A local characterisation of Kerr-Newman space-times and some applications

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Stationary Einstein-Maxwell space-times
- Geometrical set-up
- Anti-self-dual objects

Characterisation of Kerr-Newman
- Ernst potential, Ernst two-form, $\mathcal{B}_{ab}$, and $\mathcal{Q}_{abcd}$
- Main result
- Additional nice (and not nice) properties

Applications
- A unique extension theorem
- No multiple black holes

Mostly contained in W's PhD thesis (and AHP 2009 paper); remainder in a paper in preparation (parts also in P. Yu's PhD thesis).

Einstein-Maxwell space-time

- Space-time $(\mathcal{M}, g_{ab})$
  - $\mathcal{M}$ 4-dim., orientable, para-compact, simply-connected
  - g_{ab} a Lorentzian metric with signature $(-+++)$
  - $(\mathcal{M}, g_{ab})$ time-orientable
- Faraday tensor H_{ab}: real two-form on $\mathcal{M}$
- Einstein-Maxwell:

$$Ric_{ab} = (H + i^*H)_{ac}(H - i^*H)_b{}^c$$
$$0 = \nabla^a(H + i^*H)_{ac}$$

where $*$ is Hodge star: ${}^*H_{ab} = \frac{1}{2}\epsilon_{abcd}H^{cd}$

- Notice that Ric_{ab} is tracefree
- For construction no need to assume asymptotic flatness

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"Stationarity"

- Symmetry vector field t^a
  - $\mathcal{L}_t g = 0$, $\mathcal{L}_t H = 0$ (hence also $\mathcal{L}_t {}^*H = 0$)
  - For construction no assumption on causal nature of t^a
- t^a is Killing $\implies$ $\nabla_a t_b$ is anti-symmetric
- t^a is Killing $\implies$ Jacobi equation

$$\nabla_c \nabla_a t_b = R_{dcab} t^d$$

which implies

$$\nabla_{[c} \nabla_a t_{b]} = 0 \qquad \nabla^a \nabla_a t_b = -Ric_{db} t^d$$

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Kerr-Newman solution

In Boyer-Lindquist coordinates, the Kerr-Newman line element ds^2 and the associated vector potential A are

$$ds^2 = -\left(1 - \frac{2Mr - q^2}{r^2 + a^2\cos^2\theta}\right)dt^2 - \frac{2a(2Mr - q^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta d\phi dt$$

$$+\sin^2\theta\left(r^2 + a^2 + \frac{a^2(2Mr - q^2)}{r^2 + a^2\cos^2\theta}\sin^2\theta\right)d\phi^2$$

$$+\frac{r^a + a^2\cos^2\theta}{r^2 + a^2 - 2Mr + q^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2$$

$$A = -\frac{qr}{r^2 + a^2\cos^2\theta}dt + \frac{qra\sin^2\theta}{r^2 + a^2\cos^2\theta}d\phi$$

Kerr-Newman solution

- A three-paramter family of stationary, axial-symmetric, charged black holes: M the mass parameter, a the specific angular momentum, q the electromagnetic charge.
- The Kerr-Newman solution is algebraically special
  - Petrov type D $\Longrightarrow$ two repeated principal null directions for the Weyl tensor
  - Goldberg-Sachs $\Longrightarrow$ two families of geodesic shearfree null congruences
- The principal null directions of the Weyl tensor are also principal null directions of the Faraday tensor
- ...also principal null directions of the “Ernst two form” (to be described later)
- This “triple alignment” will be exploited for the local characterisation

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The bifurcate sphere

In general stationary black hole solutions:

- Stationary $\implies$ Event horizon $=$ apparent horizon
- Non-extremality $\implies$ positive surface gravity $\implies$ past and future horizons intersect in a topological 2 sphere (the bifurcate sphere) in $\mathcal{M}$
- The null generators of the horizon are principal null directions of the Weyl tensor
- On the bifurcate sphere t^a is purely space-like or 0

In Kerr-Newman, which is axially symmetric:

- On the bifurcate sphere t^a is proportional to the axial Killing vector field
- t^a vanishes on the bifurcate sphere for Schwarzschild

