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From the Geometry of Einstein-Maxwell Spacetimes in General Relativity to Gravitational Radiation

- Observing Gravitational Waves
- Investigating Spacetimes at Null Infinity
- Einstein Vacuum Equations
- From the Christodoulou-Klainerman result 'The global nonlinear stability of the Minkowski space' to Observation
- Christodoulou's Memory Effect
- * New Results for the Einstein-Maxwell Equations
- * Energy and Wave Experiments in the Presence of an Electromagnetic Field

* Joint work with PoNing Chen and Shing-Tung Yau

Gravitational Waves

What is a gravitational wave?

$\Rightarrow$ **Fluctuation of curvature of the spacetime** propagating as a wave.

Gravitational waves: Localized disturbances in the geometry propagate at the speed of light.

Spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein equations:

$$\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{1}{2}\,\mathbf{g}_{\mu\nu}\,\mathbf{R} = 2\,\mathbf{T}_{\mu\nu}\ , \tag{1}$$

where
$\mathbf{G}_{\mu\nu}$ is the **Einstein tensor**,
$\mathbf{R}_{\mu\nu}$ is the **Ricci curvature tensor**,
$\mathbf{R}$ the **scalar curvature tensor**,
$\mathbf{g}$ the **metric tensor** and
$\mathbf{T}_{\mu\nu}$ denotes the **energy-momentum tensor**.

Definition. *A* **Lorentzian metric** g *is a continuous assignment of a non-degenerate quadratic form* g_p, *of index* 1, *in* T_pM *at each* $p \in M$.

Observation of Gravitational Waves

'**We** -the **observers**- are sitting at **null infinity**.'

$\Rightarrow$ Understand **geometry of spacetimes at null infinity**: Investigate and compute **null asymptotics** of solutions of the Einstein equations, null asymptotic behavior of curvature components and geometric quantities.

$\Rightarrow$ Understand **gravitational radiation**

$\Rightarrow$ Detect **gravitational waves**

Nonlinear memory effect (D. Christodoulou, 1991) in regime of Einstein vacuum equations (with large data)

Here: Investigate the nonlinear memory effect in regime of Einstein-Maxwell equations

Christodoulou-Klainerman result
'The global nonlinear stability of the Minkowski space' ([CK])

$\Rightarrow$ **describes precisely asymptotic behavior at null and timelike infinity.**

This result established that under 'suitable' assumptions on the initial data, i.e. under a smallness assumption, the initial data yield a geodesically complete spacetime.

However, as we want to observe 'from null infinity', we need 'only' investigate the null asymptotics. The results for **null** infinity are **independent from the smallness** assumption.

$\Rightarrow$ Can have **large data**.

Solutions of the **Einstein-Vacuum (EV) equations**:

$$R_{\mu\nu} = 0 . \quad (2)$$

Spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying (2).

Is there any non-trivial, asymptotically flat initial data whose maximal development is complete?

Answer

Joint work of **D. Christodoulou** and **S. Klainerman** ([CK], 1993),
'The global nonlinear stability of the Minkowski space'.

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

- Relied on the invariant formulation of the E-V equations.
- **Precise description** of the **asymptotic behaviour at null infinity**.

Initial data set: A triplet $(H, \bar{g}, k)$ with $(H, \bar{g})$ being a three-dimensional complete Riemannian manifold and k a two-covariant symmetric tensorfield on H, satisfying the **constraint equations**:

$$\begin{aligned} \nabla^i \; k_{ij} \; - \; \nabla_j \; trk \quad &= \quad 0 \\ R - \; \mid k \mid^2 \; + \; (trk)^2 \quad &= \quad 0 \; . \end{aligned}$$

Maximal initial data set: In addition $\mathbf{trk = 0}$.
The **constraint equations** then are:

$$\begin{aligned} \nabla^i \; k_{ij} \quad &= \quad 0 \\ R \quad &= \quad \mid k \mid^2 \; . \end{aligned}$$

Evolution equations of a maximal foliation:

$$
\begin{aligned}
\frac{\partial \bar{g}_{ij}}{\partial t} &= -2\Phi k_{ij} \\
\frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \Phi + (R_{ij} - 2k_{im}k^m_{\ j})\Phi
\end{aligned}
$$

Constraint equations of a maximal foliation:

$$
\begin{aligned}
trk &= 0 \\
\nabla^i k_{ij} &= 0 \\
R &= |\, k \,|^2
\end{aligned}
$$

Lapse equation of a maximal foliation:

$$
\triangle\Phi - |\, k \,|^2 \Phi = 0
$$

The (t, u) **foliations** of the spacetime define a codimension 2 foliation by 2-surfaces

$$S_{t,u} \;=\; H_t \;\cap\; C_u \;, \tag{3}$$

the intersection between H_t (foliation by t) and a u-null-hypersurface C_u (foliation by u).

Null pairs consisting of 2 future-directed null vectors e_4 and e_3 orthogonal to $S_{t,u}$ with e_4 tangent to C_u and

$$\big\langle e_4, e_3 \big\rangle \;=\; -\,2\;. \tag{4}$$

A null pair together with an orthonormal frame e_1, e_2 on $S_{t,u}$ forms a **null frame**.

The **null decomposition** of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking **contractions** with the vectorfields e_4, e_3.

Let L and $\underline{L}$ be the outgoing, respectively incoming, null normals to the surface $S_{t,u} = H_t \cap C_u$, for which the component along T is equal to T. Also, the integral curves of L are the null geodesic generators of the null hypersurfaces C_u parametrized by t.

Then T is expressed as

$$T \ = \ \frac{1}{2} \left(L \ + \ \underline{L} \right) \ . \tag{5}$$

The generator S of scalings is defined to be:

$$S \ = \ \frac{1}{2} \left(\underline{u} \ L \ + \ u \ \underline{L} \right) \ . \tag{6}$$

And the generator K of inverted time translations is defined as:

$$K \ = \ \frac{1}{2} \left(\underline{u}^2 \ L \ + \ u^2 \ \underline{L} \right) \ . \tag{7}$$

Then the vectorfield $\bar{K} = K + T$ reads as:

$$\bar{K} \ = \ \frac{1}{2} \left(\tau_+^2 \ L \ + \ \tau_-^2 \ \underline{L} \right) \ . \tag{8}$$

We denote

$$\begin{aligned} \underline{u} \quad &= \quad u + 2r \\ \tau_- \quad &= \quad \sqrt{1 + u^2} \\ \tau_+ \quad &= \quad \sqrt{1 + \underline{u}^2} \ . \end{aligned}$$

In [CK], D. Christodoulou and S. Klainerman achieve

- **Proof of existence and uniqueness of solutions, global result**
- **Asymptotic behaviour: Precise description**

