Recent Progress on The Stability Analysis for Symmetric Hyperbolic Systems

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Outline

1. Introduction

Consider linear partial differential equations:

$$P(\partial_t, \partial_x)u = 0.$$

The characteristic equation is: $P(\lambda, i\xi) = 0.$

Let $\lambda = \lambda(i\xi)$ be the dispersion relation.

Dissipative structure: Key to the stability for $t \to \infty$.

- Dissipativity: $\mathrm{Re}\,\lambda(i\xi) \le 0$ for any ξ.
- Strict dissipativity: $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \neq 0$.

Type (I): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$

Type (II): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)^2$

Strict dissipativity

Type (I): $\operatorname{Re}\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$

$$\operatorname{Re}\lambda(i\xi) \sim -c|\xi|^2 \quad \text{for} \quad |\xi| \to 0,$$
$$\operatorname{Re}\lambda(i\xi) \sim -c \quad \text{for} \quad |\xi| \to \infty.$$

General framework:

- Symmetric hyperbolic-parabolic systems
- Symmetric hyperbolic systems
  - ◦ T. Umeda, S.K & Y. Shizuta (1984): Condition (K)
  - ◦ Y. Shizuta & S.K (1985): (SK) stability condition
  - ◦ K. Beauchard & E. Zuazua (2010): Kalman rank condition
- Symmetric hyperbolic-elliptic systems
  - ◦ S.K, Y. Nikkuni & S. Nishibata (1998)

Strict dissipativity

Type (II): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)^2$

$$\mathrm{Re}\,\lambda(i\xi) \sim -c|\xi|^2 \quad \text{for} \quad |\xi| \to 0,$$
$$\mathrm{Re}\,\lambda(i\xi) \sim -c|\xi|^{-2} \quad \text{for} \quad |\xi| \to \infty.$$

For Type (II), $\lambda(i\xi)$ may approach the imaginary axis $\mathrm{Re}\,\lambda = 0$ for $|\xi| \to \infty$.

Decay property of regularity-loss type:

- Dissipative Timoshenko system
  - ◦ J.E.M. Rivera & R. Racke (2003)
  - ◦ K. Ide, K. Haramoto & S.K (2008), ◦ K. Ide & S.K (2008)
- Euler-Maxwell system
  - ◦ Y. Ueda, S. Wang & S.K (2010 preprint)
  - ◦ R. Duan (2010 preprint)

Aim

- To survey the general theory for Type (I).
- To study the **dissipative Timoshenko system** as an example of type (II).

$$\begin{cases} w_{tt} - (w_x - \psi)_x = 0, \\ \psi_{tt} - a^2\,\psi_{xx} - (w_x - \psi) + \gamma\psi_t = 0, \end{cases}$$

  where a, $\gamma > 0$ are constants.
- To study the **Euler-Maxwell system** as an example of type (II).
- To report a new progress on the general framework for the dissipative structure of type (II).

  ◦ A joint work with Y. Ueda and R. Duan

2. Motivation of general formulation

Compressible Navier-Stokes equation: The simplest model is given by

$$\begin{cases} \rho_t + u_x = 0, \\ u_t + \rho_x = u_{xx}. \end{cases}$$

Standard energy: Multiply the first and the second equations by ρ and u, respectively, and add them. This yields

$$\frac{1}{2}(\rho^2 + u^2)_t + (\rho u - u u_x)_x + u_x^2 = 0.$$

Similarly, we have

$$\frac{1}{2}(\rho_x^2 + u_x^2)_t + (\rho_x u_x - u_x u_{xx})_x + u_{xx}^2 = 0.$$

Special technique

Special technique: The technique due to Kanal' (1968) and Matsumura & Nishida (1981) is as follows. Substitute $u_x = -\rho_t$ to the second equaion:

$$\rho_{xt} + \rho_x + u_t = 0.$$

Multiply by ρ_x:

$$\frac{1}{2}(\rho_x^2)_t + \rho_x^2 + \rho_x u_t = 0.$$

Here the last term can be rewritten as

$$\begin{aligned} \rho_x u_t &= (\rho_x u)_t - \rho_{tx} u \\ &= (\rho_x u)_t - (\rho_t u)_x + \rho_t u_x \qquad \Leftarrow \qquad \rho_t = -u_x \\ &= (\rho_x u)_t - (\rho_t u)_x - u_x^2. \end{aligned}$$

