The Global Stability of the Minkowski Spacetime Solution to the Einstein-Nonlinear Electromagnetic System in Wave Coordinates

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March 28, 2011

The “Electromagnetic Divergence Problem”

- Electric field surrounding an electron: $\mathbf{E}_{(Maxwell)} = -\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}$
- Electrostatic energy $= \int_{\mathbb{R}^3} \left|\mathbf{E}_{(Maxwell)}\right|^2 dVol$ $= 4\pi \int_{r=0}^{\infty} \frac{1}{r^2} dr = \infty$
- Lorentz force: $\mathbf{F}_{(Lorentz)} = q\mathbf{E}_{(Maxwell)} =$ Undefined (at the electron’s location)
- Analogous difficulties appear in QED

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Towards a possible resolution

- Idea (Born, Infeld, Bialynicki-Birula, Carley, Kiessling, Tahvildar-Zadeh, $\cdots$): nonlinear field equations
- Example: the Born-Infeld model ($\implies$ finite energy electrons)
- Lorentz force remains undefined; Kiessling $\to$ new law of motion for point charges
- All physically reasonable models $\sim$ Maxwell model in the weak-field limit

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The $1+3$ dimensional Einstein-nonlinear electromagnetic equations

$$
\boxed{
\begin{aligned}
\mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= T_{\mu\nu}, \\
(d\mathcal{F})_{\lambda\mu\nu} &= 0, \\
(d\mathcal{M})_{\lambda\mu\nu} &= 0.
\end{aligned}
}
$$

- $g_{\mu\nu}$ is the spacetime metric; $\mathrm{Ric}_{\mu\nu} = \mathrm{Ric}_{\mu\nu}(g, \partial g, \partial^2 g)$ is the Ricci tensor; $R \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}\mathrm{Ric}_{\kappa\lambda}$ is the scalar curvature
- $T_{\mu\nu}$ is the energy-momentum tensor
- $\mathcal{F}_{\mu\nu}$ is the Faraday tensor
- $\mathcal{M}_{\mu\nu}$ is the Maxwell tensor

Previous mathematical results in $1 + 3$ dimensions

- Fritz John: blow-up for nonlinear wave equations in three spatial dimensions (1981)
- Klainerman: global existence for nonlinear wave equations and the null condition (1984)
- Friedrich: “stability of the Minkowski spacetime solution” to the Einstein-vacuum equations for restricted data using the conformal method (1986)
- Christodoulou-Klainerman: “ ” using an invariant framework (1993)
- Nina Zipser: “ ” to the Einstein-Maxwell system (2000)
- Lindblad-Rodnianski: “ ” to the Einstein-scalar field system in wave coordinates (2005, 2010)
- Julian Loizelet: “ ” to the Einstein-scalar field-Maxwell system in Lorenz-gauge + wave coordinates (2008)
- JS: “ ” to the Einstein-nonlinear electromagnetic system in wave coordinates (2010)

The initial data on $\Sigma_0 = \mathbb{R}^3$

$\underline{\mathring{g}}$ = Riemannian metric
$\mathring{K}$ = symmetric type $\binom{0}{2}$ tensor
$\mathring{\mathfrak{D}}$ = electric displacement
$\mathring{\mathfrak{B}}$ = magnetic induction

Constraint equations ($j = 1, 2, 3$)

$$\begin{aligned}
\underline{\mathring{R}} - \mathring{K}_{ab}\mathring{K}^{ab} + \left[(\underline{\mathring{g}}^{-1})^{ab}\mathring{K}_{ab}\right]^2 &= 2T(\hat{N}, \hat{N})|_{\Sigma_0}, \\
(\underline{\mathring{g}}^{-1})^{ab}\underline{\mathring{\mathscr{D}}}_a\mathring{K}_{bj} - (\underline{\mathring{g}}^{-1})^{ab}\underline{\mathring{\mathscr{D}}}_j\mathring{K}_{ab} &= T(\hat{N}, \frac{\partial}{\partial x^j})|_{\Sigma_0}, \\
(\underline{\mathring{g}}^{-1})^{ab}\underline{\mathring{\mathscr{D}}}_a\mathring{\mathfrak{D}}_b &= 0, \\
(\underline{\mathring{g}}^{-1})^{ab}\underline{\mathring{\mathscr{D}}}_a\mathring{\mathfrak{B}}_b &= 0
\end{aligned}$$