Null decomposition of the Riemann curvature tensor of an E-V spacetime:

$$R_{A3B3} = \underline{\alpha}_{AB} \tag{9}$$

$$R_{A334} = 2\ \underline{\beta}_A \tag{10}$$

$$R_{3434} = 4\ \rho \tag{11}$$

$$^*R_{3434} = 4\ \sigma \tag{12}$$

$$R_{A434} = 2\ \beta_A \tag{13}$$

$$R_{A4B4} = \alpha_{AB} \tag{14}$$

The null components have the
decay properties:

$$\underline{\alpha} = O\ (r^{-1}\ \tau_-^{-\frac{5}{2}})$$

$$\underline{\beta} = O\ (r^{-2}\ \tau_-^{-\frac{3}{2}})$$

$$\rho = O\ (r^{-3})$$

$$\sigma = O\ (r^{-3}\ \tau_-^{-\frac{1}{2}})$$

$$\alpha,\ \beta = o\ (r^{-\frac{7}{2}})$$

From the main theorem in [CK], the authors derive the following **limiting behaviour** of the **curvature components** along the **null** hypersurfaces $\mathbf{C_u}$ as $\mathbf{t} \to \infty$.

$$
\begin{aligned}
\lim_{C_u, t \to \infty} r \underline{\alpha} &= A(u) \\
\lim_{C_u, t \to \infty} r^2 \underline{\beta} &= B(u) \\
\lim_{C_u, t \to \infty} r^3 \sigma &= Q(u) \\
\lim_{C_u, t \to \infty} r^3 \rho &= P(u) \\
\lim_{C_u, t \to \infty} r^{\frac{7}{2}} \beta &= 0 \\
\lim_{C_u, t \to \infty} r^{\frac{7}{2}} \alpha &= 0
\end{aligned}
$$

with A being a symmetric trace-free 2-covariant tensorfield,
B a 1-form, P and Q functions on S^2,
all depending on u
with decay properties as $|\, u \,| \to \infty$:

$$
\begin{aligned}
A &= o\,(|\, u \,|^{-\frac{5}{2}}) \\
B &= o\,(|\, u \,|^{-\frac{3}{2}}) \\
Q &= o\,(|\, u \,|^{-\frac{1}{2}}) \\
P - \bar{P} &= o\,(|\, u \,|^{-\frac{1}{2}})
\end{aligned}
$$

It is

$$
\begin{aligned}
\bar{P} &= o\,(|\, u \,|^{-\frac{1}{2}}) \quad \text{as} \quad u \to -\infty \\
\bar{P} + \frac{M_0}{2\pi} &= o\,(|\, u \,|^{-\frac{1}{2}}) \quad \text{as} \quad u \to \infty
\end{aligned}
$$

M_0 is the ADM mass.

Important Geometric Quantities in the Measurement of Gravitational Waves

Fundamental form χ of S relative to C:

$$\chi(X, Y) \ = \ g(D_X L, Y)$$

for any pair of vectors $X, Y \in T_p S$ and L generating vectorfield of C.

Also

$$\underline{\chi}(X, Y) \ = \ g(D_X \underline{L}, Y)$$

Shear $\hat{\chi}$ Traceless part of χ.

Torsion ζ.

$$\zeta \ (\Pi X) \ = \ g \ (Z, \ X)$$

for all X in $T_p M$, where Π denotes the projection to $T_p S$ with $p \in S$ and

$$Z \ = \ - \ \frac{1}{2} \ D_L \underline{L}$$

where $\underline{L}$ is the generator of the interior cone.

Method as introduced by D. Christodoulou and S. Klainerman in **[CK]**: **Treating propagation equations** along the cones C_u **coupled** to **elliptic systems** on the surfaces $S_{t,u}$.

This method applied in Zipser's proof **[Z]** and in Bieri's proof **[B]**.
In **[Z]** $\Rightarrow$ **electromagnetic field** is present
In **[B]** $\Rightarrow$ **details different** and **borderline cases**

Derivatives of the optical function:

One derives in the EM case the following. For the EV case, just ignore the terms involving F.

$$
\begin{aligned}
\frac{dtr\chi}{ds} + \frac{1}{2}(tr\chi)^2 &= -|\widehat{\chi}|^2 - |\alpha(F)|^2 \\
\frac{d\widehat{\chi}_{AB}}{ds} + tr\chi\widehat{\chi}_{AB} &= -\alpha(W)_{AB} \\
\frac{d\zeta_A}{ds} &= -\chi_{AB}\zeta_B + \chi_{AB}\underline{\zeta}_B - \beta(W)_A \\
&\quad -\rho(F)\alpha(F)_A - \epsilon_{AB}\sigma(F)\alpha_B(F) \\
\frac{d\omega}{ds} &= 2\underline{\zeta}\cdot\zeta - |\zeta|^2 - \rho(W) - \frac{1}{2}(\rho^2(F) + \sigma^2(F))
\end{aligned}
$$

To estimate the tangential derivatives of χ:

Commute $\nabla\!\!\!/$ with

$$\frac{dtr\chi}{ds} + \frac{1}{2}(tr\chi)^2 = -|\widehat{\chi}|^2 - |\alpha(F)|^2$$

And couple with Codazzi equation.

However: one term that does not have fast enough decay. To solve this problem, define the 1-form $\chi\!\!\!/$

$$\chi\!\!\!/_A = \nabla\!\!\!/_A tr\chi + tr\chi\zeta_A.$$

Taking into account the propagation equation for ζ, calculate a propagation equation for $\chi\!\!\!/$. Derive the equation for $\chi\!\!\!/$:

$$\begin{aligned}\frac{d}{ds}\chi\!\!\!/_A + \frac{3}{2}tr\chi\chi\!\!\!/ &= -\widehat{\chi}_{AB}\chi\!\!\!/_B - 2\widehat{\chi}_{BC}\nabla\!\!\!/_A\widehat{\chi}_{BC} \\ &\quad - tr\chi\beta(W)_A - tr\chi\mathbf{R}_{A4} \\ &\quad + tr\chi\widehat{\chi}_{AB}\underline{\zeta}_B - 2|\widehat{\chi}|^2\zeta_A \\ &\quad + \nabla\!\!\!/_A|\alpha(F)|^2 + \left(\zeta_A + \underline{\zeta}_A\right)|\alpha(F)|^2.\end{aligned}$$

The Gauss equation for the Gauss curvature K of $S_{t,u}$ is represented by

$$\begin{aligned}K &= -\frac{1}{4}tr\chi tr\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \rho(W) \\ &\quad -\frac{1}{2}\left(\rho^2(F) + \sigma^2(F)\right).\end{aligned} \quad (15)$$

Null Asymptotics $\Rightarrow$ Gravitational Radiation

Limit for the shear $\hat{\chi}$

$$\lim_{C_u, t \to \infty} r^2 \widehat{\chi} \; = \; \Sigma(u)$$

Σ symmetric trace-free 2-covariant tensorfield on S^2 depending on u.