Thus we obtain

$$\frac{1}{2}(\rho_x^2 + 2\rho_x u)_t - (\rho_t u)_x + \rho_x^2 - u_x^2 = 0.$$

Simpler technique: The above computation is perfect. But it is too technical and not suitable for the general framework. A simpler computation is as follows. Multiply the first and the second equations by $-u_x$ and ρ_x, and add them. This yields

$$(\rho_x u_t - \rho_t u_x) + \rho_x^2 - \rho_x u_{xx} - u_x^2 = 0$$

with $\rho_x u_t - \rho_t u_x = (\rho_x u)_t - (\rho_t u)_x$. This computation can be expressed as

$$\begin{pmatrix} -u_x \\ \rho_x \end{pmatrix} \cdot \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} \cdot \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} = \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \rho_t \\ u_t \end{pmatrix},$$

which suggests a formulation in terms of a skew-symmetric matrix.

3. General theory for type (I)

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{1}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for each j,

(c) L is symmetric and nonnegative definite.

(c) L is nonnegative definite (not necessarily symmetric) such that

$$\ker(L) = \ker(L_1),$$

where L_1 is the symmetric part of L.

Condition (K)

Apply the Fourier transform:

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0, \tag{2}$$

where $A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}$.

- Dispersion relation $\lambda = \lambda(i\xi)$:

$$\det(\lambda A^0 + i|\xi| A(\omega) + L) = 0.$$

Condition (K): Umeda, S.K. & Shizuta (1984)

There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is skew-symmetric.

(ii) $(K(\omega)A(\omega))_1 + L$ is positive definite,

where X_1 is the symmetric part of X.

Decay property

Theorem 1 (Pointwise estimate) Umeda, S.K. & Shizuta (1984)

Under the condition (K), we have

$$|\hat{u}(\xi,t)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{3}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Corollary (Decay estimate) Umeda, S.K. & Shizuta (1984)

Under the condition (K), we have

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \tag{4}$$

where $k \ge 0$.

- Decay estimate of the standard type (without loss of regularity)

Lyapunov function:

$$E[\,\hat{u}\,] = \langle A^0\hat{u},\, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle,$$

where $\alpha > 0$ is a small constant. We have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0,$$

where P is the orthogonal projection onto $\ker(L)$. Therefore we have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + c\rho(\xi)E[\,\hat{u}\,] \le 0,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. This is solved as

$$E[\,\hat{u}\,](\xi,t) \le e^{-c\rho(\xi)t}E[\,\hat{u}_0\,](\xi),$$

which gives the desired pointwise estimate (3).

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \leq 0,$$

we have the energy estimate of the form

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 + \|(I-P)u(\tau)\|_{H^s}^2\,d\tau \leq C\|u_0\|_{H^s}^2,$$

where $s \geq 0$.

- Energy estimate of the standard type (without loss of regularity)

Dissipative structure

(SK) stability condition: Shizuta & S.K. (1985)

Let $\varphi \in \mathbb{R}^m$, $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$.

$$\text{If} \quad L\varphi = 0 \quad \text{and} \quad \mu A^0 \varphi + A(\omega)\varphi = 0, \quad \text{then} \quad \varphi = 0.$$

Theorem 2 (Characterization of dissipativity)
Shizuta & S.K. (1985), Beauchard & Zuazua (2010)

The following five conditions are equivalent.

(a) (SK) stability condition.

(b) Condition (K).

(c) $\mathrm{Re}\,\lambda(i\xi) \leq -c|\xi|^2/(1+|\xi|^2)$ for any $\xi \in \mathbb{R}^n$.

(d) $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \neq 0$.

(d) Kalman rank condition.