Positive mass theorem of Schoen-Yau, Witten

Decay assumptions on the data ($\kappa > 0$, $M > 0$)

- $\underline{\mathring{g}}_{jk} = \delta_{jk} + \underline{\mathring{h}}_{jk}^{(0)} + \underline{\mathring{h}}_{jk}^{(1)}$,
- $\underline{\mathring{h}}_{jk}^{(0)} = \chi(r)\frac{2M}{r}\delta_{jk}$,
- $\underline{\mathring{h}}_{jk}^{(1)} = o(r^{-1-\kappa})$, as $r \to \infty$,
- $\mathring{K}_{jk} = o(r^{-2-\kappa})$, as $r \to \infty$,
- $\mathring{\mathfrak{D}}_j = o(r^{-2-\kappa})$, as $r \to \infty$,
- $\mathring{\mathfrak{B}}_j = o(r^{-2-\kappa})$, as $r \to \infty$

Splitting the spacetime metric

- $g_{\mu\nu} = \overbrace{m_{\mu\nu}}^{\text{Minkowski metric}} + h_{\mu\nu}$
- $h_{\mu\nu} = \underbrace{h^{(0)}_{\mu\nu}}_{\text{Schwarzschild tail}} + \underbrace{h^{(1)}_{\mu\nu}}_{\text{remainder piece}}$
- $h^{(0)}_{\mu\nu} = \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu}$

$$\chi \in C^\infty, \qquad \chi \equiv 1 \text{ for } z \geq 3/4, \qquad \chi \equiv 0 \text{ for } z \leq 1/2.$$

Note: $\chi'(\frac{r}{t})$ is supported in the interior region $\{[1/2 \leq r/t \leq 3/4]\}$

Assumptions on the electromagnetic Lagrangian

$$\boxed{{}^{\star}\mathscr{L} = \underbrace{-\frac{1}{2}\digamma_{(1)}}_{{}^{\star}\mathscr{L}_{(Maxwell)}} + O(|(\digamma_{(1)}, \digamma_{(2)})|^2)} \qquad \text{(A1)}$$

- $\digamma_{(1)} \stackrel{\text{def}}{=} \frac{1}{2}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu} \simeq$ "$|\mathbf{B}|^2 - |\mathbf{E}|^2$,"
- $\digamma_{(2)} \stackrel{\text{def}}{=} \frac{1}{4}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathcal{F}_{\kappa\lambda}{}^{\star}\mathcal{F}_{\mu\nu} = \frac{1}{8}\epsilon^{\#\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu} \simeq$ "$\mathbf{E}\cdot\mathbf{B}$"

$$\boxed{T_{\kappa\lambda}X^{\kappa}Y^{\lambda} \geq 0 \qquad \text{(Dominant energy condition)}} \qquad \text{(A2)}$$

whenever

- X, Y are both timelike (i.e., $g_{\kappa\lambda}X^{\kappa}X^{\lambda} < 0$, $g_{\kappa\lambda}Y^{\kappa}Y^{\lambda} < 0$)
- X, Y are future-directed

Example: the Born-Infeld Lagrangian

$$
\begin{aligned}
{}^{\star}\mathscr{L}_{(BI)} &= 1 - \sqrt{1 + \lightning_{(1)} - \lightning_{(2)}^2} \\
&= -\frac{1}{2}\lightning_{(1)} + O(|(\lightning_{(1)}, \lightning_{(2)})|^2)
\end{aligned}
$$

The Maxwell tensor and the energy-momentum tensor

- $$\mathcal{M}_{\mu\nu} = 2\frac{\partial^\star\mathscr{L}}{\partial\lightning_{(1)}}{}^\star\mathcal{F}_{\mu\nu} - \frac{\partial^\star\mathscr{L}}{\partial\lightning_{(2)}}\mathcal{F}_{\mu\nu}$$
$$= \underbrace{{}^*\mathcal{F}_{\mu\nu}}_{\text{Minkowskian Hodge dual}} + O(|h||\mathcal{F}|) + O(|\mathcal{F}|^3; h),$$