Moreover,

$$\begin{aligned} \lim_{C_u, t \to \infty} r \; tr\chi \quad &= \quad - \lim_{C_u, t \to \infty} r \; tr\underline{\chi} \; = \; 2 \\ \lim_{C_u, t \to \infty} r \widehat{\underline{\chi}} \quad &= \quad \Xi(u) \end{aligned}$$

Ξ symmetric trace-free 2-covariant tensorfield on S^2 depending on u.

$$\Xi \; = \; o \, (| \, u \, |^{-\frac{3}{2}}) \quad \text{as} \quad | \, u \, | \to \; \infty \quad .$$

Σ and Ξ are related by

$$\frac{\partial \Sigma}{\partial u} \; = \; -\Xi \quad . \tag{16}$$

A, B and Ξ are related according to the formulas

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4} A \tag{17}$$

$$\nabla\!\!\!/^{B} \Xi_{AB} = B_A \tag{18}$$

with $\nabla\!\!\!/$ relative to an arbitrary local frame on S^2.

ADM mass M_0 enters the
asymptotic expansion of the area radius of the sections $S_{t,u}$ **of null hypersurface** C_u **as** $t \to \infty$:

$$r(t,u) = t - 2M_0 \log t + O(1) \tag{19}$$

at constant u as $t \to \infty$.

Hawking mass $m(t,u)$ contained in a surface $S_{t,u}$ defined as:

$$m(t,u) = \frac{r}{2} \left(1 + \frac{1}{16\pi} \int_{S_{t,u}} tr\chi \; tr\underline{\chi}\right) \tag{20}$$

Bondi mass $M(u)$ contained in C_u defined as:

$$M(u) = \lim_{t \to \infty} m(t,u) \tag{21}$$

[CK] derived the **Bondi mass loss formula**

$$\frac{\partial M}{\partial u} = \frac{1}{8} \int_{S^2} | \Xi |^2 \; d\mu_{\gamma_{S^2}} \tag{22}$$

Obtained the **limits**

$$\lim_{u \to \infty} M(u) = M_0 \tag{23}$$

$$\lim_{u \to -\infty} M(u) = 0 \tag{24}$$

M_0 is the total mass.

Important difference between the limits Σ^+ and Σ^- as $u \to \infty$ resp. $u \to -\infty$

$\Rightarrow$

Yielding equation for **nonlinear memory effect [Christodoulou]**

Define the **total energy radiated to infinity in a given direction, per unit solid angle** as

$$\frac{\mathbf{F}}{4\pi} \quad \text{for} \quad F = \frac{1}{8} \int_{-\infty}^{\infty} | \Xi(u) |^2 \; du \tag{25}$$

Consider equation

$$di\!\!/v \; (\Sigma^+ - \Sigma^-) = \nabla\!\!\!/ \, f$$

with f being a solution of

$$\triangle\!\!\!/ \, f = 2\,(F - \bar{F}) \quad , \quad \bar{f} = 0$$

$\nabla\!\!\!/ \, , di\!\!/v \, , \triangle\!\!\!/$ on S^2.

Integrability condition of the last two equations is that F is L^2-orthogonal to the first eigenspace of $\not\triangle$:

$$F_{(1)} = 0 \quad .$$

Derive

$$\Sigma^{+} \; - \; \Sigma^{-} \; = \; \frac{1}{2} \int_{-\infty}^{\infty} \Xi(u) \; du \tag{26}$$

and

$$\Sigma(u) \; = \; \Sigma^{-} \; + \; \frac{1}{2} \int_{-\infty}^{u} \Xi(u') \; du'$$

$$\mathbf{\Sigma(u) - \Sigma^{-}}$$

related to

instantaneous displacements of faraway test masses w.r.t. reference test mass, relative to which they are initially at rest.

$$\mathbf{\Sigma^{+} - \Sigma^{-}}$$

yields

permanent displacement of the test masses.
Non-linear effect.
An effect observable in principle.

Gravitational Radiation in Different Situations

In **[Christodoulou-Klainerman] , [Zipser]**: strongly asymptotically flat initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that, as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$:

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\, \delta_{ij} + o_4\, (r^{-\frac{3}{2}}) \tag{27}$$

$$k_{ij} = o_3\, (r^{-\frac{5}{2}})\ , \tag{28}$$

where M denotes the mass.

In **[Bieri]**: Asymptotically flat initial data $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\, (r^{-\frac{1}{2}}) \tag{29}$$

$$k_{ij} = o_2\, (r^{-\frac{3}{2}})\ . \tag{30}$$

$\Rightarrow$ **Can compute gravitational radiation for the cases [CK], [Z], but not for [B].**
Decay!

Einstein-Maxwell Case

What happens in the presence of an electromagnetic field?

Einstein-Maxwell equations:

$$\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{1}{2}\,\mathbf{g}_{\mu\nu}\,\mathbf{R} = 8\pi\;\mathbf{T}_{\mu\nu}\;, \tag{31}$$

setting $G = c = 1$, $\mu, \nu = 0, 1, 2, 3$, where
$\mathbf{G}_{\mu\nu}$ is the **Einstein tensor**,
$\mathbf{R}_{\mu\nu}$ is the **Ricci curvature tensor**,
R the **scalar curvature tensor**,
g the **metric tensor** and
$\mathbf{T}_{\mu\nu}$ denotes the **stress-energy tensor of the electromagnetic field**.

In particular, $\mathbf{F}$ denoting the **electromagnetic field**, the tensor $\mathbf{T}_{\alpha\beta}$ reads:

$$T_{\alpha\beta} = \frac{1}{8\pi}\Big(F_{\alpha}{}^{\rho}F_{\beta\rho} - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}\Big) \tag{32}$$

F is an antisymmetric covariant 2-tensor.