4. Dissipative Timoshenko system

Dissipative Timoshenko system:

$$\begin{cases} w_{tt} - (w_x - \psi)_x = 0, \\ \psi_{tt} - a^2\,\psi_{xx} - (w_x - \psi) + \gamma\psi_t = 0, \end{cases}$$

where $a > 0$, $\gamma > 0$ are constants. The equivalent 1st order system is

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \\ y_t - a z_x - v + \gamma y = 0, \end{cases} \tag{5}$$

where

$$u = w_t,\ v = w_x - \psi,\ y = \psi_t,\ z = a\psi_x.$$

Dissipative Timoshenko system

The system is written as

$$U_t + AU_x + LU = 0,$$

where

$$U = \begin{pmatrix} v \\ u \\ z \\ y \end{pmatrix}, \quad A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}.$$

Claim:

L is <u>not symmetric</u> but satisfies the (SK) stability condition:

$$\text{If} \quad L\varphi = 0 \quad \text{and} \quad \mu\varphi + A\varphi = 0, \quad \text{then} \quad \varphi = 0.$$

In this case, however, $\ker(L) \neq \ker(L_1)$.

Dissipative structure:

- If $a = 1$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^2/(1+\xi^2)$.
- If $a \ne 1$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^2/(1+\xi^2)^2$.

When $a \ne 1$, the asymptotic expansion of $\lambda(i\xi)$ for $|\xi| \to \infty$ is:

$$\lambda(i\xi) = \pm\, i\xi \pm \frac{\sigma}{2}(i\xi)^{-1} + \sigma^2\gamma(i\xi)^{-2} + O(|\xi|^{-3}),$$

$$\lambda(i\xi) = \pm\, ai\xi - \frac{\gamma}{2} + O(|\xi|^{-1}),$$

where $\sigma = 1/(a^2-1)$.

Decay property

Theorem 3 (Pointwise estimate) Ide, Haramoto & S.K (2008)

When $a \neq 1$, we have

$$|\hat{U}(\xi,t)| \leq Ce^{-c\eta(\xi)t}|\hat{U}_0(\xi)|, \tag{6}$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$.

Corollary (Decay estimate) Ide, Haramoto & S.K (2008)

When $a \neq 1$, we have

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l} U_0\|_{L^2}, \tag{7}$$

where k, $l \geq 0$.

- Decay estimate of the regularity-loss type

Lyapunov function

Lyapunov function: When $a \neq 1$,

$$E[\,\hat{U}\,] = |\hat{U}|^2 + \frac{\alpha_1}{1+\xi^2}\Big\{-\operatorname{Re}\big(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}}\big) + \frac{\alpha_2\xi}{1+\xi^2}\operatorname{Re}\big(i\hat{v}\bar{\hat{u}} + i\hat{y}\bar{\hat{z}}\big)\Big\},$$

where α_1, $\alpha_2 > 0$ are small constants. We have

$$\frac{\partial}{\partial t}E[\,\hat{U}\,] + cD[\,\hat{U}\,] \leq 0,$$

where

$$D[\,\hat{U}\,] = \frac{\xi^2}{(1+\xi^2)^2}\big(|\hat{u}|^2 + |\hat{z}|^2\big) + \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2.$$

Therefore we obtain

$$\frac{\partial}{\partial t}E[\,\hat{U}\,] + c\eta(\xi)E[\,\hat{U}\,] \leq 0,$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$. This yields the desired pointwise estimate (6).

Energy estimate: When $a \neq 1$, as a simple corollary of

$$\frac{\partial}{\partial t} E[\,\hat{U}\,] + cD[\,\hat{U}\,] \leq 0,$$

we have the following energy estimate:

$$\|U(t)\|_{H^s}^2 + \int_0^t \|\partial_x(u,z)(\tau)\|_{H^{s-2}}^2 + \\ + \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 \, d\tau \leq C\|U_0\|_{H^s}^2,$$

where $s \geq 0$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for the component (v, u, z).