- $$T_{\mu\nu} = -2\frac{\partial^\star\mathscr{L}}{\partial\lightning_{(1)}}(g^{-1})^{\kappa\lambda}\mathcal{F}_{\mu\kappa}\mathcal{F}_{\nu\lambda} - \lightning_{(2)}\frac{\partial^\star\mathscr{L}}{\partial\lightning_{(2)}}g_{\mu\nu} + g_{\mu\nu}{}^\star\mathscr{L}$$
$$= \underbrace{(m^{-1})^{\kappa\lambda}\mathcal{F}_{\mu\kappa}\mathcal{F}_{\nu\lambda} - \frac{1}{4}m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\zeta}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\eta\zeta}}_{\text{Minkowskian/linear energy-momentum tensor}}$$
$$+ O(|h||\mathcal{F}|^2) + O(|\mathcal{F}|^3; h)$$

The wave coordinate gauge

Wave coordinates (à la Choquet-Bruhat & Lindblad-Rodnianski):

- $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu} = 0$

$$\implies \mathrm{Ric}_{\mu\nu} = -\frac{1}{2}\widetilde{\Box}_g g_{\mu\nu} + \mathcal{N}_{\mu\nu}(g, \nabla g)$$

- $\widetilde{\Box}_g = (g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$
- ∇ = Minkowski connection

The reduced equations

$$\widetilde{\Box}_g h^{(1)}_{\mu\nu} = \mathfrak{H}_{\mu\nu} - \widetilde{\Box}_g h^{(0)}_{\mu\nu},$$
$$\nabla_\lambda \mathcal{F}_{\mu\nu} + \nabla_\mu \mathcal{F}_{\nu\lambda} + \nabla_\nu \mathcal{F}_{\lambda\mu} = 0,$$
$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathcal{F}_{\kappa\lambda} = \mathfrak{F}^\nu.$$

- $\mathfrak{H}_{\mu\nu} = \mathscr{P}(\nabla_\mu h, \nabla_\nu h) + \mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h) + \mathscr{Q}^{(2;h)}_{\mu\nu}(\mathcal{F}, \mathcal{F}) + \mathfrak{H}^{\triangle}_{\mu\nu},$
- $\mathfrak{F}^\nu = \mathscr{Q}^\nu_{(2;\mathcal{F})}(\nabla h, \mathcal{F}) + \mathfrak{F}^\nu_\triangle,$
- $N^{\#\mu\nu\kappa\lambda} = -\frac{1}{2} \frac{\partial^{2\star}\mathscr{L}}{\partial \mathcal{F}_{\mu\nu} \partial \mathcal{F}_{\kappa\lambda}} + \frac{1}{2} \frac{\partial^\star \mathscr{L}}{\partial ↯_{(2)}} \epsilon^{\#\mu\nu\kappa\lambda}$

$$= \frac{1}{2}\Big\{(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa} - 2h^{\mu\kappa}(m^{-1})^{\nu\lambda} - 2(m^{-1})^{\mu\kappa} h^{\nu\lambda}\Big\} + N^{\#\mu\nu\kappa\lambda}_\triangle.$$

The energy of the reduced solution

- $\mathcal{E}^2_{\ell;\gamma;\mu}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \sum_{|I|\leq\ell} \int_{\Sigma_\tau} \left\{ |\nabla\nabla^I_{\mathcal{Z}} h^{(1)}|^2 + |\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|^2 \right\} w(q)\, d^3x,$

$q \stackrel{\text{def}}{=} |x| - t,$

- $w = w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & \text{if } q > 0, \\ 1 + (1+|q|)^{-2\mu}, & \text{if } q < 0, \end{cases},$

$0 < \gamma < 1/2,\ 0 < \mu < 1/2,$

- $\underbrace{\mathcal{Z} \stackrel{\text{def}}{=} \left\{ \frac{\partial}{\partial x^\mu}, x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, x^\kappa \frac{\partial}{\partial x^\kappa} \right\}_{0\leq\mu<\nu\leq 3}}_{\text{11 Minkowskian conformal Killing fields}}$

Continuation principle

Lemma

If the solution blows up at time $T_{max} > 0$, then one of the following blow-up scenarios must occur:

- *The hyperbolicity of the equations breaks down as $t \uparrow T_{max}$*
- $\lim_{t \uparrow T_{max}} \mathcal{E}_{\ell;\gamma;\mu}(t) = \infty$

Moral conclusion: You can show global existence by proving that the energy $\mathcal{E}_{\ell;\gamma;\mu}(t)$ never blows up.

