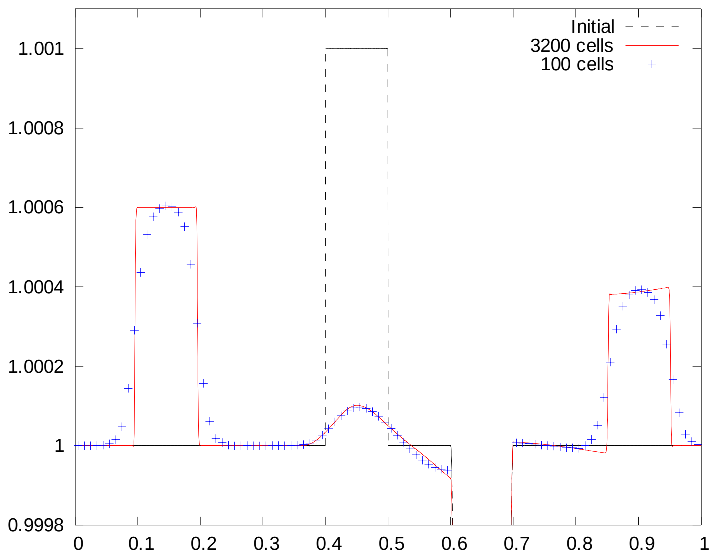
- Left: h, Right: q = rho u
- Solid: 3200 cells, Crosses: 100 cells

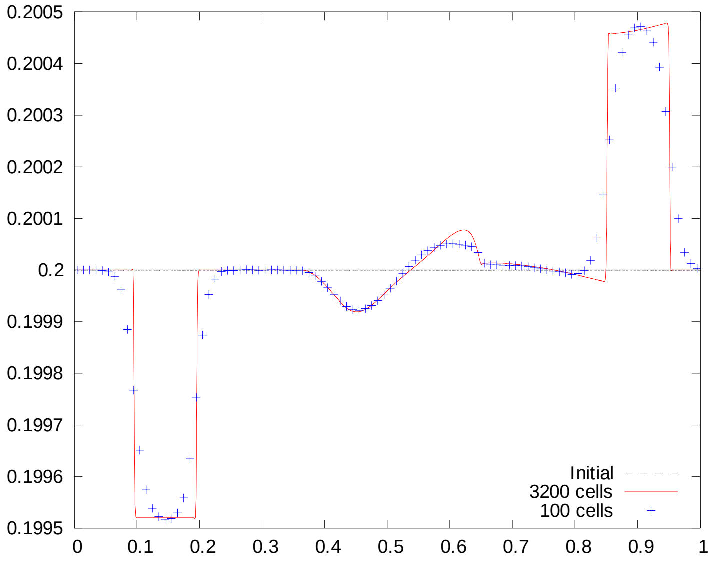




4^{th} order scheme

- Left: h, Right: q = rho u
- Solid: 3200 cells, Crosses: 100 cells





Accuracy Tests

- Tests performed:
- 100, 200, 400, 800, 1600, 3200 cells

h	q
0.879	0.980
0.934	0.984
0.970	0.995
0.984	0.997

h	q
1.837	1.886
1.913	1.886
1.899	1.955
1.764	1.797

h	q
3.980	3.944
4.060	4.046
4.136	4.123
4.094	4.087

Nozzle flow

Euler equations on a channel of variable cross section

Cross section of the channel

$$A(x) = 1 - 0.1\cos(2\pi x)$$

Initial condition

$$u = 0,$$
$$\rho = \cos\sin(2\pi x),$$
$$p = \begin{cases} 1.001, & \text{if } x \in (0.45, 0.55), \\ 1, & \text{otherwise.} \end{cases}$$

Periodic B.C.
CFL = 0.2
Final time T=0.26

Conservative and Equilibrium variables

Conservative

$$\begin{aligned} R &= \rho A \\ Q &= \rho u A \\ E &= \mathcal{E} A \end{aligned}$$

Equilibrium

$$\begin{aligned} Q &= \rho u A \\ S &= A p^{1/\gamma} \\ L &= p^{-1/\gamma}\left(\frac{\gamma}{\gamma-1}p + \frac{1}{2}\rho u^2\right) \end{aligned}$$

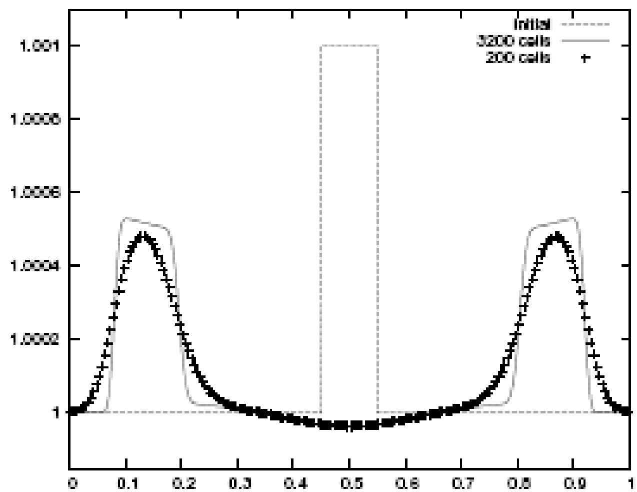
A new equilibrium variable $\tilde{S} = S / S_0(x)$ is used in place of S.
The new variable is $\tilde{S} = 1$ at equilibrium.

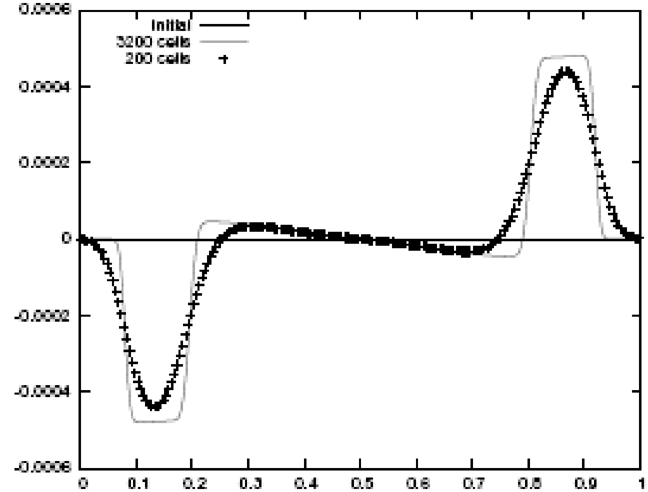
Calculations are performed with staggered grid