The **Einstein-Maxwell (EM) equations** are given by

$$
\begin{aligned}
R_{\mu\nu} &= 8\pi T_{\mu\nu} && (33)\\
D^\alpha F_{\alpha\beta} &= 0 && (34)\\
D^\alpha\ {}^*F_{\alpha\beta} &= 0. && (35)
\end{aligned}
$$

Whereas in the EV case, the **Weyl tensor** satisfies the homogeneous equations

$$D^\alpha W_{\alpha\beta\gamma\delta} = 0 \ ,$$

in the EM case the corresponding equations are inhomogeneous

$$D^\alpha W_{\alpha\beta\gamma\delta} = \frac{1}{2}(D_\gamma R_{\beta\delta} - D_\delta R_{\beta\gamma}) \quad . \tag{36}$$

Zipser works with the same conditions as [CK] on the metric, second fundamental form and curvature,
in addition she imposes a decay condition on the electromagnetic field, namely

$$F|_H = o_3\left(r^{-\frac{5}{2}}\right). \tag{37}$$

The null components of the electromagnetic field are written as

$$\begin{array}{ll} F_{A3} = \underline{\alpha}(F)_A & F_{A4} = \alpha(F)_A \\ F_{34} = 2\rho(F) & F_{12} = \sigma(F). \end{array} \tag{38}$$

The corresponding null decomposition $\{\underline{\alpha}({}^*F), \alpha({}^*F), \rho({}^*F), \sigma({}^*F)\}$ of ${}^*F$ is given by

$$\begin{array}{rcll} \underline{\alpha}({}^*F)_A & = & -\underline{\alpha}(F)^B \epsilon_{BA} & \alpha({}^*F)_A = \alpha(F)^B \epsilon_{BA} \\ \rho({}^*F) & = & \sigma(F) & \sigma({}^*F) = -\rho(F) \end{array} \tag{39}$$

where the Hodge dual of a tensor u tangent to $S_{t,u}$, is defined by

$${}^*u_A = \epsilon_A{}^B u_{B.}$$

The estimates in Zipser yield the decay behaviour:

$$\begin{array}{rcl} \underline{\alpha}(F) & = & O\ (r^{-1}\ \tau_-^{-\frac{3}{2}}) \\ \rho(F), \sigma(F) & = & O\ (r^{-2}\ \tau_-^{-\frac{1}{2}}) \\ \alpha(F) & = & o\ (r^{-\frac{5}{2}}) \end{array}$$

Null Asymptotics $\Rightarrow$ Gravitational Radiation

The parameters of the foliations and the components of the Weyl tensor behave exactly as in [CK].

Along the null hypersurfaces C_u as $t \to \infty$, one finds

$$\lim_{C_u, t \to \infty} \phi = 1, \qquad \lim_{C_u, t \to \infty} a = 1 \tag{40}$$

and

$$\lim_{C_u, t \to \infty} (rtr\chi) = 2, \qquad \lim_{C_u, t \to \infty} \left(rtr\underline{\chi}\right) = -2 \tag{41}$$

Furthermore, we let

$$H = \lim_{C_u, t \to \infty} \left(r^2 (tr\chi' - \frac{2}{r}) \right) . \tag{42}$$

Zipser makes the following conclusions:

Theorem 1. *On any null hypersurface C_u, the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$, $r^3\rho(W)$, $r^3\sigma(W)$, $r\underline{\alpha}(F)$, $r^2\rho(F)$, $r^2\sigma(F)$ have limits as $t \to \infty$, that is*

$$\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = B_W(u,\cdot)$$
$$\lim_{C_u, t\to\infty} r^3\rho(W) = P_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^3\sigma(W) = Q_W(u,\cdot)$$
$$\lim_{C_u, t\to\infty} r\underline{\alpha}(F) = A_F(u,\cdot),$$
$$\lim_{C_u, t\to\infty} r^2\rho(F) = P_F(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\sigma(F) = Q_F(u,\cdot)$$

where A_W is a symmetric traceless covariant 2-tensor, B_W and A_F are 1-forms and P_W, Q_W, P_F, Q_F are functions on S^2 depending on u and having the following decay properties:

$$\begin{aligned}
|A_W(u,\cdot)| &\leq C(1+|u|)^{-5/2} \\
|B_W(u,\cdot)| &\leq C(1+|u|)^{-3/2} \\
\left|P_W(u,\cdot) - \overline{P}_W(u)\right| &\leq (1+|u|)^{-1/2} \\
\left|Q_W(u,\cdot) - \overline{Q}_W(u)\right| &\leq (1+|u|)^{-1/2} \\
|A_F(u,\cdot)| &\leq C(1+|u|)^{-3/2} \\
|P_F(u,\cdot)| &\leq (1+|u|)^{-1/2} \\
|Q_F(u,\cdot)| &\leq (1+|u|)^{-1/2}
\end{aligned}$$

and

$$\lim_{u\to-\infty} \overline{P}_W(u) = 0, \qquad \lim_{u\to-\infty} \overline{Q}_W(u) = 0.$$

Pointwise norms $|\quad|$ of the tensors on S^2 relate to metric $\overset{\circ}{\gamma}$, being the limit of the induced metrics on $S_{t,u}$ $\forall u$ as $t \to \infty$.

More Structure:

Have

$$g(L, \underline{L}) = -2$$

Null second fundamental form χ and
conjugate null second fundamental form $\underline{\chi}$ of S:

$$\begin{aligned} \chi(X, Y) &= g(\nabla_X L, Y) \\ \underline{\chi}(X, Y) &= g(\nabla_X \underline{L}, Y) \quad \forall\ X, Y \in T_p S \end{aligned}$$

Shear: traceless part $\hat{\chi}$ of χ

Torsion: ζ defined by

$$\zeta\ (X) = \frac{1}{2}\, g\ (\nabla_X L, \underline{L})$$

for all X in $T_p S$.

Mass aspect function μ and
conjugate mass aspect function $\underline{\mu}$ of S:

$$\begin{aligned} \mu &= K + \frac{1}{4}\, tr\chi\ tr\underline{\chi} - d\not{i}v\,\zeta \\ \underline{\mu} &= K + \frac{1}{4}\, tr\chi\ tr\underline{\chi} - d\not{i}v\,\underline{\zeta} \end{aligned}$$

The **Hawking mass**:

$$m(t,u) = \frac{r}{2}\left(1 + \frac{1}{16\pi}\int_{S_{t,u}} tr\chi tr\underline{\chi}\ d\mu_\gamma\right)$$

If S has the topology of S^2, then the following holds by Gauss-Bonnet

$$\int_S \underline{\mu}\ d\mu_\gamma \ = \ \frac{8\pi m}{r}$$

Null Codazzi equation and its **conjugate**:

(EV)

$$(\not{d}\!\not{i}\!v\,\hat{\chi})_A \ + \ \hat{\chi}_{AB}\zeta_B \ = \ \frac{1}{2}\big(\not\nabla_A tr\chi \ + \ tr\chi\zeta_A\big) \ - \ \beta_A(W)$$

$$(\not{d}\!\not{i}\!v\,\hat{\underline{\chi}})_A \ - \ \hat{\underline{\chi}}_{AB}\zeta_B \ = \ \frac{1}{2}\big(\not\nabla_A tr\underline{\chi} \ - \ tr\underline{\chi}\zeta_A\big) \ + \ \underline{\beta}_A(W)$$