Main results

Theorem

Let $\ell \geq 8$ and assume the wave coordinate condition. There exists a constant $\varepsilon_\ell > 0$ such that if $\mathcal{E}_{\ell;\gamma;\mu}(0) + M \leq \varepsilon \leq \varepsilon_\ell$, then the reduced solution exists for $(t, x) \in (-\infty, \infty) \times \mathbb{R}^3$ and the resulting spacetime is geodesically complete. Furthermore, there exist constants $c_\ell > 0$, $\widetilde{c}_\ell > 0$ such that

$$\mathcal{E}_{\ell;\gamma;\mu}(t) \leq c_\ell \varepsilon (1 + |t|)^{\widetilde{c}_\ell \varepsilon}$$

for $t \in (-\infty, \infty)$.

The bootstrap argument

Goal: assume that there exist constants $0 < \varepsilon$ and $0 < \delta < 1/4$ such that for $t \in [0, T)$

- $\mathcal{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^{\delta}, \qquad 0 \leq k \leq \ell,$

and prove that for $0 \leq k \leq \ell$ and $t \in [0, T)$:

$$\mathcal{E}^2_{k;\gamma;\mu}(t) \lesssim \mathcal{E}^2_{\ell;\gamma;\mu}(0) + M^2 + \varepsilon^3 + \varepsilon \int_0^t (1+\tau)^{-1} \mathcal{E}^2_{k;\gamma;\mu}(\tau)\, d\tau + \varepsilon \int_0^t (1+\tau)^{-1+C\varepsilon} \mathcal{E}^2_{k-1;\gamma;\mu}(\tau)\, d\tau$$

Conclusion (Gronwall's lemma):
$\mathcal{E}^2_{\ell;\gamma;\mu}(t) \leq c_\ell \{\mathcal{E}^2_{\ell;\gamma;\mu}(0) + M^2 + \varepsilon^3\}(1+t)^{\widetilde{c}_\ell \varepsilon}$

Fundamental difficulty

The behavior of $\mathcal{E}^2_{k;\gamma;\mu}(t)$ is not directly accessible. We only know how to study $\mathcal{E}^2_{k;\gamma;\mu}(t)$ through the use of auxiliary quantities, e.g. energy currents.

The EOV and the canonical stress

The equations of variation (EOV)

$$\nabla_\lambda \dot{\mathcal{F}}_{\mu\nu} + \nabla_\mu \dot{\mathcal{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathcal{F}}_{\lambda\mu} = \overbrace{\dot{\mathfrak{F}}_{\lambda\mu\nu}}^{0 \text{ for us}},$$

$$N^{\#\mu\nu\kappa\lambda}[h, \mathcal{F}]\nabla_\mu \dot{\mathcal{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu$$

- To estimate $\mathcal{F}$, we can use $(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa T_{\mu\lambda} = 0$.
- But how do we obtain useful differential identities for $\dot{\mathcal{F}}$?
- Answer: the canonical stress tensor (Christodoulou)

$$\dot{Q}^\mu_{\ \nu} \overset{\text{def}}{=} N^{\#\mu\zeta\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4}\delta^\mu_\nu N^{\#\zeta\eta\kappa\lambda}\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}_{\kappa\lambda}$$

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Expansion of the canonical stress

$$\dot{Q}^{\mu}{}_{\nu} = \overbrace{\dot{\mathcal{F}}^{\mu\zeta}\dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4}\delta^{\mu}_{\nu}\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}^{\zeta\eta}}^{\text{linear energy-momentum tensor in flat spacetime}}$$

$$\overbrace{-h^{\mu\kappa}\dot{\mathcal{F}}_{\kappa\zeta}\dot{\mathcal{F}}_{\nu}{}^{\zeta} - h^{\kappa\lambda}\dot{\mathcal{F}}^{\mu}{}_{\kappa}\dot{\mathcal{F}}_{\nu\lambda} + \frac{1}{2}\delta^{\mu}_{\nu}h^{\kappa\lambda}\dot{\mathcal{F}}_{\kappa\eta}\dot{\mathcal{F}}_{\lambda}{}^{\eta}}^{\text{corrections to linear theory arising from } h}$$

$$\underbrace{+ N_{\triangle}^{\#\mu\zeta\kappa\lambda}\dot{\mathcal{F}}_{\kappa\lambda}\dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4}\delta^{\mu}_{\nu}N_{\triangle}^{\#\zeta\eta\kappa\lambda}\dot{\mathcal{F}}_{\zeta\eta}\dot{\mathcal{F}}_{\kappa\lambda}.}_{\text{quartic error terms}}$$