Nozzle flow – shock capturing and WB

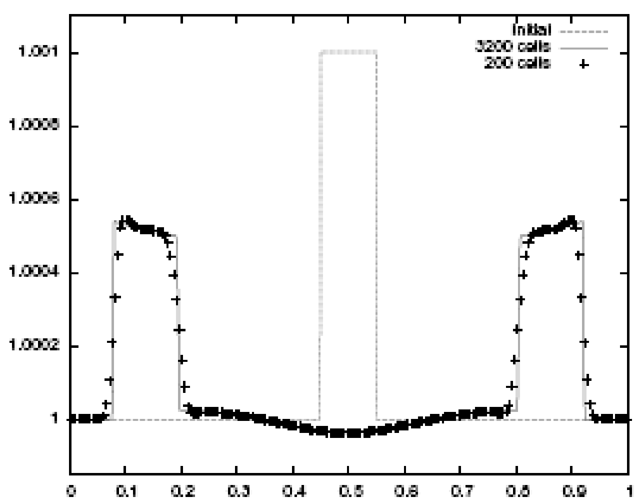
Pressure | Velocity

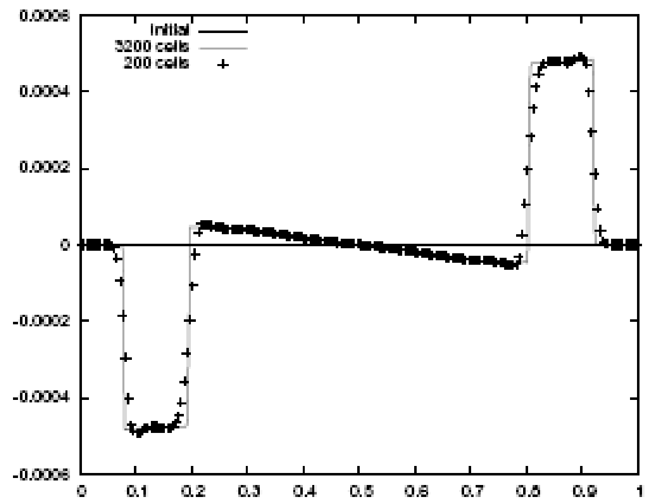
First order





Second order





Equilibrium preservation

Nozzle flow, Moving equilibrium, 1st order scheme
L1 errors in conservative variables, dt/dx = 0.2

Cells: 100:	2.2272E-016	1.9529E-015	6.7946E-016
Cells: 200:	9.9143E-016	3.5666E-015	4.7939E-015
Cells: 400:	3.4360E-015	1.6227E-014	3.1442E-015

Nozzle flow, Moving equilibrium, 2nd order scheme
L1 errors in conservative variables, dt/dx = 0.2

Cells: 100:	3.5016E-016	3.7648E-015	7.9936E-016
Cells: 200:	7.6102E-016	4.7445E-015	3.1175E-015
Cells: 400:	1.5455E-015	2.7246E-014	1.4070E-014

Nozzle flow: accuracy test

Cells	q		r		e	
	L_1 error	Order	L_1 error	Order	L_1 error	Order
200	2.28E-02		2.31E-02		2.00E-03	
400	1.28E-02	0.84	1.29E-02	0.84	1.09E-03	0.88
800	6.78E-03	0.91	6.84E-03	0.92	5.75E-04	0.92
1600	3.50E-03	0.96	3.52E-03	0.96	2.96E-04	0.96
3200	1.77E-03	0.98	1.78E-03	0.98	1.50E-04	0.98

Cells	q		r		e	
	L_1 error	Order	L_1 error	Order	L_1 error	Order
200	1.30E-03		1.23E-03		6.31E-05	
400	2.64E-04	2.30	2.59E-04	2.25	1.19E-05	2.41
800	4.82E-05	2.45	4.82E-05	2.42	2.81E-06	2.08
1600	7.96E-06	2.60	8.00E-06	2.59	6.90E-07	2.02
3200	1.16E-06	2.78	1.13E-06	2.82	1.70E-07	2.03

Application to ADER schemes

The procedure can be used to construct WB ADER schemes. ADER is a technique introduced by Toro and developed by Toro, Titarev, Dumbser, etc. (recall talk of Castedo Ruiz on monday). In classical ADER schemes, the numerical solution of a system of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u,x)}{\partial x} = g(u,x)$$

is obtained as follows: integrating over a cell one obtains

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x}\int_0^{\Delta t}\left[f(u_{j+1/2}(t^n+\tau)) - f(u_{j-1/2}(t^n+\tau))\right] d\tau$$
$$+\int_0^{\Delta t} < g(u(x,t^n+\tau),x) >_j \; d\tau$$

Skip ADER

summary of ADER schemes

- The solution at cell edges may be computed by Taylor expansion in time at cell edges.
- Taylor expansion contains time derivative of the solution evaluated at the initial time.
- Differentiating the original equation in space, time derivatives can be computed from space derivatives (Cauchy-Kovalewsky procedure).
- The initial value at the cell edge is evaluated by the solution of the Riemann problem (or by an approximate Riemann solver)
- The initial value of the space derivatives at cell edges is evaluated by the solution of a linear Riemann problem.

Remark: Taylor expansion can be replaced by Runge-Kutta procedure (G.R., Titarev, Toro, 2006)

ADER WB schemes

Using the property

$$A\frac{\partial U}{\partial x} = g$$

the evolution equation on cell edges becomes:

$$\frac{\partial u}{\partial t} + A\frac{\partial U}{\partial v}\frac{\partial v}{\partial x} = 0$$

Now, if $\{\bar{u}_j\}$ are cell averages of an equilibrium solution then one has

$$\frac{\partial v}{\partial x} = 0 \Rightarrow u_{j+1/2}(t^n + \tau) = U(v, x_{j+1/2})$$

Which implies $$f(u_{j+1/2}) - f(u_{j-1/2}) = < g(U(v,x)) >_j$$

And therefore $$\bar{u}_j^{n+1} = \bar{u}_j^n$$

Numerical reconstruction of equilibrium states

(A.Khe, G.R., in preparation)

The approach used so far requires the explicit knowledge of the mapping between conservative and equilibrium variables.

Such a mapping is not always available or known.

It would be desirable to formulate the method without relying on the explicit knowledge of the mapping.

This can be obtained by a numerical reconstruction of equilibrium states

Equilibrium states

Defined by $A(u)\, u_x = g(u, x)$ (1)

where $A = \nabla_u f(u)$

Local equilibrium $u_j^e(x)$ in cell I_j is defined by (1) and

$$\langle u \rangle_j \equiv \frac{1}{\Delta x} \int_{I_j} u(x)\, dx = \bar{u}_j \qquad (2)$$

A state which is formed by piacewise local equilibrium states

$$\sum_j \chi_j(x) u_j^e(x)$$

is the natural generalization of a piecewise constant state for systems without source

($\chi_j(x)$: char. function of I_j , $u_j^e(x)$ solution of (1) and (2))

Numerical local equilibria

Solve Eq.(1) and (2) numerically, e.g. by collocation. Look for an approximate solution in a finite dimensional space, say polynomial of degree p

$$A(u)_k \frac{d}{dx} u(x_k) = g\big(u(x_k), x_k\big) \qquad x_k \in I_j, k = 1, \ldots, p$$

which gives p+1 equations for p+1 unknowns

Remarks

- By piacewise local equilibria one can construct WB schemes of order one.
- The quality of WB property depends on the accuracy of the solution of local equilibria (at equilibrium there are jumps $O(\Delta x^{p+1})$)

First order scheme

Staggered scheme

$$\bar{u}^{n+1}_{j+1/2} = \bar{u}^{n}_{j+1/2} - \frac{\Delta t}{\Delta x}\left(f(u(x_{j+1}),t^n) - f(u(x_j,t^n))\right) + \Delta t < g >_{j+1/2}(t^n)$$

It is

$\bar{u}^{n+1}_{j+1/2} = \bar{u}^{n}_{j+1/2}$ because =0 (a part from $O(\Delta x^{p+1})$)