Proof of decay estimate

Proof of Corollary: We have

$$\|\partial_x^k U(t)\|_{L^2}^2 = \int |\xi|^{2k} |\hat{U}(\xi,t)|^2 \, d\xi \le C \int |\xi|^{2k} e^{-c\eta(\xi)t} \, |\hat{U}_0(\xi)|^2 \, d\xi$$
$$= \int_{|\xi|\le 1} + \int_{|\xi|\ge 1} := I_1 + I_2.$$

Low frequency term I_1 is estimated as

$$I_1 \le C \int_{|\xi|\le 1} |\xi|^{2k} e^{-c|\xi|^2 t} \, |\hat{U}_0(\xi)|^2 \, d\xi$$
$$\le C \sup_{|\xi|\le 1} |\hat{U}_0(\xi)|^2 \int_{|\xi|\le 1} |\xi|^{2k} e^{-c|\xi|^2 t} \, d\xi$$
$$\le C(1+t)^{-1/2-k} \|U_0\|_{L^1}^2.$$

Proof of decay estimate

High frequency term I_2 can be estimated as

$$\begin{aligned} I_2 &\le C\int_{|\xi|\ge 1} |\xi|^{2k} e^{-ct/|\xi|^2}\, |\widehat{U}_0(\xi)|^2\, d\xi \\ &\le C \sup_{|\xi|\ge 1} \frac{e^{-ct/|\xi|^2}}{|\xi|^{2l}} \int_{|\xi|\ge 1} |\xi|^{2(k+l)}\, |\hat{U}_0(\xi)|^2\, d\xi \\ &\le C(1+t)^{-l} \|\partial_x^{k+l} U_0\|_{L^2}^2. \end{aligned}$$

This shows the desired decay estimate (7).

5. Euler-Maxwell system

Euler-Maxwell system in $\mathbb{R}^3$:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ (nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) = -n(E + u \times B) - nu, \\ E_t - \operatorname{rot} B = nu, \\ B_t + \operatorname{rot} E = 0, \end{cases} \tag{8}$$

$$\operatorname{div} E = n_\infty - n, \qquad \operatorname{div} B = 0, \tag{9}$$

Here $n > 0$: mass density, $u \in \mathbb{R}^3$: velocity, $E \in \mathbb{R}^3$: electric field, and $B \in \mathbb{R}^3$: magnetic induction; $p(n)$: pressure satisfying $p'(n) > 0$ for $n > 0$, and $n_\infty > 0$: a constant.

Solutions of (8) satisfy (9) for $t > 0$ if the initial data verify (9).

Euler-Maxwell system

The system (8) is written as

$$A^0(w)w_t + \sum_{j=1}^{3} A^j(w)w_{x_j} + L(w)w = 0,$$

where $w = (n, u, E, B)^T$ and

$$A^0(w) = \begin{pmatrix} p'(n)/n & & & \\ & nI & & \\ & & I & \\ & & & I \end{pmatrix}, \quad L(w) = \begin{pmatrix} 0 & & & \\ & n(I - \Omega_B) & nI & \\ & -nI & O & \\ & & & O \end{pmatrix},$$

$$\sum_{j=1}^{3} A^j(w)\xi_j = \begin{pmatrix} (p'(n)/n)(u \cdot \xi) & p'(n)\xi & & \\ p'(n)\xi^T & n(u \cdot \xi)I & & \\ & & O & -\Omega_\xi \\ & & \Omega_\xi & O \end{pmatrix}.$$

Here I and O denotes the 3×3 identity matrix and the zero matrix, respectively, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

for $\xi \in \mathbb{R}^3$. Note that $\Omega_\xi E = \xi \times E$ as a column vector in $\mathbb{R}^3$.

Constant equilibrium: The Euler-Maxwell system admits a constant equilibrium state

$$w_\infty = (n_\infty, 0, 0, B_\infty)^T,$$

where $n_\infty > 0$ and $B_\infty \in \mathbb{R}^3$ are constant states. Note that $L(w)w_\infty = 0$ for each w.

Linearized Euler-Maxwell system:

$$A^0 z_t + \sum_{j=1}^{3} A^j z_{x_j} + Lz = 0, \tag{10}$$

$$\operatorname{div} E = -\rho, \qquad \operatorname{div} h = 0, \tag{11}$$

where $z = (\rho, u, E, h)^T$ with $\rho = n - n_\infty$ and $h = B - B_\infty$, and $A^0 = A^0(w_\infty)$, $A^j = A^j(w_\infty)$ and $L = L(w_\infty)$.

Apply the Fourier transform:

$$A^0 \hat{z}_t + i|\xi| A(\omega)\hat{z} + L\hat{z} = 0, \tag{12}$$

$$i|\xi|\hat{E} \cdot \omega = -\hat{\rho}, \qquad i|\xi|\hat{h} \cdot \omega = 0, \tag{13}$$

where $A(\omega) = \sum_{j=1}^{3} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^2$.