An expression for $\nabla_\mu \dot{Q}^\mu_{\ \nu}$

Lemma

If $\dot{\mathcal{F}}_{\mu\nu}$ verifies the equations of variation with inhomogeneous terms $\dot{\mathfrak{F}}_{\nu\kappa\lambda}$ and $\dot{\mathfrak{F}}^\eta$, then

$$\begin{aligned}
\nabla_\mu \dot{Q}^\mu_{\ \nu} = & -\frac{1}{2} N^{\#\zeta\eta\kappa\lambda} \dot{\mathcal{F}}_{\zeta\eta} \dot{\mathfrak{F}}_{\nu\kappa\lambda} + \dot{\mathcal{F}}_{\nu\eta} \dot{\mathfrak{F}}^\eta \\
& - (\nabla_\mu h^{\mu\kappa}) \dot{\mathcal{F}}_{\kappa\zeta} \dot{\mathcal{F}}_\nu^{\ \zeta} - (\nabla_\mu h^{\kappa\lambda}) \dot{\mathcal{F}}^\mu_{\ \kappa} \dot{\mathcal{F}}_{\nu\lambda} + \frac{1}{2} (\nabla_\nu h^{\kappa\lambda}) \dot{\mathcal{F}}_{\kappa\eta} \dot{\mathcal{F}}_\lambda^{\ \eta} \\
& + (\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}) \dot{\mathcal{F}}_{\kappa\lambda} \dot{\mathcal{F}}_{\nu\zeta} - \frac{1}{4} (\nabla_\nu N_\triangle^{\#\zeta\eta\kappa\lambda}) \dot{\mathcal{F}}_{\zeta\eta} \dot{\mathcal{F}}_{\kappa\lambda}
\end{aligned}$$

A current for $\dot{\mathcal{F}}$

Set $X^\nu = w(q)\delta_0^\nu$ and define

$$\boxed{\dot{J}^\mu_{(h,\mathcal{F})}[\dot{\mathcal{F}}] \overset{\text{def}}{=} -\dot{Q}^\mu{}_\nu X^\nu = -w(q)\dot{Q}^\mu{}_0}$$

$$w(q) \overset{\text{def}}{=} \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & \text{if } q > 0, \\ 1 + (1+|q|)^{-2\mu}, & \text{if } q < 0, \end{cases}$$

$0 < \gamma < 1/2, \quad 0 < \mu < 1/2.$

Lemma

$$\dot{J}^0_{(h,\mathcal{F})}[\dot{\mathcal{F}}] \approx |\dot{\mathcal{F}}|^2 w(q).$$

The divergence theorem

Lemma

Let $\dot{\mathcal{F}}_{\mu\nu}$ be a solution to the electromagnetic equations of variation with inhomogeneous terms $\dot{\mathfrak{F}}^{\nu}$. Assume that "$h \lesssim \varepsilon$" is suitably small. Then

$$\int_{\Sigma_t} |\dot{\mathcal{F}}|^2 w(q)\, d^3x \;+\; \int_0^t \int_{\Sigma_\tau} \left(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2\right) w'(q)\, d^3x\, d\tau$$
$$\lesssim \int_{\Sigma_0} |\dot{\mathcal{F}}|^2 w(q)\, d^3x \;+\; \varepsilon \int_0^t \int_{\Sigma_\tau} \frac{|\dot{\mathcal{F}}|^2}{1+\tau} w(q)\, d^3x\, d\tau$$
$$+ \int_0^t \int_{\Sigma_\tau} |\dot{\mathcal{F}}_{0\kappa} \dot{\mathfrak{F}}^{\kappa}|\, w(q)\, d^3x\, d\tau$$