Unstaggered scheme

$$u^{-}_{j+1/2} = u^{+}_{j+1/2} + O(\Delta x^{p+1})$$

which at equilibrium gives

$$\bar{u}^{n+1}_{j} = \bar{u}^{n}_{j} + O(\Delta x^{p+1})$$

High order schemes

Are obtained by high order reconstructions from local equilibria (rather than from constant states)

Second (and third) order recons. (equivalent to WENO2-3)

In cell I_j define function $u_j(x)$ as

$$u_j(x) = w_j^l \, R_{j-1}(x) + w_j^r \, R_j(x)$$

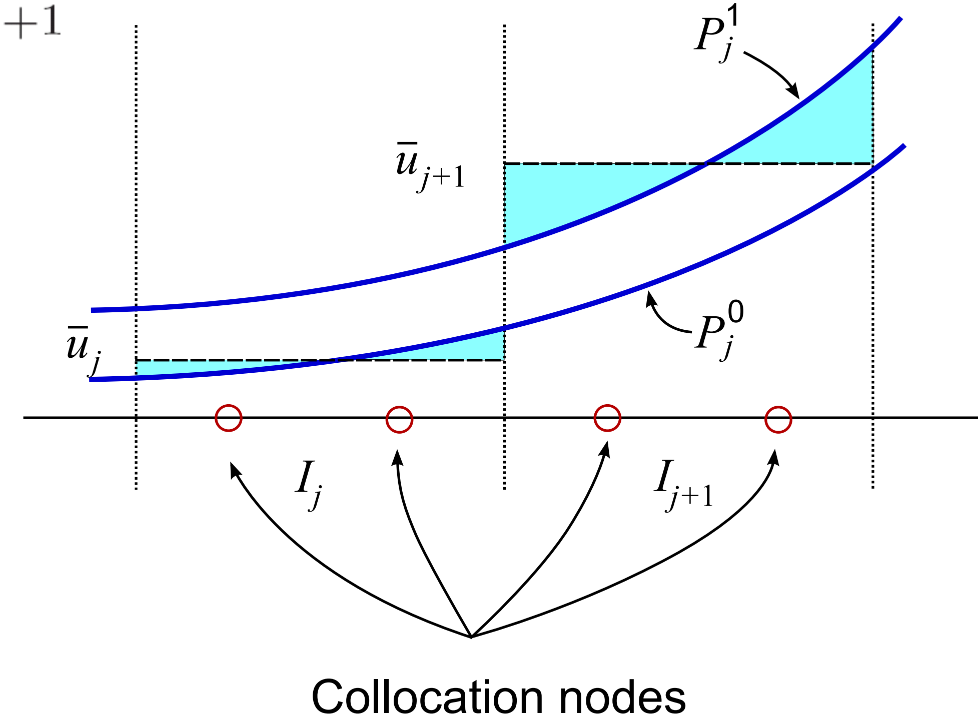
where

$$R_j(x) = q_j^0(x) \, P_j^0(x) + q_j^1(x) \, P_j^1(x)$$

$w_j^l, \; w_j^r$ are the WENO weight

and the local equilibria $P_j^0(x)$, $P_j^1(x)$ are defined in $I_j \cup I_{j+1}$:

They satisfy the collocation equation in *p* points in $I_j \cup I_{j+1}$



and the cell average conditions

$$\langle P_j^i \rangle_{j+i} = \bar{u}_{j+i}, \quad i = 0, 1$$

Linear polynomials

$$q_j^0(x),\ q_j^1(x)$$

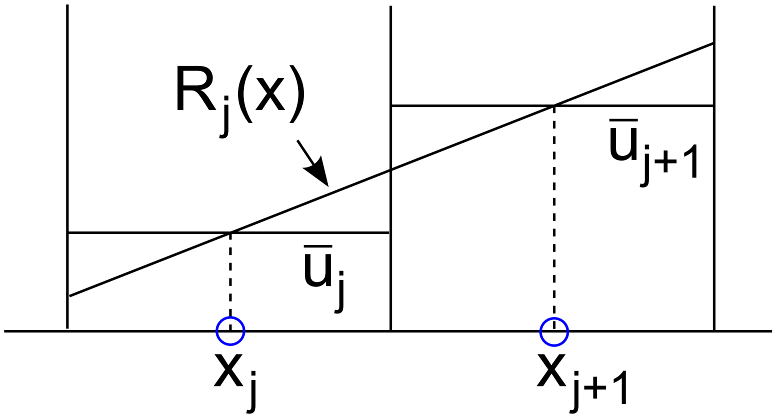
are obtained by imposing that the reconstruction is conservative:

4x4 linear system

$$\frac{1}{\Delta x} \int_{I_{j+k}} q_j^i(x)\, P_j^i(x)\, dx = \bar{u}_{j+k}\, \delta_{ik}\,, \quad i, k = 0, 1$$

Remark

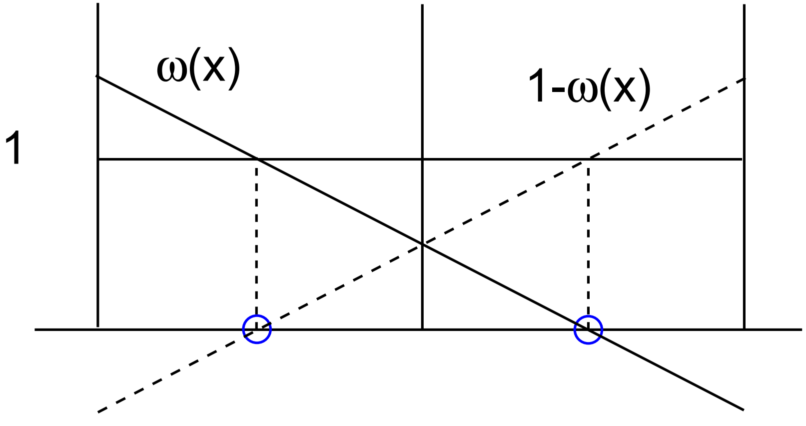
This property is satisfied by the first order polynomials that matches cell averages on two adjacent cells in classical FV schemes:



$$R_j(x) = \omega(x)\bar{u}_j + (1-\omega(x))\bar{u}_{j+1}$$

$$q_j^0(x) = \omega(x)$$

$$q_j^1(x) = 1 - \omega(x)$$



Time advancement (staggered version)

Staggered cell average computed from the reconstruction

Initial values at cell center computed by from the reconstruction

⇨ numerical solution does not change (to $O(\Delta x^{p+1})$)

if stage values do not change.

