Gluing of TT-tensors

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The constraints equations

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$$(C) \quad \begin{cases} {}^{g}\nabla^i K_{ij} - {}^{g}\nabla_j(tr_g\, K) = 0 \\ R(g) - |K|_g^2 + (tr_g\, K)^2 = 2\Lambda \end{cases} .$$

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- Reciprocally if (M, g, K) solves (C) then there exist $(\mathcal{M}^{n+1}, \mathcal{G})$ solution of (E) by Choquet-Bruhat (1952) and Choquet-Bruhat Geroch (1969).

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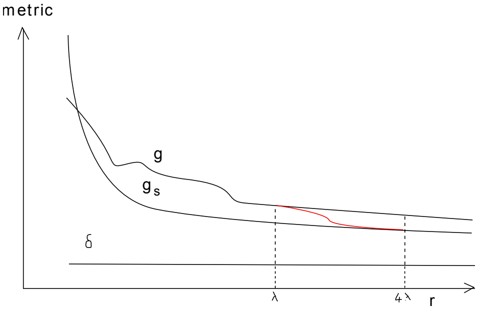
- Gluing of an AF metric g, $R(g) = 0$ with a Schwarzschild (slice) metric g_S on an annulus $B(4\lambda)\backslash\overline{B}(\lambda)$, to a $R = 0$ metric.



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- Compactly supported solutions of some PDE and gluing, D 2010.

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- Note : the linearized scalar curvature operator is an under determined elliptic operator.

$$Lh := DR(g)h = \text{div div } h + \Delta \text{ Tr } h - \langle \text{Ric}(g), h \rangle$$

$$L^* u = \text{Hess } u + \Delta u \, g - u \, \text{Ric}(g)$$

Remark : The kernel of L^* on $(\mathbb{R}^n, \delta)$ consists of affine functions so is $n+1$ dimensional.

Here some examples

- div $: T^{p+1} \longrightarrow T^p$. Kernel : divergence free tensors. On $\mathbb{R}^3 \backslash \{0\}$, with $p = 0$ a model in the kernel is a point charge field :
$$E_Q = \frac{1}{4\pi} \frac{Q}{r^2} \frac{\vec{r}}{r} \,, \quad \vec{r} = (x, y, z) \,, \quad r = |\vec{r}|.$$

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$$E = \frac{m}{|x - c|^{n-2}} \delta.$$
- $\mathrm{div}^m : T^{p+m} \longrightarrow T^p$.

The method

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with U vanishing at any order at $\partial\Omega$, so U can be extended by zero across $\partial\Omega$.

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- Look for U of the form

$$U = \zeta P^* u,$$

with u allowed to blow up a the boundary, but ζ vanishes more !

Isomorphism theorem

Let P be an under determined elliptic operator of order m.
Define the weighted spaces on Ω :

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- Then

$$\Pi_{K^\perp}\ e^{2/x}\ P\ x^{4m} e^{-2/x}\ P^* : K^\perp \cap H^{k+2m} \longrightarrow H^k \cap K^\perp$$

is an isomorphism.

Proof

- By contradiction there exist C' s. t. for $u \in K^\perp \cap H^m$

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- Injectivity :

$$0 = \langle e^{2/x} \ P \ x^{4m} e^{-2/x} \ P^* u, u \rangle_{H^0} = \langle x^{2m} P^* u, x^{2m} P^* u \rangle_{H^0} \ .$$

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- Surjectivity : minimize on $u \in K^\perp \cap H^m$ the coercive functionnal

$$\mathcal{F}(u) := \int_M (\frac{1}{2} | x^{2m} P^* u |_g^2 - \langle u, f \rangle_g) e^{-2/x} d\mu_g \ .$$

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- Integrations by part shows that projection onto the kernel is trivial using (KRC).

Example of gluing

Let $\Omega_1 \subset \Omega_2 \subset \Omega_3$ with $\Omega = \Omega_2 \backslash \overline{\Omega}_1$ relatively compact. Let V and W (defined in Ω_3) in the kernel of P. One wants to glue V and W on Ω. Let χ equal 1 near $\partial\Omega_1$ and 0 near $\partial\Omega_2$. Let

$$T := \chi V + (1 - \chi) W$$

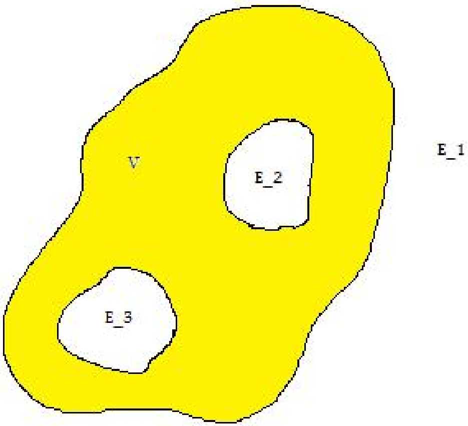
We then solve

$$PU = -PT =: f.$$

The glued solution is $T + U$. The necessary condition that f is orthogonal to K corresponds to the fact that V and W induce the same "flux" on, say, $\partial\Omega_2$.

Note : If the flux is zero, one can take V or W to be zero (this can be used for quotients or connected sum).

Gluing with some models on $\mathbb{R}^n$



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- $\widetilde{K}$ is TT : $\widetilde{\nabla}^i \widetilde{K}_{ij} = 0$, $tr_{\widetilde{g}} \widetilde{K} = 0$.
- u solve the Lichnerowicz equation :

$$\frac{4(n-1)}{n-2} \Delta_{\widetilde{g}} u + R(\widetilde{g}) u - |\widetilde{K}|^2_{\widetilde{g}} u^{-q-3} + [n(n-1)\tau^2 - 2\Lambda] u^{q+1} = 0.$$

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- Take $B(r)$ a ball and h a smooth TT tensor on $B(r)$. The flux of h on $\partial B(\epsilon r)$ $(0 < \epsilon < 1)$ is zero then one can glue h with zero near ∂B to define a new TT-tensors with compact support in B.

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- Conclusion : On any open set, the set of smooth compactly supported TT-tensors is infinite dimensional.