Weighted Klainerman-Sobolev inequality (with $q \overset{\text{def}}{=} |x| - t$)

Lemma

$$(1 + t + |x|)[(1 + |q|)w(q)]^{1/2}|\phi(t,x)| \lesssim \sum_{|I| \le 2} \left\| w^{1/2} \nabla_{\mathcal{Z}}^{I} \phi(t,\cdot) \right\|_{L^2},$$

$$(1 + t + |x|)[(1 + |q|)w(q)]^{1/2}|\mathcal{F}(t,x)| \lesssim \sum_{|I| \le 2} \left\| w^{1/2} \mathcal{L}_{\mathcal{Z}}^{I} \mathcal{F}(t,\cdot) \right\|_{L^2},$$

$$(1 + t + |x|)^2[(1 + |q|)w(q)]^{1/2}|\overline{\nabla}\phi(t,x)| \lesssim \sum_{|I| \le 3} \left\| w^{1/2} \nabla_{\mathcal{Z}}^{I} \phi(t,\cdot) \right\|_{L^2}$$

Not sufficient to close the estimates; we need to upgrade these inequalities using the **special null structure** of the equations.

Weighted Klainerman-Sobolev inequality (with $q \overset{\text{def}}{=} |x| - t$)

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Heuristics

Moral reason for stability: the “worst possible” quadratic terms are absent from the equations

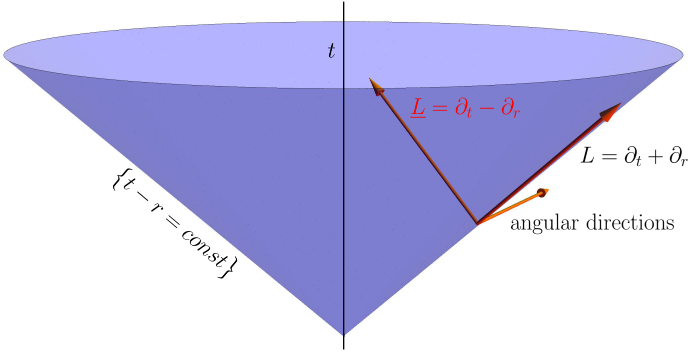
weak null condition

Important surfaces in Minkowski space

$t \in \mathbb{R},\ x \in \mathbb{R}^3$

- $C_s^- \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}$ are the ingoing null cones
- $C_q^+ \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\}$ are the outgoing null cones
- $\Sigma_t \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t\}$ are the constant time slices
- $S_{r,t} \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t, |y| = r\}$ are the Euclidean spheres

The Minkowskian null frame



Minkowskian null frame and null coordinates

Null frame: $\mathcal{N} \overset{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$, $\mathcal{T} \overset{\text{def}}{=} \{L, e_1, e_2\}$, $\mathcal{L} \overset{\text{def}}{=} \{L\}$

- $\underline{L} \overset{\text{def}}{=} \partial_t - \partial_r$ is tangent to the ingoing cones
- $L \overset{\text{def}}{=} \partial_t + \partial_r$ is tangent to the outgoing cones
- e_1, e_2 are orthonormal, & tangent to the spheres

Null coordinates (useful for expressing decay rates)

- $q = r - t$ (constant on outgoing cones)
- $s = r + t$ (constant on ingoing cones)

Null decomposition of $\mathcal{F}$

With $\not{m}_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + \frac{1}{2}(L_\mu \underline{L}_\nu + \underline{L}_\mu L_\nu)$, $\not{\upsilon}_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}[\mu\nu\kappa\lambda]\underline{L}^\kappa L^\lambda$,

The 6 components of $\mathcal{F}$

$$\begin{cases} \underline{\alpha}_\mu & \stackrel{\text{def}}{=} \not{m}_\mu^{\ \nu} \mathcal{F}_{\nu\lambda} \underline{L}^\lambda & = \text{BAD}, \\ \alpha_\mu & \stackrel{\text{def}}{=} \not{m}_\mu^{\ \nu} \mathcal{F}_{\nu\lambda} L^\lambda & = \text{good}, \\ \rho & \stackrel{\text{def}}{=} \frac{1}{2} \mathcal{F}_{\kappa\lambda} \underline{L}^\kappa L^\lambda & = \text{good}, \\ \sigma & \stackrel{\text{def}}{=} \frac{1}{2} \not{\upsilon}^{\kappa\lambda} \mathcal{F}_{\kappa\lambda} & = \text{good}. \end{cases}$$