Stage values: computed from $u_t = g - Au_x$

⇨ RHS = 0 at equilibrium

Higher order schemes

Can be constructed with a similar procedure

Here we construct up to a fourth order method

Application to scalar equation

$$u_t + uu_x + uB' = 0$$

$$u(x) + B(x) \equiv v = \text{const}$$

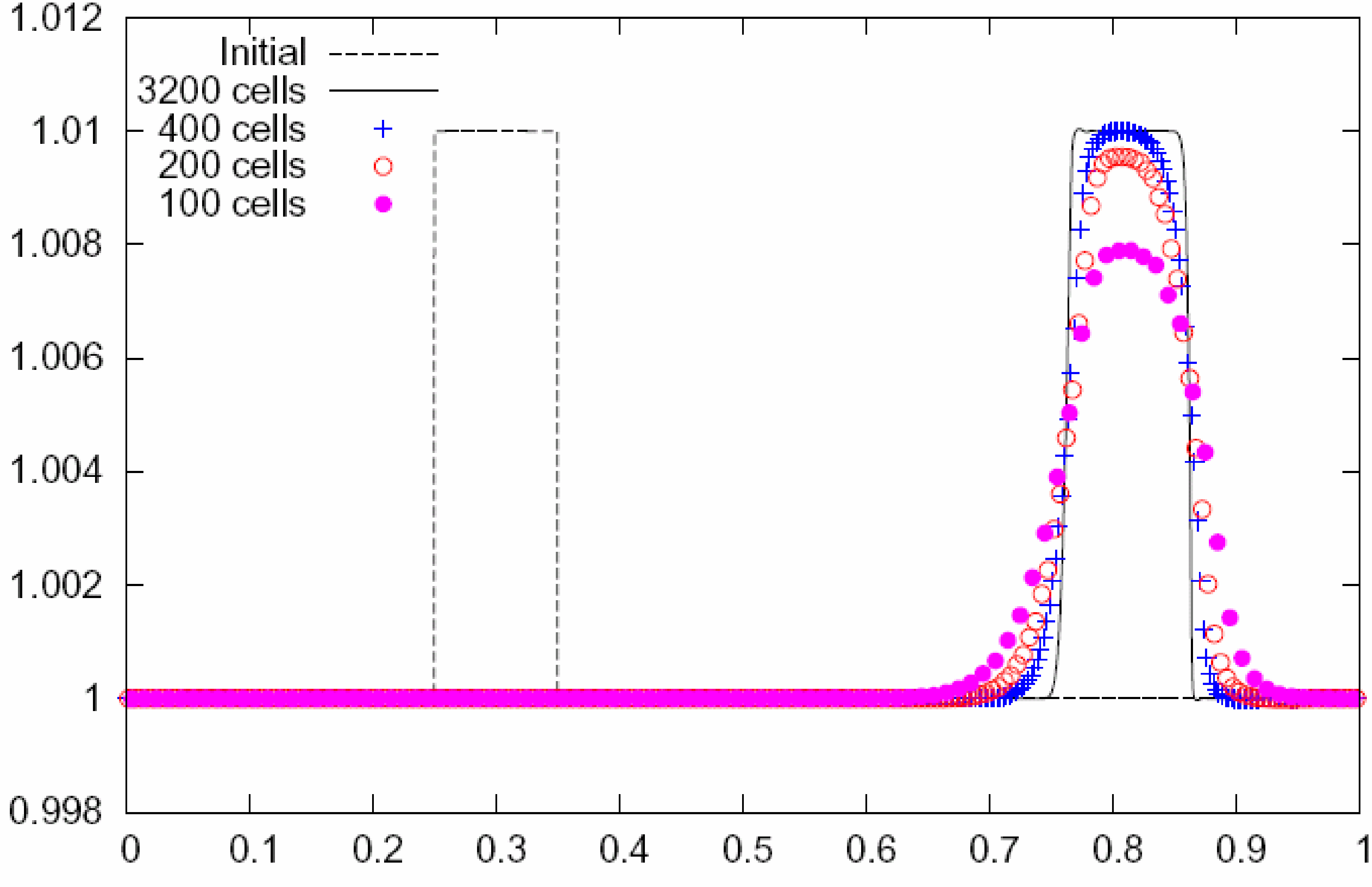
$$u' = g \equiv -B'(x)$$

$$P_j'(x_k) = g(x_k), \quad k = 1, \ldots, 4$$

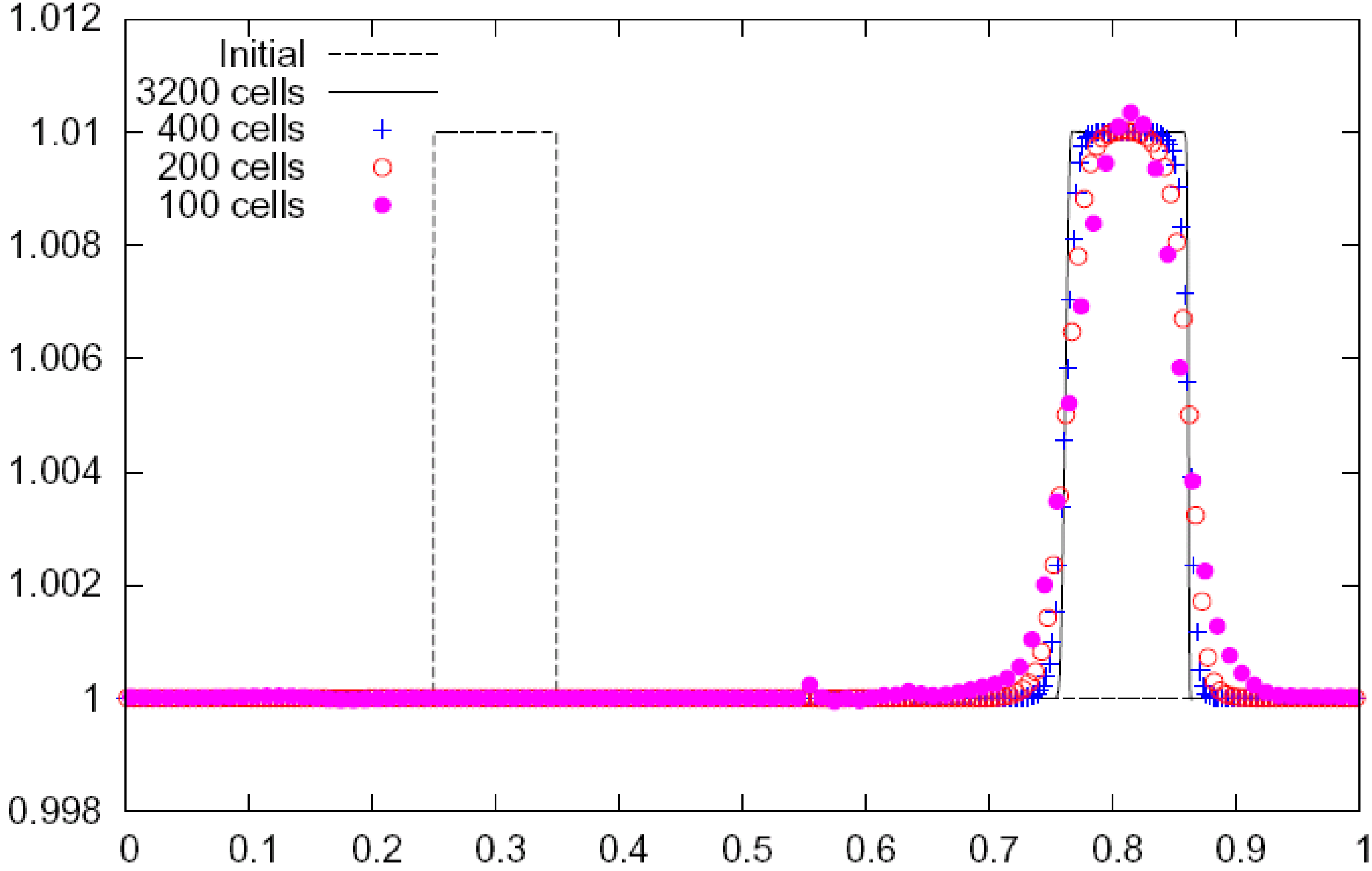
Bottom profile: hump centered in 0.6

$$B(x) = \begin{cases} 0.25\left(1 + \cos 20\pi(x - 0.6)\right), & x \in [0.55, 0.65] \\ 0, & \text{elsewhere} \end{cases}$$