Claim: Ueda, Wang & S.K (2010 preprint)

L is not symmetric but satisfies the following modified (SK) stability condition: Let $\varphi = (\rho, u, E, h)^T \in \mathbb{R}^{10}$.

$$\text{If} \quad L\varphi = 0, \quad \mu A^0 \varphi + A(\omega)\varphi = 0 \quad \text{and} \quad h \cdot \omega = 0, \quad \text{then} \quad \varphi = 0.$$

In this case, however, $\ker(L) \neq \ker(L_1)$.

Let $\varphi = (\rho, u, E, h)^T$. Then $L\varphi = 0$ gives

$$n_\infty(u - B_\infty \times u) + n_\infty E = 0, \qquad -n_\infty u = 0,$$

which shows that $u = E = 0$. Then $\varphi = (\rho, 0, 0, h)^T$. For this φ, we suppose that $\mu A^0 \varphi + A(\omega)\varphi = 0$ and $h \cdot \omega = 0$. This implies

$$\mu a_\infty \rho = 0, \quad b_\infty \rho\, \omega = 0, \quad h \times \omega = 0, \quad \mu h = 0, \quad h \cdot \omega = 0,$$

where $a_\infty = p'(n_\infty)/n_\infty$ and $b_\infty = p'(n_\infty)$ are positive constants. This shows that $\rho = h = 0$ and hence $\varphi = 0$.

Decay property

Theorem 4 (Pointwise estimate)
Ueda & S.K (2010 preprint), Duan (2010 preprint)

We have

$$|\hat{z}(\xi,t)| \le Ce^{-c\eta(\xi)t}|\hat{z}_0(\xi)|, \tag{14}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$.

Corollary (Decay estimate) Ueda & S.K (preprint), Duan (preprint)

We have

$$\|\partial_x^k z(t)\|_{L^2} \le C(1+t)^{-3/4-k/2}\|z_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l} z_0\|_{L^2}, \tag{15}$$

where k, $l \ge 0$.

- Decay estimate of the regularity-loss type

Lyapunov function:

$$E[\,\hat{z}\,] = H_0 + \frac{\alpha_1}{1+|\xi|^2}\Big(H_1 + \frac{\alpha_2|\xi|}{1+|\xi|^2}H_2\Big).$$

Here

$$H_0 = \langle A^0\hat{z},\,\hat{z}\rangle = a_\infty|\hat{\rho}|^2 + n_\infty|\hat{u}|^2 + |\hat{E}|^2 + |\hat{h}|^2,$$

$$H_1 = \mathrm{Re}\{a_\infty i|\xi|\langle\hat{\rho}\omega\,|\,\hat{u}\rangle + \langle\hat{u}\,|\,\hat{E}\rangle\}$$

$$H_2 = \mathrm{Re}\langle\hat{E}\,|\,\hat{h}\times i\omega\rangle$$

where α_1, $\alpha_2 > 0$ are small constants, $a_\infty = p'(n_\infty)/n_\infty$, and $\langle\cdot\,|\,\cdot\rangle$ is the inner product of $\mathbb{C}^3$.

Lyapunov function

We have

$$\frac{\partial}{\partial t}E[\,\hat{z}\,] + cD[\,\hat{z}\,] \leq 0,$$

where

$$D[\,\hat{z}\,] = |\hat{\rho}|^2 + |\hat{u}|^2 + \frac{1}{1+|\xi|^2}|\hat{E}|^2 + \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2.$$

Therefore we obtain

$$\frac{\partial}{\partial t}E[\,\hat{z}\,] + c\eta(\xi)E[\,\hat{z}\,] \leq 0,$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. This yields the desired pointwise estimate (14).

Energy estimate

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\,\hat{z}\,] + cD[\,\hat{z}\,] \leq 0,$$

we have the following energy estimate:

$$\|z(t)\|_{H^s}^2 + \int_0^t \|(\rho, u)(\tau)\|_{H^s}^2 + \|E(\tau)\|_{H^{s-1}}^2 + \|\partial_x h(\tau)\|_{H^{s-2}}^2 \, d\tau \leq C\|z_0\|_{H^s}^2,$$

where $s \geq 0$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for the component (E, h).