(EM)

$$(\not{d}\!\not{i}\!v\,\hat{\chi})_A \ + \ \hat{\chi}_{AB}\zeta_B \ = \ \frac{1}{2}\big(\not\nabla_A tr\chi \ + \ tr\chi\zeta_A\big) \ - \ \beta_A(W) \ - \ \rho(F)\alpha(F)_A \ - \ \epsilon_{AB}\sigma(F)\alpha_B(F)$$

$$(\not{d}\!\not{i}\!v\,\hat{\underline{\chi}})_A \ - \ \hat{\underline{\chi}}_{AB}\zeta_B \ = \ \frac{1}{2}\big(\not\nabla_A tr\underline{\chi} \ - \ tr\underline{\chi}\zeta_A\big) \ + \ \underline{\beta}_A(W) \ + \ \rho(F)\underline{\alpha}_A(F) \ + \ \epsilon_{AB}\sigma(F)\underline{\alpha}_B(F)$$

On the null hypersurface C_u, the normalized **shear** $r^2\chi'$ has **limit** as $t \to \infty$:

$$\lim_{C_u, t\to\infty} r^2\chi' = \Sigma\,(u,\cdot)$$

where Σ is a symmetric traceless covariant 2-tensor on S^2 depending on u.

On any null hypersurface C_u, the **limit** of $r\widehat{\eta}$ exists as $t \to \infty$, i.e.

$$\lim_{C_u, t\to\infty} r\widehat{\eta} = \Xi\,(u,\cdot)$$

where Ξ is a symmetric traceless 2-covariant tensor on S^2 depending on u and having the decay property

$$|\Xi\,(u,\cdot)|_{\overset{\circ}{\gamma}} \leq C\,(1+|u|)^{-3/2}\,.$$

Moreover,

$$\lim_{C_u, t\to\infty} r\widehat{\theta} = -\frac{1}{2}\lim_{C_u, t\to\infty} r\widehat{\underline{\chi}}' = \Xi$$

and

$$\frac{\partial\Sigma}{\partial u} = -\Xi \tag{43}$$

$$\frac{\partial\Xi}{\partial u} = -\frac{1}{4}A_W. \tag{44}$$

Bondi mass loss formula

The **Hawking mass** is defined as follows

$$m(t,u) = \frac{r}{2}\left(1 + \frac{1}{16\pi}\int_{S_{t,u}} tr\chi tr\underline{\chi}\right). \tag{45}$$

To calculate the propagation equation for m, let

$$\underline{\mu} = -\not{d}iv\,\underline{\zeta} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}} - \rho(W) - \frac{1}{2}\left(\rho^2(F) + \sigma^2(F)\right). \tag{46}$$

Using the null structure equations with respect to the l-pair,

$$\begin{aligned}
\frac{dtr\underline{\chi}}{ds} + \frac{1}{2}tr\chi tr\underline{\chi} &= -2\underline{\mu} + 2|\zeta|^2 \\
\frac{dtr\chi}{ds} + \frac{1}{2}(tr\chi)^2 &= -|\widehat{\chi}|^2 - |\alpha(F)|^2.
\end{aligned}$$

Derive that

$$\begin{aligned}
\frac{\partial}{\partial t}m(t,u) &= -\frac{r}{16\pi}\int_{S_{t,u}}\left(a\phi tr\chi - \overline{\phi a tr\chi}\right)\underline{\mu} \\
&\quad + \frac{r}{8\pi}\int_{S_{t,u}} a\phi\left(\frac{1}{2}tr\chi|\zeta|^2 - \frac{1}{4}tr\underline{\chi}|\widehat{\chi}|^2 - \frac{1}{4}tr\underline{\chi}|\alpha(F)|^2\right).
\end{aligned} \tag{47}$$

Because $K + \frac{1}{4} tr\chi tr\underline{\chi} = O\left(r^{-3}\right)$, $\underline{\mu} = O\left(r^{-3}\right)$. Given the asymptotic behavior of the right-hand side terms of 47, we conclude that

$$\frac{\partial}{\partial t} m\left(t, u\right) = O\left(r^{-2}\right).$$

$\Rightarrow \mathbf{m}\left(\mathbf{t}, \mathbf{u}\right)$ has a **limit** $M\left(u\right)$ for any fixed u as $t \to \infty$.

$\mathrm{M}\left(\mathbf{u}\right)$: **Bondi mass of the null hypersurface** $\mathrm{C}_{\mathbf{u}}$.

Only **difference** between EM and EV case:
terms appearing due to the presence of the **electromagnetic field**.

However, these terms **decay fast enough** so that the mass decays at the same rate as in [CK]. In particular, as $t \to \infty$ on C_u, we find:

$$m(t,u) = M(u) + O\left(r^{-1}\right)$$

Calculate a Bondi mass loss formula by considering

$$\frac{\partial}{\partial u} m(t,u)$$

where

$$\frac{\partial}{\partial u} m(t,u) = \frac{1}{2}\overline{atr\theta} m + \frac{r}{32\pi}\int_{S_{t,u}} a\left(\nabla_N \underline{\mu} + tr\theta \underline{\mu}\right).$$

One can then prove

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi}\int_{S^2}\left(|\Xi|^2 + \frac{1}{2}|A_F|^2\right) d\mu_{\overset{\circ}{\gamma}}. \qquad (48)$$

RHS of (48) **positive and integrable** in u.

$\Rightarrow \mathbf{M}(\mathbf{u})$ is a **non-decreasing function** of u and has **finite limits** $\mathrm{M}(-\infty)$ for $u \to -\infty$ and $\mathrm{M}(\infty)$ for $u \to \infty$.

$\Rightarrow \mathbf{M}(-\infty) = \mathbf{0}$, and $\mathrm{M}(\infty)$ is the **total mass**.

Theorem 2. [Z] *The* **Hawking mass** $\mathbf{m}(\mathbf{t},\mathbf{u})$ **tends** *to the* **Bondi mass** $\mathbf{M}(\mathbf{u})$ *as* $t \to \infty$ *on any null hypersurface* C_u. *More precisely,*

$$m(t,u) = M(u) + O(r^{-1}).$$

And $M(u)$ *verifies the* **Bondi mass loss formula**

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + \frac{1}{2} |A_F|^2 \right) d\mu_{\overset{\circ}{\gamma}}$$

where $d\mu_{\overset{\circ}{\gamma}}$ *is the area element of the standard unit sphere* S^2.

In the **Bondi mass loss formula**
$\Rightarrow$ limiting term $\mathbf{A_F}$ of **electromagnetic field** comes in.

Comparison with the **Bondi mass loss formula obtained in [CK]**:

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} |\Xi|^2 \, d\mu_{\overset{\circ}{\gamma}} \ .$$

We find: the **electromagnetic field contributes** to the **change of the Bondi mass** by

$$\frac{1}{16\pi}\int_{S^2} |A_F|^2 \, d\mu_{\overset{\circ}{\gamma}} \ .$$

[BCY]
The **decay behaviour** of $\mathbf{A_F}$ is the **same** as for Ξ.