Second order scheme



Fourth order scheme



Test with a smooth solution

Check for convergence rate (L_1 error)

Second order scheme

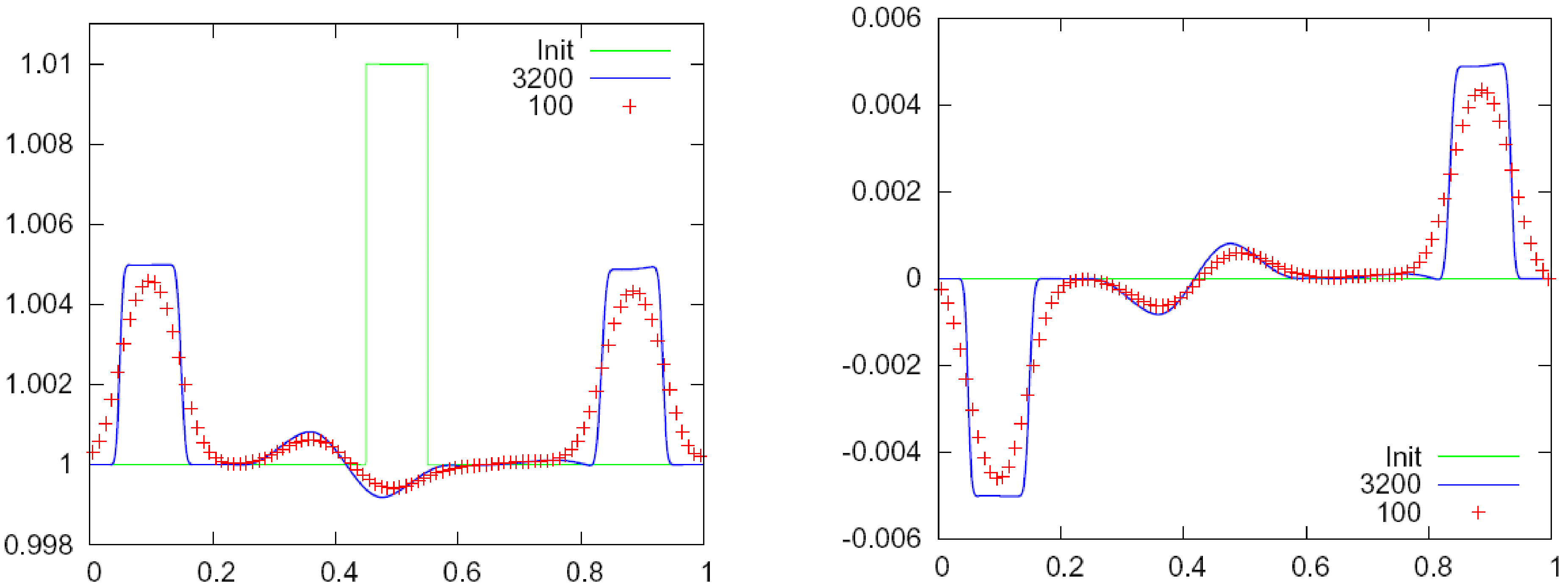
Grid	Error	Order
200	$1.268 \cdot 10^{-4}$	
400	$3.212 \cdot 10^{-5}$	1.981
800	$8.079 \cdot 10^{-6}$	1.991
1600	$2.027 \cdot 10^{-6}$	1.994
3200	$5.078 \cdot 10^{-7}$	1.997

Fourth order scheme

Grid	Error	Order
200	$6.032 \cdot 10^{-7}$	
400	$3.635 \cdot 10^{-8}$	4.053
800	$2.208 \cdot 10^{-9}$	4.041
1600	$1.275 \cdot 10^{-10}$	4.114
3200	$6.810 \cdot 10^{-12}$	4.226