- $\not{m}_\mu^{\ \nu}$ projects $m-$orthogonally onto the $S_{r,t}$
- $\overline{\pi}_\mu^{\ \nu}$ projects $m-$orthogonally onto the C_q^+
- $\overline{\nabla}_\mu \stackrel{\text{def}}{=} \overline{\pi}_\mu^{\ \kappa} \nabla_\kappa$ are the good derivatives

Upgraded decay estimates via wave coordinates

Null decompose the wave coordinate condition $\nabla_\mu\big(\sqrt{|\det g|}(g^{-1})^{\mu\nu}\big) = 0$ to obtain (Lindblad-Rodnianski)

$$\boxed{|\nabla h|_{\mathcal{LT}} \lesssim |\overline{\nabla} h| \ + \ |h||\nabla h|}$$

Conclusion: $|h|_{\mathcal{LT}}$ decays *better* than "expected"

Upgraded decay estimates for $\mathcal{F}$

With $\underline{\alpha}_\mu \stackrel{\text{def}}{=} \not{m}_\mu^{\ \nu} \mathcal{F}_{\nu\lambda} \underline{L}^\lambda$, $\Lambda \stackrel{\text{def}}{=} L + \frac{1}{4} h_{LL} \underline{L}$,
Null decomposing the electromagnetic equations $\Longrightarrow$

$$
\begin{aligned}
r^{-1} |\nabla_\Lambda (r\underline{\alpha})| \lesssim r^{-1} |h|_{\mathcal{L}\mathcal{L}} |\underline{\alpha}| \ &+ \sum_{|I| \leq 1} r^{-1} \big(|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}} \big) \\
&+ \sum_{|I_1| + |I_2| \leq 1} r^{-1} |\nabla_{\mathcal{Z}}^{I_1} h| |\mathcal{L}_{\mathcal{Z}}^{I_2} \mathcal{F}| \\
&+ \sum_{|I| \leq 1} (1 + |q|)^{-1} |h| \big(|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}} \big) \\
&+ \text{cubic error terms.}
\end{aligned}
$$

- Conclusion: $r\underline{\alpha}$ satisfies an “ODE” with favorable sources
- Integration yields $|\underline{\alpha}| \lesssim \epsilon (1 + t)^{-1}$
- Klainerman-Sobolev yields only $|\underline{\alpha}| \lesssim \epsilon (1 + t)^{-1+\delta}$

Post-upgraded-decay-estimate integral inequalities

$$\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\Big|(\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{0\nu})\Big\{N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\kappa\lambda}-\hat{\mathcal{L}}_{\mathcal{Z}}^I\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathcal{F}_{\kappa\lambda}\big)\Big\}\Big|w(q)\,d^3x\,d\tau$$

$$\lesssim \varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2w(q)\,d^3x\,d\tau$$

$$+\ \varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla[\nabla_{\mathcal{Z}}^I h^{(1)}]|^2w(q)\,d^3x\,d\tau$$

$$+\ \varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\Big\{|\overline{\nabla}[\nabla_{\mathcal{Z}}^I h^{(1)}]|^2+|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{L}\mathcal{N}}+|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{T}\mathcal{T}}\Big\}w'(q)\,d^3x\,d\tau$$

$$+\ \underbrace{\varepsilon\sum_{|J'|\leq k-2}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1+C\varepsilon}|\nabla[\nabla_{\mathcal{Z}}^{J'}h^{(1)}]|^2w(q)d^3x\,d\tau}_{\text{absent if } k=1}$$

$$+\ \varepsilon\sum_{|J|\leq k-1}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1+C\varepsilon}|\mathcal{L}_{\mathcal{Z}}^J\mathcal{F}|^2w(q)d^3x\,d\tau+\varepsilon^3$$

Future directions

- The structure of regularly hyperbolic equations (with Willie Wong)
- Global stability of solutions to the Euler-Einstein system
- Formation of singularities in nonlinear electromagnetic equations

Thank you