6. General framework for type (II)

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{16}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for each j,

(c) L is nonnegative definite (not symmetric) such that

$$\ker(L) \neq \ker(L_1),$$

where L_1 is the symmetric part of L.

Structural conditions

Apply the Fourier transform:

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0. \tag{17}$$

Condition (S): Ueda, Duan & S.K

There exits S with the following properties:

(i) SA^0 is symmetric.

(ii) $(SL)_1 + L_1$ is nonnegative definite and $\ker((SL)_1 + L_1) = \ker(L)$.

(iii) $i(SA(\omega))_2$ is nonnegative on $\ker(L_1)$, where X_2 is the skew-symmetric part of X.

Modified condition (K): Ueda, Duan & S.K

There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is skew-symmetric.

(ii) $(K(\omega)A(\omega))_1 + (SL)_1 + L_1$ is positive definite.

Theorem 5 (Pointwise estimate) Ueda, Duan & S.K

Under the conditions (S) and modified (K), we have

$$|\hat{u}(\xi,t)| \le Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \tag{18}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$.

Corollary (Decay estimate) Ueda, Duan & S.K

Under the conditions (S) and modified (K), we have

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}u_0\|_{L^2}, \tag{19}$$

where k, $l \ge 0$.

- Decay estimate of the regularity-loss type

Lyapunov function

Lyapunov function:

$$E[\,\hat{u}\,] = \langle A^0\hat{u},\, \hat{u}\rangle + \frac{\alpha_1}{1+|\xi|^2}\Big\{\langle SA^0\hat{u},\, \hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle\Big\},$$

where $\alpha_1,\ \alpha_2 > 0$ are small constants. We have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + \frac{c|\xi|^2}{(1+|\xi|^2)^2}\,|\hat{u}|^2 + \frac{c}{1+|\xi|^2}\,|(I-P)\hat{u}|^2 + c|(I-P_1)\hat{u}|^2 \le 0,$$

where P and P_1 are the orthogonal projections onto $\ker(L)$ and $\ker(L_1)$, respectively. Therefore we have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + c\eta(\xi)E[\,\hat{u}\,] \le 0,$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. This gives the desired pointwise estimate (18).

Energy estimate

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + \frac{c|\xi|^2}{(1+|\xi|^2)^2}\,|\hat{u}|^2 + \frac{c}{1+|\xi|^2}\,|(I-P)\hat{u}|^2 + c|(I-P_1)\hat{u}|^2 \le 0,$$

we have the energy estimate of the form

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|(I-P)u(\tau)\|_{H^{s-1}}^2 + \|(I-P_1)u(\tau)\|_{H^s}^2\,d\tau \le C\|u_0\|_{H^s}^2,$$

where $s \ge 0$.

- Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for the component $P_1 u$.

Theorem 6 (Pointwise estimate) Ueda, Duan & S.K

Under the conditions (S) with $(SA(\omega))_2 = 0$ and modified (K), we have

$$|\hat{u}(\xi,t)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \quad (20)$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Corollary (Decay estimate) Ueda, Duan & S.K

Under the conditions (S) with $(SA(\omega))_2 = 0$ and modified (K), we have

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \quad (21)$$

where $k \ge 0$.

- Decay estimate of the standard type

Lyapunov function: When $(SA(\omega))_2 = 0$,

$$E[\,\hat{u}\,] = \langle A^0\hat{u},\, \hat{u}\rangle + \alpha_1\Big\{\langle SA^0\hat{u},\, \hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle\Big\},$$

where α_1, $\alpha_2 > 0$ are small constants. We have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0,$$

where P is the orthogonal projection onto $\ker(L)$. Therefore we have

$$\frac{\partial}{\partial t}E[\,\hat{u}\,] + c\rho(\xi)E[\,\hat{u}\,] \le 0,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. This gives the desired pointwise estimate (20).

Open questions

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- General framework which covers the Euler-Maxwell system
- Characterization of (S) + modified (K)
- Relation with Kalman rank condition
- General framework for nonlinear problems of type (II)

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Thank You for Your Attention