Application to shallow water

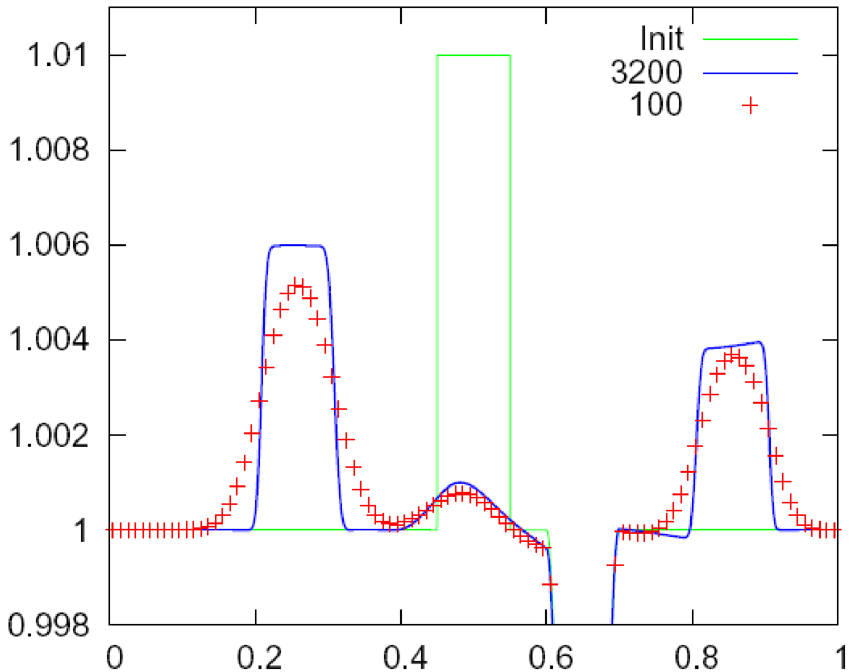
First Order Scheme. Lake at Rest



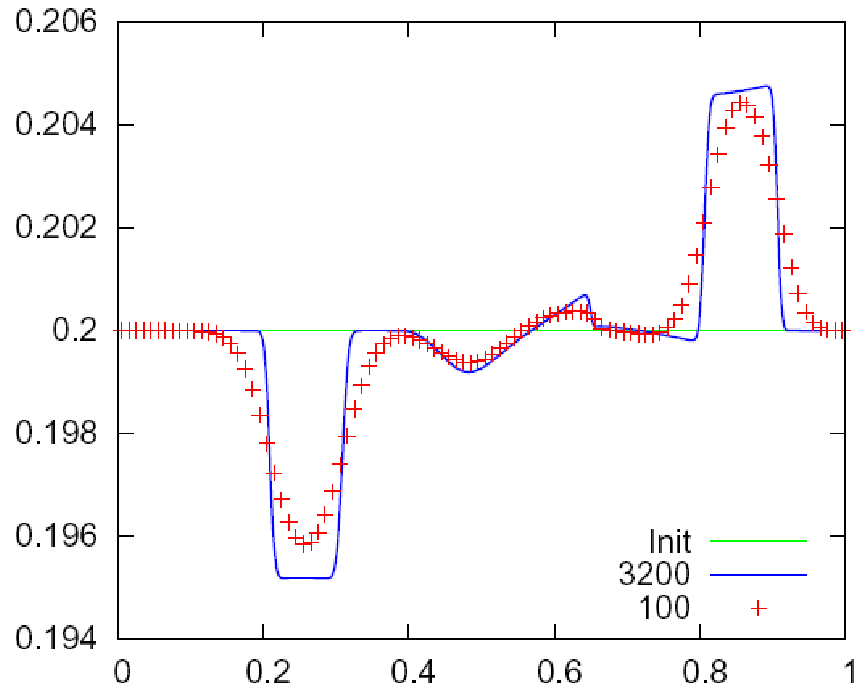
Free surface

Discharge q

First Order Scheme. Moving equilibrium



Free surface

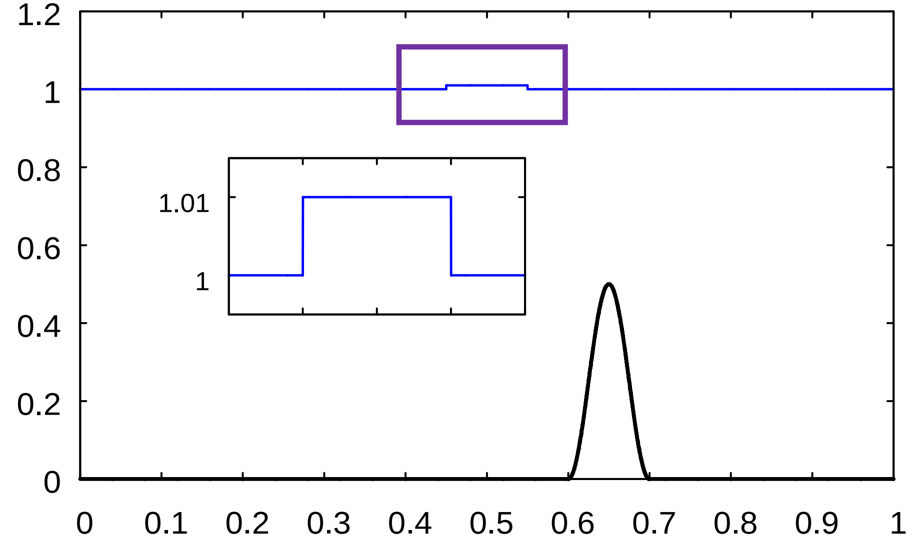


Discharge q

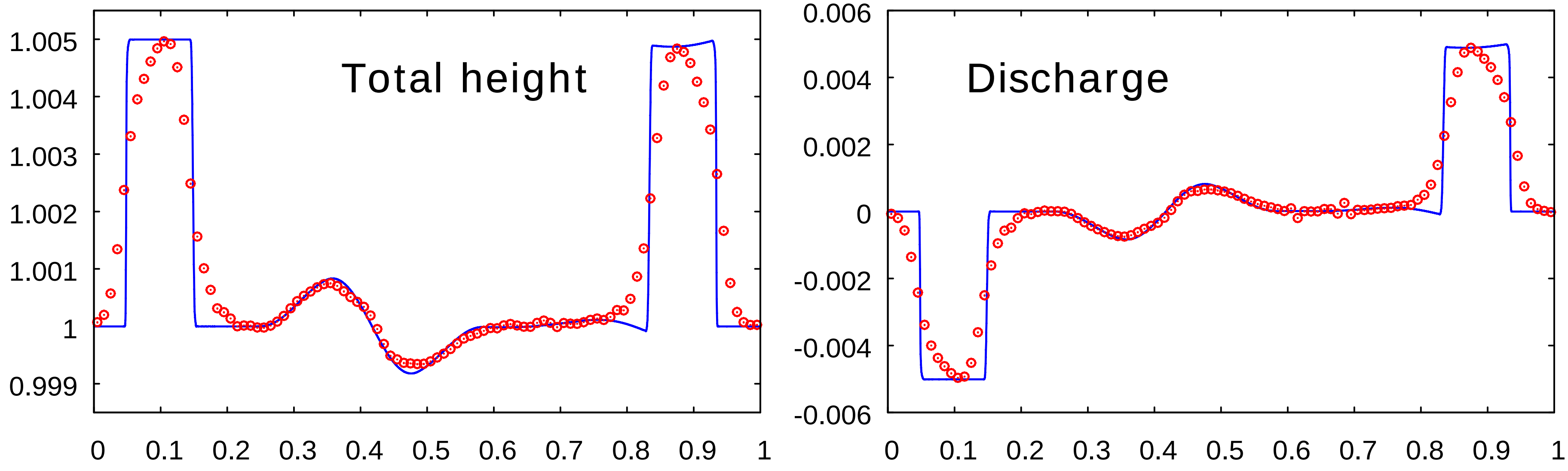
Shallow water system

4^{th} Order Scheme

I. Lake at rest



Initial configuration



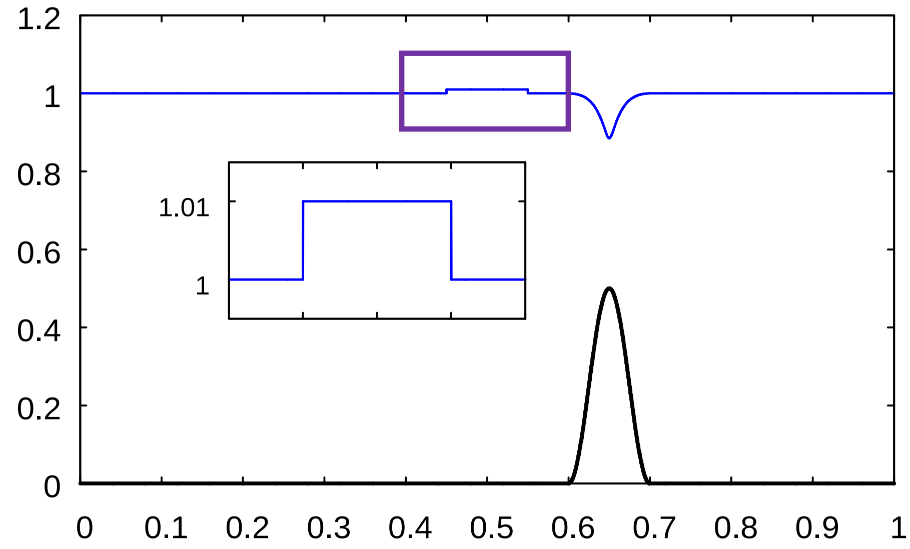
Solid — 4000 cells, Circles — 100 cells

One can notice small perturbations over [0.6, 0.7] due to numerical reconstruction