[Bieri-Chen-Yau] Define the new function

$$F = \frac{1}{8}\int_{-\infty}^{+\infty} \left(| \ \Xi \ |^2 + \frac{1}{2} \mid A_F \mid^2 \right) du \ . \qquad (49)$$

$\frac{\mathbf{F}}{\mathbf{4\pi}}$ is the **total energy radiated to infinity in a given direction per unit solid angle**.

Thus: the **integrand** in (49) is **proportional** to the **power radiated to infinity at a given retarded time** u**, in a given direction, per unit area on** S^2 **(per unit solid angle)**.

Already **Christodoulou** tells us how to adapt the formula for F when matter radiation is present, that is also in the EM case.

The Bondi mass loss formula from [Z] agrees with the one in Bondi coordinates derived by van der Burg.

Permanent Displacement Formula

Christodoulou's Memory Effect $\Rightarrow$ governed by the permanent displacement formula $\Sigma^+ - \Sigma^-$.

We consider $\Sigma^+ - \Sigma^-$ in the **EM** case.

Theorem 3. [BCY] *Let* $\Sigma^+(\cdot) = \lim_{u\to\infty} \Sigma(u,\cdot)$ *and* $\Sigma^-(\cdot) = \lim_{u\to-\infty} \Sigma(u,\cdot)$. *Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \left(\mid \Xi(u,\cdot) \mid^2 + \frac{1}{2} \mid A_F(u,\cdot) \mid^2 \right) du \quad . \tag{50}$$

Moreover, let Φ *be the solution with* $\bar{\Phi} = 0$ *on* S^2 *of the equation*

$$\overset{\circ}{\not{\Delta}} \, \Phi = F - \bar{F} \quad .$$

Then $\Sigma^+ - \Sigma^-$ *is given by the following equation on* S^2*:*

$$\overset{\circ}{\not{div}} \, (\Sigma^+ - \Sigma^-) = \overset{\circ}{\not{\nabla}} \, \Phi \quad . \tag{51}$$

Proof - Sketch: We have

$$\Sigma(u) = \Sigma^- - \int_{-\infty}^{u} \Xi(u')du'$$

and

$$\Sigma^{+} - \Sigma^{-} = - \int_{-\infty}^{\infty} \Xi(u')du' \quad .$$

We work with the following as derived in [Z]

$$\begin{aligned}
\triangle \Psi &= r \mid \widehat{\eta} \mid^2 - \frac{r}{4} \mid \underline{\alpha}(F) \mid^2 \\
\triangle \Psi' &= -ra^{-1}\lambda\Big(\mid \widehat{\eta} \mid^2 - \overline{\mid \widehat{\eta} \mid^2}\Big) \\
&\quad + \frac{r^2 a^{-1}}{4}\Big(a \not{D}_4 \mid \underline{\alpha}(F) \mid^2 - \overline{a \not{D}_4 \mid \underline{\alpha}(F) \mid^2}\Big)
\end{aligned}$$

whereas in [CK] it is

$$\begin{aligned}
\triangle \Psi &= r \mid \widehat{\eta} \mid^2 \\
\triangle \Psi' &= -ra^{-1}\lambda\Big(\mid \widehat{\eta} \mid^2 - \overline{\mid \widehat{\eta} \mid^2}\Big) \quad .
\end{aligned}$$

We compute the following limits in the new case.

$$\lim_{C_u, t \to \infty} \Psi = \boldsymbol{\Psi} \qquad \lim_{C_u, t \to \infty} \Psi' = \boldsymbol{\Psi}'$$

$$\lim_{C_u, t \to \infty} r \nabla_N \Psi = \Omega(u, \cdot) \qquad \lim_{C_u, t \to \infty} r \nabla_N \Psi' = \Omega'(u, \cdot).$$

Hodge system for ϵ

$$\begin{aligned}
\not{div}\, \epsilon &= -\nabla_N \delta - \frac{3}{2} tr\theta\delta + \widehat{\eta} \cdot \widehat{\theta} \\
&\quad -2(a^{-1} \not{\nabla} a) \cdot \epsilon + \frac{1}{4} \mid \alpha(F) \mid^2 - \frac{1}{4} \mid \underline{\alpha}(F) \mid^2 \\
\not{curl}\ \epsilon &= \sigma(W) + \widehat{\theta} \wedge \widehat{\eta} \quad .
\end{aligned}$$

We derive

$$\begin{aligned}
\not{div}\, \epsilon &= \rho - \bar{\rho} + \widehat{\chi} \cdot \widehat{\eta} - \overline{\widehat{\chi} \cdot \widehat{\eta}} + r^{-1} \not{\triangle}\, \Psi - r^{-2} \nabla_N \Psi' \\
&\quad - r^{-3} a^{-1} \lambda \Psi' + l.o.t. \qquad (52) \\
\not{curl}\ \epsilon &= \sigma(W) + \widehat{\theta} \wedge \widehat{\eta} \qquad (53)
\end{aligned}$$

Let

$$E = \lim_{C_u, t \to \infty} \left(r^2 \epsilon \right)$$

Multiply equations (52) and (53) by r^3 and take the limits on C_u as $t \to \infty$. This yields:

$$\overset{\circ}{c\not{u}rl} \;\; E \quad = \quad Q + \Sigma \wedge \Xi \tag{54}$$

$$\begin{aligned} \overset{\circ}{d\not{i}v} \;\; E \quad &= \quad P - \bar{P} + \Sigma \cdot \Xi - \overline{\Sigma \cdot \Xi} \\ &\quad + \overset{\circ}{\not{\triangle}} \; \Psi - \Psi' - \Omega' \quad . \end{aligned} \tag{55}$$

Investigate limits as $u \to +\infty$ and $u \to -\infty$.

$\Rightarrow$

E tends to a limit E^+ as $u \to +\infty$ and to E^- as $u \to -\infty$.

Obtain

$$\overset{\circ}{c\not{u}rl} \;\; (E^+ - E^-) \; = \; 0$$

Now, compute $\overset{\circ}{d\not{i}v} \; (E^+ - E^-)$.

Have to consider the corresponding limits for the terms involving $\mathbf{\Psi}$ and $\mathbf{\Psi}'$, that is also Ω'.