Anti-self-duality

- Hodge star $* : \wedge^2 T^*\mathcal{M} \to \wedge^2 T^*\mathcal{M}$, $** = -\mathrm{id}$.
- $\mathbb{C} \otimes \wedge^2 T^*\mathcal{M} = \Lambda_+ \oplus \Lambda_-$, eigenspace decomposition with eigenvalues $\pm i$.
- A complex-valued two-form $\mathcal{X}$ is *anti-self-dual* if ${}^*\mathcal{X} = -i\mathcal{X}$
- $\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2}(H + i^*H)$ is ASD; $\bar{\mathcal{H}}$ is SD
- The projection operator to ASD forms is given by

$$\mathcal{I}_{abcd} \stackrel{\text{def}}{=} \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} + i\epsilon_{abcd})$$

(so $\mathcal{H}_{ab} = \mathcal{I}_{abcd}H^{cd}$, $\mathcal{I} + \bar{\mathcal{I}} = \mathrm{id}$)

- ASD forms have nice multiplication properties, e.g.

$$\mathcal{X}_{ac}\mathcal{Y}_b{}^c + \mathcal{Y}_{ac}\mathcal{X}_b{}^c = \frac{1}{2}g_{ab}\mathcal{X}_{cd}\mathcal{Y}^{cd} \implies \mathcal{X}_{ac}\mathcal{X}_b{}^c = \frac{1}{4}g_{ab}\mathcal{X}^2$$

(throughout $\mathcal{Z}^2$ will denote the full tensor contraction, via the metric, of $\mathcal{Z}$ with itself)

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Curvature decomposition

- Treating the Riemann curvature tensor R_{abcd} as a mapping from $\wedge^2 T^* M$ to itself, we can factor it through the ASD and SD projections:

$$R_{abcd} = (\mathcal{I} R \mathcal{I})_{abcd} + (\bar{\mathcal{I}} R \mathcal{I})_{abcd} + (\mathcal{I} R \bar{\mathcal{I}})_{abcd} + (\bar{\mathcal{I}} R \bar{\mathcal{I}})_{abcd}$$

- On the other hand, we have the standard decomposition

$$R_{abcd} = W_{abcd} + \frac{1}{2}(g \owedge S)_{abcd} + \frac{R}{12}(g \owedge g)_{abcd}$$

where R is the scalar curvature, S the tracefree part of Ric, and $\owedge$ the Kulkarni-Nomizu product.

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Curvature decomposition

- Applying a theorem of Singer and Thorpe (1969), and using that Einstein-Maxwell solutions are scalar-flat, we get that

$$\mathcal{C}_{abcd} \stackrel{\text{def}}{=} \mathcal{I}_{abef} R^{efgh} \mathcal{I}_{ghcd} = (W + i^*W)_{abcd} = (W + iW^*)_{abcd}$$

and

$$\frac{1}{2}(g \owedge S)_{abcd} = \mathcal{I}_{abef} R^{efgh} \bar{\mathcal{I}}_{ghcd} + \bar{\mathcal{I}}_{abef} R^{efgh} \mathcal{I}_{ghcd}$$

- In other words, the Weyl tensor sends (A)SD forms to (A)SD forms, while the traceless Ricci part of the curvature sends ASD forms to SD forms and vice versa

Symmetric spinor product

- Two-form fields correspond to spin-1 waves, Weyl fields to spin-2 waves, here we give a method to combine two ASD forms to get a Weyl field
- (A Weyl field is a (0,4) tensor with algebraic symmetries of the Weyl tensor)
- Define the *symmetric spinor product* of two ASD forms $\mathcal{X}_{ab}$ and $\mathcal{Y}_{cd}$ by

$$(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} \stackrel{\text{def}}{=} \frac{1}{2}(\mathcal{X}_{ab}\mathcal{Y}_{cd} + \mathcal{Y}_{ab}\mathcal{X}_{cd}) - \frac{1}{3}\mathcal{I}_{abcd}\mathcal{X}_{ef}\mathcal{Y}^{ef}$$

- $(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd}$ is an ASD Weyl field
- A word on the nomenclature: the spinor representation of spin-1 fields are symmetric in the indices, the above corresponds to full symmetrisation of all indices of the product of spinor representations.