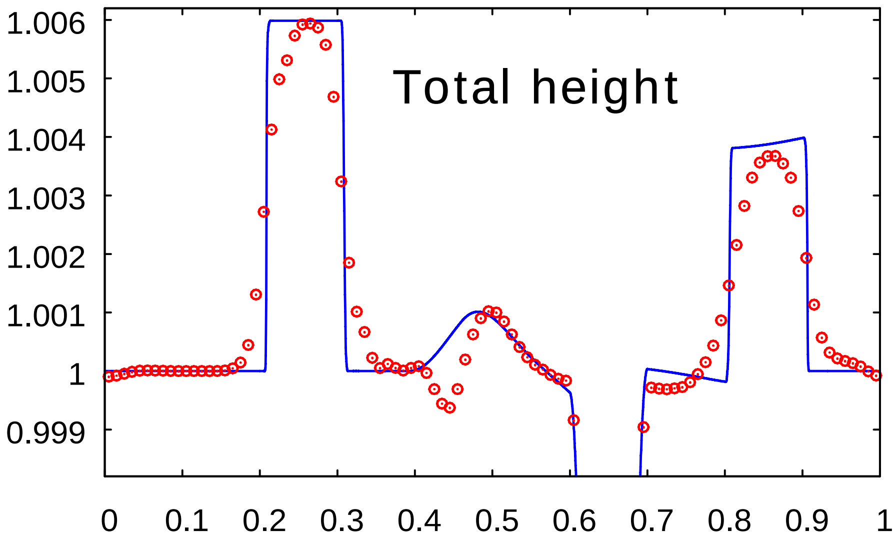
Shallow water system

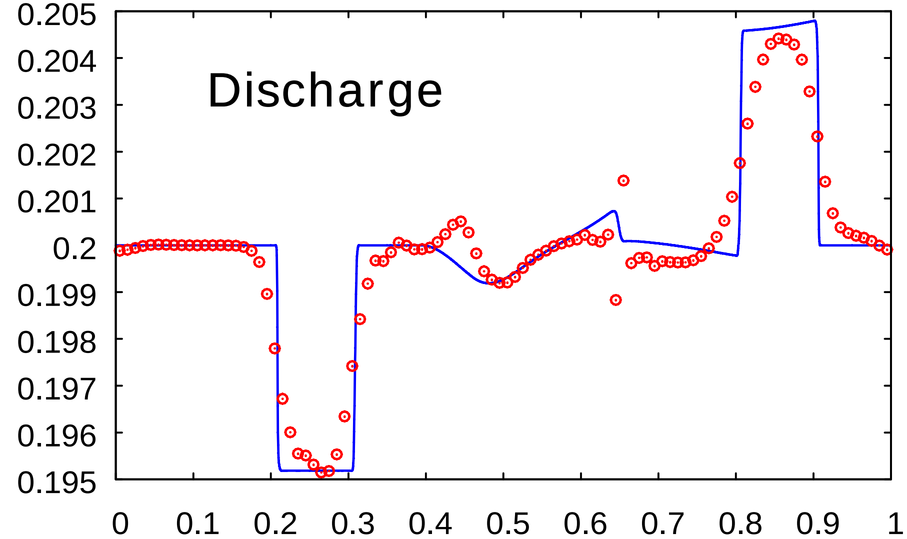
4^{th} Order Scheme

II. Moving Equilibrium



Initial configuration



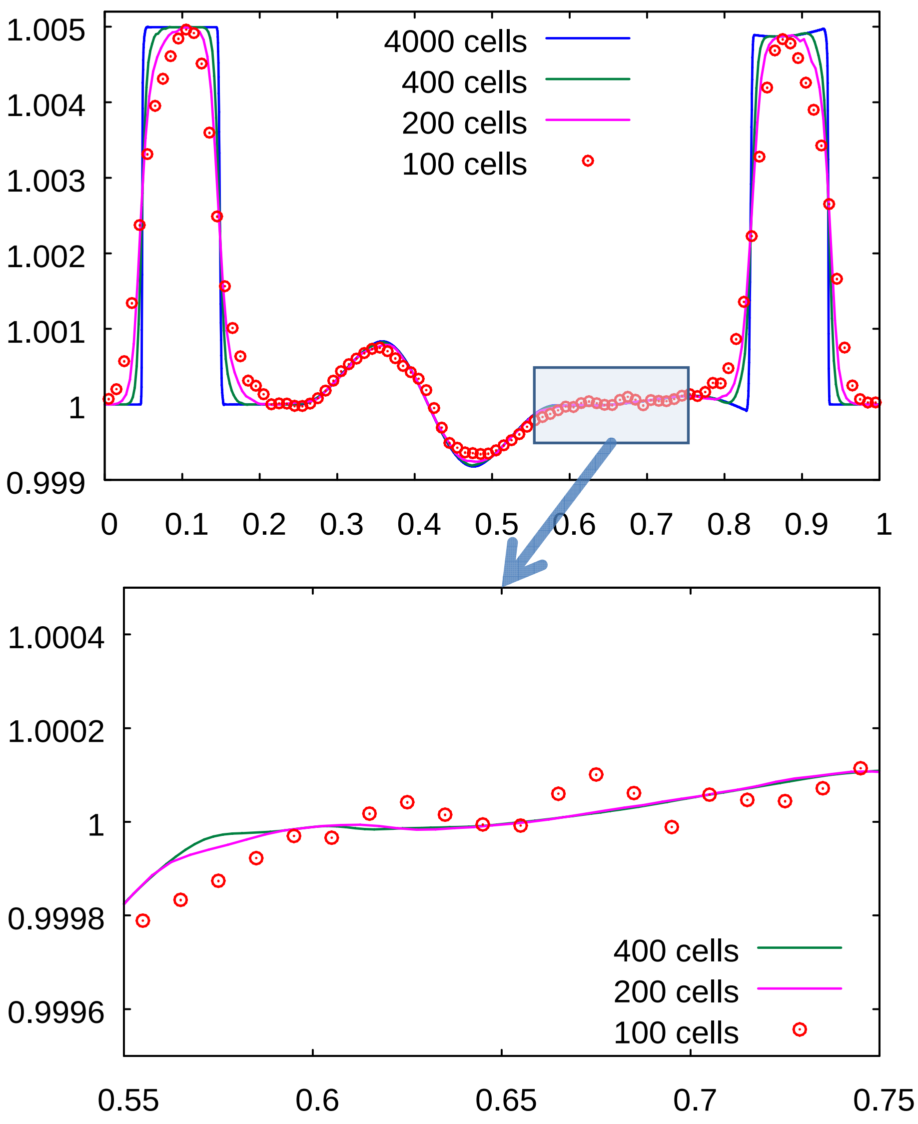


Solid — 4000 cells, Circles — 100 cells

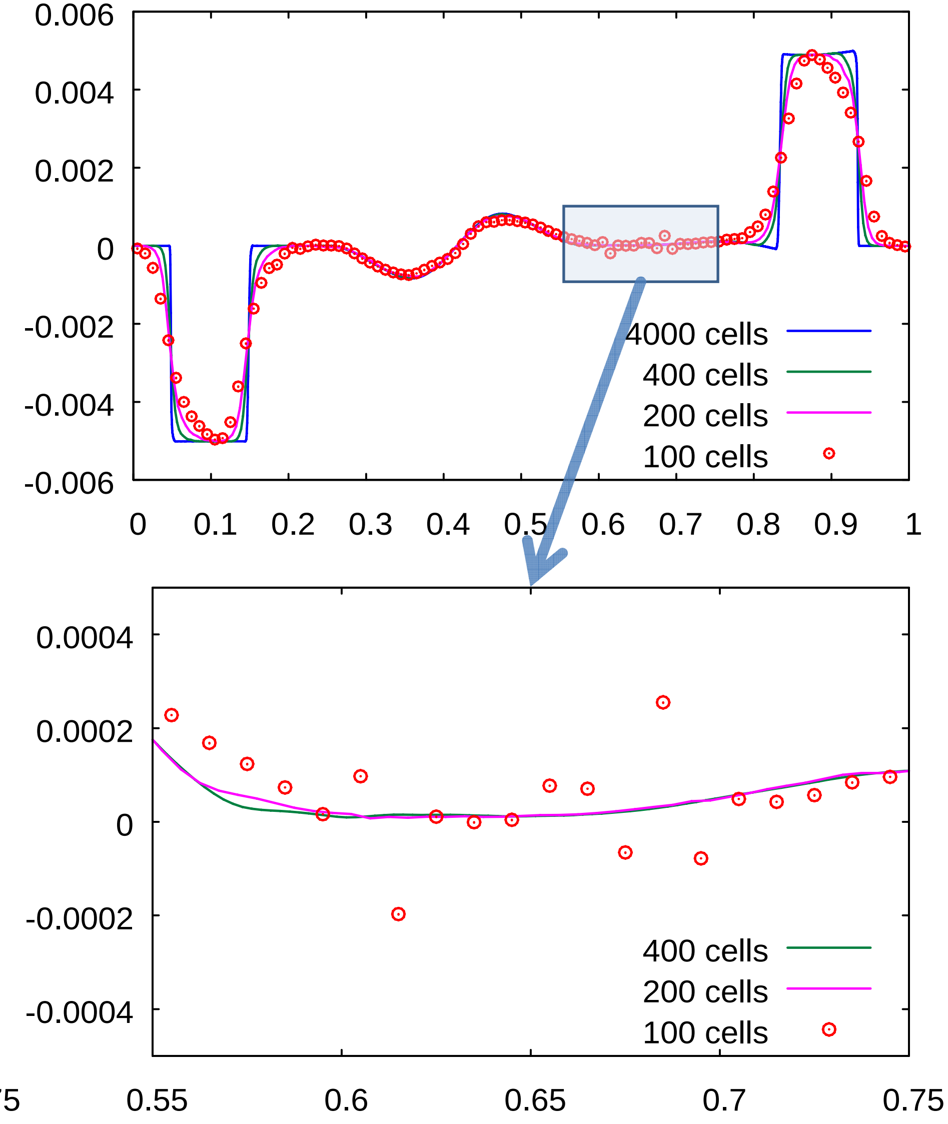
One can notice small perturbations over [0.6, 0.7] due to numerical reconstruction

Reconstruction: rest

Total height

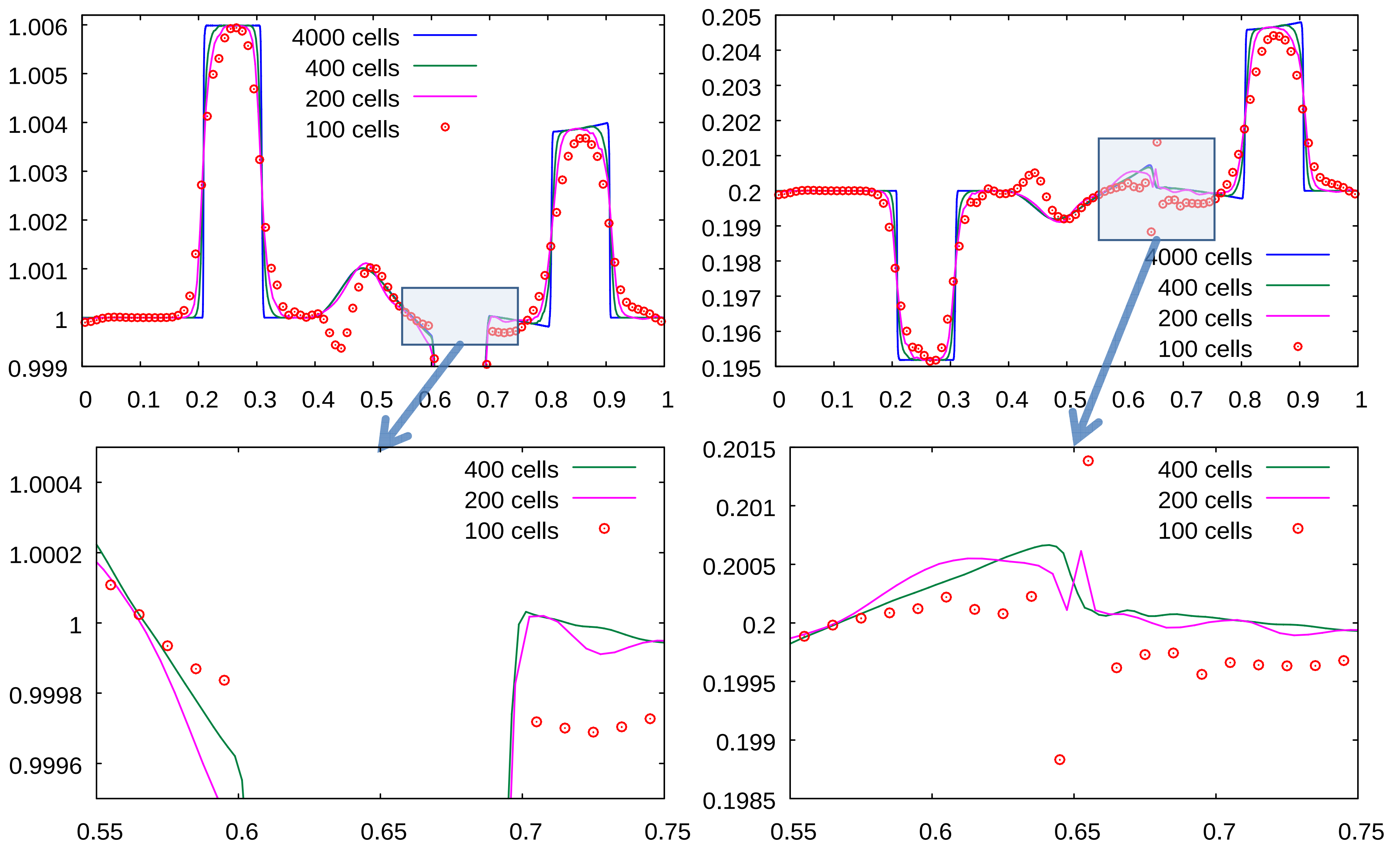


Discharge



Reconstruction: moving

Total height / Discharge



Numerical Tests. Order

	L1-norm of error		**Order**	
Cells	***h***	***q***	***h***	***q***
64				
128	1.42e-5	1.63e-5		
256	5.90e-7	5.86e-7	4.59	4.80
512	2.55e-8	2.32e-8	4.53	4.66
1024	1.19e-9	9.83e-10	4.42	4.56
2048	6.17e-11	4.80e-11	4.27	4.36
4096	3.58e-12	2.84e-12	4.11	4.08
8192	2.20e-13	1.76e-13	4.03	4.01

Conclusion

High order WB can be obtained using:
conservative variables for the evolution
equilibrium variables used in the reconstruction
analytical knowledge of equilibrium variables is not necessary

Work in progress

- Include other effects (dry zones, transcritical flow, etc.)
- Apply to other models (stratified athmosphrere, …)
- Apply numerical reconstruction to problems with more space dimensions
- Connections with other approaches