We use the fact that

$$\not{D}_4 \underline{\alpha}(F)_A = -\frac{1}{2} tr\chi \underline{\alpha}(F)_A + l.o.t. \tag{56}$$

Now, considering (52) and using (56), we find that

$$\not{D}_4 \mid \underline{\alpha}(F) \mid^2 = -tr\chi \mid \underline{\alpha}(F) \mid^2 + l.o.t. \tag{57}$$

Using (57), (52) and (52), we deduce formulas for $\mathbf{\Psi}$, $\mathbf{\Psi}'$, Ω, Ω' by computing the limits (52). We give the formulas qualitatively:

$$
\begin{aligned}
\mathbf{\Psi} &= \int_{-\infty}^{+\infty} \Big\{ \int_{S^2} \Big(\cdots \text{involving}(\mid \Xi \mid^2 (u', \omega')) \cdots \Big) d\omega' \\
&\quad + \int_{S^2} \Big(\cdots \text{involving}(\mid A_F \mid^2 (u', \omega')) \cdots \Big) d\omega' \Big\} du'
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Psi}' &= \int_{-\infty}^{+\infty} \Big\{ \int_{S^2} \Big(\cdots \text{involving}(\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')}) \cdots \Big) d\omega' \\
&+ \int_{S^2} \Big(\cdots \text{involving}(\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')}) \cdots \Big) d\omega' \Big\} du'
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= \int_{-\infty}^{+\infty} \Big\{ \int_{S^2} \Big(\cdots \text{involving}(\mid \Xi \mid^2 (u', \omega')) \cdots \Big) d\omega' \\
&\quad + \int_{S^2} \Big(\cdots \text{involving}(\mid A_F \mid^2 (u', \omega')) \cdots \Big) d\omega' \\
&\quad + \cdots \text{involving}\Big(\mid \Xi \mid^2 (u', \omega') + \mid A_F \mid^2 (u', \omega') \Big) \cdots \Big\} du'
\end{aligned}
$$

$$
\begin{aligned}
\Omega' &= \int_{-\infty}^{+\infty} \Big\{ \int_{S^2} \Big(\cdots \text{involving}(\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')}) \cdots \Big) d\omega' \\
&\quad + \int_{S^2} \Big(\cdots \text{involving}(\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')}) \cdots \Big) d\omega' \\
&\quad + \cdots \text{involving}\Big(\Big(\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')} \Big) \\
&\quad\ + \frac{1}{2} \Big(\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')} \Big) \Big) \cdots \Big\} du'
\end{aligned}
$$

Evaluating the difference of the limits as $u \to +\infty$ and $u \to -\infty$ in (55), the contribution of $\overset{\circ}{\not\Delta}\ \Psi$, Ψ' and Ω' comes only from terms in Ω'. We find that Ω' tends to limits $\Omega'^{+}(\cdot)$ and $\Omega'^{-}(\cdot)$ as $t \to \infty$ and $t \to -\infty$, respectively. Thus, we conclude

$$\Omega'^{+}(\cdot) - \Omega'^{-}(\cdot) = \int_{-\infty}^{+\infty} \Big(\mid \Xi(u,\cdot) \mid^2 - \overline{\mid \Xi(u,\cdot) \mid^2} + \frac{1}{2} \mid A_F(u,\cdot) \mid^2 - \frac{1}{2}\overline{\mid A_F(u,\cdot) \mid^2} \Big) du \quad .$$

Finally, we obtain

$$\overset{\circ}{\not{div}}\ (E^{+} - E^{-}) = -\Omega'^{+} + \Omega'^{-} \tag{58}$$

$$= \int_{-\infty}^{+\infty} \Big(- \mid \Xi(u,\cdot) \mid^2 + \overline{\mid \Xi(u,\cdot) \mid^2} - \frac{1}{2} \mid A_F(u,\cdot) \mid^2 + \frac{1}{2}\overline{\mid A_F(u,\cdot) \mid^2} \Big) du \quad . \tag{59}$$

We know that

$$(E^{+} - E^{-}) = \overset{\circ}{\not\nabla}\ \Phi \tag{60}$$

with Φ being the solution of vanishing mean of

$$\overset{\circ}{\not\Delta}\ \Phi = -\Omega'^{+} + \Omega'^{-} \quad \text{on } S^2 \quad .$$

We derive

$$\overset{\circ}{d\!\!/iv}\ \Sigma = \overset{\circ}{\nabla\!\!\!/}\ H + E$$

We conclude (as $\frac{\partial H}{\partial u} = 0$)

$$\overset{\circ}{d\!\!/iv}\ (\Sigma^{+} - \Sigma^{-}) = E^{+} - E^{-} \quad . \tag{61}$$

This proves theorem 3.

Limit for $\mathbf{r}$ as $\mathbf{t} \to \infty$ on Null Hypersurface $\mathbf{C_u}$

Constraint on the spacelike scalar curvature, which is given by

$$R = |k|^2 + \mathbf{R}_{TT},$$

differs from the constraint in the vacuum case only by the term $\mathbf{R}_{TT}$, which is a quadratic in F.

We also prove the following theorem.

Theorem 4. [Bieri-Chen-Yau] *As $t \to \infty$ we obtain on any null hypersurface C_u*

$$r = t - 2M(\infty) \log t + O(1) \quad .$$

Gravitational Wave Experiments

How do these results relate to experiment?

Christodoulou discusses idea and setup for a **laser interferometer gravitational-wave detector**.
He shows how the **theoretical result on** $\Sigma^+ - \Sigma^-$ leads to an **effect measurable** by such detectors.

This effect manifests itself in a **permanent displacement** of the test masses of the detector after a wave train has passed.

Permanent displacement of the test masses in the EM case:

Electromagnetic field comes into the formula $\Sigma^+ - \Sigma^-$.

Instantaneous displacement of the test masses in the EM case:

Nothing changes at highest order.

3 test masses $\mathbf{m_0}$, $\mathbf{m_1}$, $\mathbf{m_2}$ suspended by equal length pendulums.

$\mathbf{m_0}$: reference mass.

Measure by laser interferometry the distance of $\mathbf{m_1}$ and $\mathbf{m_2}$ from the reference mass $\mathbf{m_0}$

The beam splitter is at m_0.

Motion of masses on the horizontal plane: considered free for timelike scales much shorter than the period of the pendulums.

Any **difference** in the **light travel times** between m_0 and m_1 and m_2, respectively, results in a **difference of phase of the laser light** at m_0.

m_0, m_1, m_2 move along **geodesics** γ_0, γ_1, γ_2 in spacetime.
T: unit future-directed tangent vectorfield of γ_0
t: arc length along γ_0.
Let H_t for each t be the spacelike, geodesic hyperplane through $\gamma_0(t)$ orthogonal to T.

Consider the **orthonormal frame field** (T, E_1, E_2, E_3) along γ_0, where (E_1, E_2, E_3) is an orthonormal frame for H_0 at $\gamma_0(0)$, parallely propagated along γ_0.