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Potential functions relative to a Killing vector

- The complex Faraday tensor $\mathcal{H}$ is a closed two-form by Maxwell's equations
- Recall Cartan's identity

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega$$

$\implies$ $\iota_t \mathcal{H}$ is a closed one-form

$\implies$ $\exists$ complex function Ξ (defined up to constant) s.t. $d\Xi = \iota_t \mathcal{H}$

- Call it the "electromagnetic potential"
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Ernst two-form and Ernst potential

- $\hat{\mathcal{F}}_{ab} \stackrel{\text{def}}{=} 2\mathcal{I}_{abcd}\nabla^c t^d$ is ASD two-form, but not divergence free
- The *Ernst two-form* is defined by

$$\mathcal{F}_{ab} \stackrel{\text{def}}{=} \hat{\mathcal{F}}_{ab} - 4\Xi\mathcal{H}_{ab}$$

$\Longrightarrow$ ASD two-form

- Jacobi equation for Killing vector t^a and Einstein equation

$\Longrightarrow$ $\mathcal{F}_{ab}$ is closed (slide 2)

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Characterisation tensors

Definition

Up to four normalisation constants — the two complex constants implicit in the definition of Ξ and σ, a complex constant κ, and a real constant μ — we define the *characterisation* (or *error*) tensors

$$\mathcal{B}_{ab} \overset{\text{def}}{=} \kappa \mathcal{F}_{ab} + 2\mu \mathcal{H}_{ab}$$

$$\mathcal{Q}_{abcd} \overset{\text{def}}{=} \mathcal{C}_{abcd} + \frac{6\kappa\Xi - 3\mu}{2\mu\sigma} (\mathcal{F} \tilde{\otimes} \mathcal{F})_{abcd}$$

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Characterisation tensors

$$\mathcal{B} = \kappa\mathcal{F} + 2\mu\mathcal{H} \qquad \mathcal{Q} = \mathcal{C} + \frac{6\kappa\Xi - 3\mu}{2\mu\sigma}(\mathcal{F}\tilde{\otimes}\mathcal{F})$$

- $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ are natural generalisations of the Mars-Simon tensor for Kerr spacetimes (Mars 2000)
  - In vacuum, $\mathcal{H} \equiv 0 \implies$ set $\kappa = 0$ and so $\mathcal{B} = \kappa\mathcal{F} - 2\mu\mathcal{H} \equiv 0$
  - Normalize $\Xi \equiv 0$, hence $\mathcal{F}_{ab} \equiv \hat{\mathcal{F}}_{ab} = (dt)_{ab}$
  - $\mathcal{Q} = \mathcal{C} - \frac{3}{2}\sigma^{-1}(\mathcal{F}\tilde{\otimes}\mathcal{F})$ is precisely the Mars-Simon tensor
- As seen above, the parameter κ takes the role of electromagnetic charge; μ will take the role of mass
- $\mathcal{B}$ and $\mathcal{Q}$ measures the simultaneous alignment of the principal null directions between $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{C}$, while also keeping track of the proportionality factor
- Note: definition given here is a modification of the original one givin by W (2009)

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  - In vacuum, $\mathcal{H} \equiv 0 \implies$ set $\kappa = 0$ and so $\mathcal{B} = \kappa\mathcal{F} - 2\mu\mathcal{H} \equiv 0$
  - Normalize $\Xi \equiv 0$, hence $\mathcal{F}_{ab} \equiv \hat{\mathcal{F}}_{ab} = (dt)_{ab}$
  - $\mathcal{Q} = \mathcal{C} - \frac{3}{2}\sigma^{-1}(\mathcal{F}\tilde{\otimes}\mathcal{F})$ is precisely the Mars-Simon tensor
- As seen above, the parameter κ takes the role of electromagnetic charge; μ will take the role of mass
- $\mathcal{B}$ and $\mathcal{Q}$ measures the simultaneous alignment of the principal null directions between $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{C}$, while also keeping track of the proportionality factor
- Note: definition given here is a modification of the original one givin by W (2009)

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Local Characterisation Theorem

Theorem (W (2009))

Let $(\mathcal{M}, g_{ab}.\mathcal{H}_{ab})$ be a solution to the Einstein-Maxwell equations admitting a continuous symmetry t^a. Let $U \subset \mathcal{M}$ be a connected open subset, and suppose that there exists a choice of the four normalisation constants such that, restricted to U, we have $\sigma \neq 0$, $\mathcal{B}_{ab} = 0$, $\mathcal{Q}_{ab} = 0$. Then

$$t^2 + 2\Re\sigma + \frac{|\kappa\sigma|^2}{\mu^2} + 1 = const. \qquad \text{and} \qquad \mu^2\mathcal{F}^2 + 4\sigma^4 = const.$$