$\Rightarrow$ at each t, (E_1, E_2, E_3) is an orthonormal frame for H_t at $\gamma_0(t)$.

Measure distances of m_1 and m_2 from m_0.

Under corresponding physical assumption, **differences in phase of the laser light will reflect differences in distance of m_1 and m_2 from m_0.**

The same assumption allows us to replace the **geodesic equation** for γ_1 and γ_2 by the **Jacobi equation** (geodesic deviation from γ_0).

$$\frac{d^2 x^k}{dt^2} = - R_{kTlT} \; x^l \tag{62}$$

with

$$R_{kTlT} = R \left(E_k, T, E_l, T \right) . \tag{63}$$

Now, assume for simplicity that the source is in the E_3-direction.

Investigate the formula (62) for the Einstein-Maxwell situation:

Non-charged test masses: formula (62) stays the same, but the electromagnetic field comes in.
However, it enters at lower order.

It is:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) \\ &\quad -\frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) R \quad . \end{aligned} \tag{64}$$

Using the EM equations:

$$R_{00} = 8\pi T_{00} \quad ,$$

and in particular,

$$R_{00} = \frac{1}{2}(\mid \underline{\alpha}(F) \mid^2 + \mid \alpha(F) \mid^2) + \rho(F)^2 + \sigma(F)^2 \tag{65}$$

we can investigate the components of the Ricci curvature. R_{00} includes the term $\mid \underline{\alpha}(F) \mid^2$. **Worst decay behavior. However** it enters as a **quadratic** the formula for R_{00}.

Consider $L = T - E_3$, $\underline{L} = T + E_3$.

The **leading components of the curvature** are

$$\underline{\alpha}_{AB}(W) \quad = \quad R\ (E_A,\ \underline{L},\ E_B,\ \underline{L}) \tag{66}$$

$$\underline{\alpha}_{AB}(W) \quad = \quad \frac{A_{AB}(W)}{r}\ +\ o\ (r^{-2})\ . \tag{67}$$

The **leading components of the electromagnetic field** are

$$\underline{\alpha}_A(F) \quad = \quad F(E_A, \underline{L}) \tag{68}$$

$$\underline{\alpha}_A(F) \quad = \quad \frac{A_A(F)}{r}\ +\ o\ (r^{-2})\ . \tag{69}$$

Denote the kth Cartesian coordinate of the mass m_A for $A = 1, 2$ by ${x^k}_{(A)}$.

Then the **Jacobi equation** becomes

$$\frac{d^2\ {x^k}_{(A)}}{d\ t^2} = -\frac{1}{4}\,r^{-1}\ A_{AB}\ {x^l}_{(B)} - \frac{1}{8} r^{-2} \mid A_F \mid^2 {x^l}_{(B)} + O\ (r^{-2})$$

that is

$$\frac{d^2\ {x^3}_{(C)}}{d\ t^2} = 0$$

$$\frac{d^2\ {x^A}_{(C)}}{d\ t^2} = -\frac{1}{4}\,r^{-1}\ A_{AB}\ {x^B}_{(D)} - \frac{1}{8} r^{-2} \mid A_F \mid^2 {x^B}_{(D)} + O(r^{-2})$$

From the Jacobi equation $\Rightarrow$ see that the electromagnetic field enters on the right hand side at order (r^{-2}) only.

$\Rightarrow$ **The electromagnetic field does not contribute at leading order to the deviation measured by the Jacobi equation.**

$\Rightarrow$ At leading order, results for the Einstein vacuum case apply.

Christodoulou derived:

$$\frac{d^2\ x^k{}_{(A)}}{d\ t^2} = -\ \frac{1}{4}\ r^{-1}\ A_{AB}\ x^l{}_{(B)} + O\ (r^{-2})$$

Obtain: In the vertical direction there is no acceleration to leading order (r^{-1}).

Initially m_1 and m_2 are at rest at equal distance d_0 and at right angles from m_0. This implies the following initial conditions, as $t \to -\infty$:

$$x^3{}_{(A)} = 0\ ,\ \dot{x}^3{}_{(A)} = 0\ ,\ x^B{}_{(A)} = d_0 \delta^B_A\ ,\ \dot{x}^B{}_{(A)} = 0.$$

The right hand side being very small, one can substitute the initial values on the right hand side. Then the motion is confined to the horizontal plane. One has to leading order:

$$\ddot{x}^A{}_{(B)} = -\ \frac{1}{4}\ r^{-1}\ d_0\ A_{AB} \quad . \tag{70}$$

One obtains

$$\dot{x}^A{}_{(B)}\ (t) = -\ \frac{1}{4}\ d_0\ r^{-1} \int_{-\infty}^{t} A_{AB}\ (u)\ du\ . \tag{71}$$

In view of equation
$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}\ A$ and $\lim_{|u| \to \infty} \Xi = 0$

we obtain

$$- \int_{-\infty}^{t} A_{AB}\ (u)\ du \ = \ \Xi\ (t) \qquad (72)$$

and

$$\dot{x}^{A}{}_{(B)}\ (t) \ = \ \frac{d_0}{r}\ \Xi_{AB}\ (t)\ . \qquad (73)$$

As $\Xi \to 0$ for $u \to \infty$, the test masses return to rest after the passage of the gravitational wave.

Taking into account
$\frac{\partial \Sigma}{\partial u} = -\Xi$,

and integrating again:

$$x^{A}{}_{(B)}\ (t) \ = \ -\ (\frac{d_0}{r})\ (\Sigma_{AB}\ (t) \ - \ \Sigma^{-})\ . \qquad (74)$$

The limit $t \rightarrow \infty$ is taken and it follows that the test masses experience **permanent displacements**.

Thus
$\Sigma^{+} - \Sigma^{-}$
is equivalent to an overall displacement of the test masses:

$$\triangle \; x^{A}{}_{(B)} \; = \; - \; (\frac{d_0}{r}) \; (\Sigma^{+}_{AB} \; - \; \Sigma^{-}_{AB}) \; . \tag{75}$$

The right hand side of (75) includes terms from the **electromagnetic field** at highest order
as given in our theorem 3.

Even though the form of (75) is as in the EV case investigated by Christodoulou, the **nonlinear contribution from the electromagnetic field** is present in $\Sigma^{+}_{AB} - \Sigma^{-}_{AB}$.

Recall also: total energy $\frac{\mathbf{F}}{4\pi}$ radiated to infinity in a given direction per unit solid angle:

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(\mid \Xi \mid^2 + \frac{1}{2} \mid A_F \mid^2 \right) du \quad .$$

Open Questions

• What happens, when inserting other fields on the right hand side of Einstein equations?

• What are the patterns in the gravitational radiation for typical sources such as mergers of binary black holes, binary neutron stars?

⇒ Leads to another challenge of GR: **2-body problem**.