...(cont'd next page)

Local Characterisation Theorem

Theorem (W (2009))

(cont'd from previous page) If, furthermore, both of the expressions evaluate to 0, that is,

$$t^2 + 2\Re\sigma + \frac{|\kappa\sigma|^2}{\mu^2} + 1 = \mu^2\mathcal{F}^2 + 4\sigma^4 = 0 \ ,$$

and t^a is time-like somewhere in U, then U is locally isometric to a domain in a Kerr-Newman spacetime with charge κ, mass μ, and angular momentum $\mu\sqrt{\mathfrak{A}}$, where

$$\mathfrak{A} \stackrel{def}{=} \left|\frac{\mu}{\sigma}\right|^2 \left(\Im\nabla\frac{1}{\sigma}\right)^2 + \left(\Im\frac{1}{\sigma}\right)^2$$

is a constant on U.

Remarks

- In the present formulation, the case $\mathcal{H}_{ab} \equiv 0$ and $\kappa = 0$ reduces exactly to the characterisation of Kerr space-times given by Mars (2000).
- The condition that t^a is timelike somewhere is not sharp: mainly requires t^a is not everywhere orthogonal to the principal null directions of $\mathcal{F}_{ab}$

$\Longrightarrow$ Part of the proof implies that this should only happen "on the bifurcate sphere"; in constructing the local isometry we also construct the "axial Killing vector field", and it can be shown that t^a is orthogonal to both principal null directions only it is tangent to the axial Killing vector field

- Globalisation: if $\mathcal{M}$ is asymptotically flat, U an end, and t^a a time-translation at infinity, can choose Ξ and σ to vanish at infinity. That the algebraic expressions vanish (and are not equal to other constants) follows

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Ideas used in the proof

- Using that t^a is Killing, and $\mathcal{B}_{ab} = 0$, we show that $\mathcal{F}^2 \neq 0$ on a dense subset of U. On this set $\mathcal{F}_{ab}$ has two distinct principal null vectors, which it shares with $\mathcal{H}_{ab}$
- Using $\mathcal{Q}_{abcd} = 0$, this now implies $\mathcal{C}_{abcd}$ is algebraically special (Petrov type D)
- Einstein's equations and the Bianchi equations greatly simplify, the assumed proportionality factors allow us to directly integrate to get conditions on the rotation coefficients relative to a null tetrad, and that $\mathfrak{A}$ is constant
- Using the rotation coefficients, construct the axial Killing vector field (assuming $\mathfrak{A} \neq 0$; $\mathfrak{A} = 0$ corresponds to the Reissner-Nordström case to be treated separately)
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Wave equation

- Unsurprisingly, $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ obey wave equations
- $\mathcal{B}_{ab}$ is a linear combination of Maxwell fields $\mathcal{F}_{ab}$ and $\mathcal{H}_{ab}$ $\implies$ is Maxwell
- $\mathcal{Q}_{abcd}$ formed from $\mathcal{C}_{abcd}$ and $\mathcal{F}_{ab}$, so must obey *some* wave equation
- Somewhat surprising: $\mathcal{Q}_{abcd}$ has a wave equation *without* external forcing terms, that is, schematically we have

$$\Box_g \mathcal{Q} = \mathcal{A} * \mathcal{Q} + \mathcal{B} * \nabla \mathcal{Q}$$

- Allows us to study the system using wave equation techniques

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Drawbacks

- Despite solving a wave equation, the tensors $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ are *not dynamical*

$\Longrightarrow$ Finite speed of propagation result mooted by existence of transversal Killing vector field

- Their constructions using the Ernst two-form and Ernst potential strongly depend on existence of a Killing vector field:
  - The potentials σ and Ξ are defined using Cartan's relation, and requires t^a be Killing
  - $\hat{\mathcal{F}}_{ab}$ (and hence $\mathcal{F}_{ab}$) has good properties from vanishing of deformation tensor
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As error tensors

- In the proof of local isometry, we used an explicit integration to obtain control on the rotation coefficients.
- If $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ are not vanishing, but "suitably small" in some sense, we can establish a moving null frame that is locally "close" to a Kerr-Newman frame. The difference in rotation coefficients can be estimated.
- Furthermore, we also obtain control on the scalar quantities $\mathfrak{A}$, σ and Ξ.
- So in a vague sense we can say that $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ being small $\implies$ we are "close" to being Kerr-Newman.

The functions y and z

- That we can control σ is particularly important: in Kerr-Newman, $\sigma \neq 0$ and in fact

$$\sigma^{-1} = r + ia\cos\theta$$

  in Boyer-Lindquist coordinates.
- In general, whenever $\sigma \neq 0$, we define

$$y \stackrel{\text{def}}{=} \Re\sigma^{-1} \qquad z \stackrel{\text{def}}{=} \Im\sigma^{-1}$$

- As part of the previous theorem, z can be bounded by $\mathfrak{A}$
- y locally behaves like the Boyer-Lindquist r, and in particular has nice *convexity* properties.

Unique extension

- Method of Ionescu and Klainerman (2009) for Kerr
- Based on study of unique continuation properties of PDEs
- With an eye toward “no hair theorem” without the assumption of real analyticity or axi-symmetry.
  - Real analyticity allows for reduction to axial symmetric space-time (Hawking 1973)
  - With axial symmetry we can use elliptic techniques (since a linear combination of the axial symmetry vector field and t^a is time-like) (Bunting, Carter, Robinson, Mazur; all in 1970s and 80s)

$\Longrightarrow$ Without the symmetry we need wave equation techniques

The theorem

Theorem (W 2009)

Given a stationary black hole solution to the Einstein-Maxwell equations with non-degenerate event horizon, if we further assume that the bifurcate sphere "looks identical" to the bifurcate sphere of a Kerr-Newman solution, then the space-time must be everywhere locally isometric to said Kerr-Newman solution.

Looks identical?

- To understand the "looks identical" technical requirement, we need to consider the method of proof.
- The proof uses a unique continuation technique based on existence of suitable *pseudo-convex* weights (a function with nice properties relative to a linear partial differential operator)
- On a Kerr and Kerr-Newman background, for a *stationary solution* to *linear wave equations* (with possible lower order terms), the Boyer-Lindquist r coordinate is a good pseudo-convex weight
  - Stationarity is crucial for outside of ergosphere
  - Related to the statement that, on a Kerr-Newman background, "all null geodesics orthogal to t^a must exit through the event horizon"

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Looks identical!

- In our problem, we do not a priori fix the background metric to be Kerr-Newman, so we need to construct the pseudo-convex weight *via a bootstrap from the bifurcate sphere*
- To do so requires that $\mathcal{B}_{ab} = 0$, $\mathcal{Q}_{abcd} = 0$, and the two expressions in the statement of the characterisation theorem take the value of 0 *on the bifurcate sphere*.
- Einstein-Maxwell equation trivializes along stationary horizon, so $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ will vanish along horizon.
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Perturbative rigidity

- Results below based on collaboration with Pin Yu
- Whereas unique extension maybe thought of as “large data with constraint” result, we can also consider “small data without constraint”
- An example, Yu (2010) has the following theorem

Theorem (Yu 2010)

Any stationary black hole solution to the Einstein-Maxwell equations with non-degenerate horizon that is “sufficiently close” to a Kerr-Newman solution must be axi-symmetric.

where “sufficiently close” means that a weighted norm of $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ is sufficiently small (the weight is to ensure decay near infinity that allows us to “integrate back”)

Perturbative rigidity

- Observe that now inserting in the classical no hair theorems, we have a perturbative rigidity statement in the stationary class (notice that trivially, small smooth perturbations of the metric and vector potential with sufficient decay implies smallness of $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ with decay)
- Yu's theorem is a weak generalisation of Carter's original "no hair theorem" (1973), which states that "non-degenerate stationary axi-symmetric charged black holes form a 3 parameter continuous family"
- Here we have that there's still no "branching" allowed if we remove the axial symmetry

Only one black hole

An interesting preliminary version to Yu's theorem is the following lemma (see Yu's 2010 thesis)

Lemma (W and Yu)

A stationary solution of the Einstein-Maxwell equations with sufficiently small $\mathcal{B}_{ab}$ and $\mathcal{Q}_{abcd}$ cannot have a non-degenerate event horizon with multiple connected components.

Only one black hole

Sketch of the proof:

- Smallness condition allows us to integrate back from infinity

$\Longrightarrow$ The function y can be defined up to some small error term, and behaves largely the same as y in the case when $\mathcal{B}$ and $\mathcal{Q}$ vanishes

- In particular, ∇y is non-vanishing
- Einstein's equation implies that y must be constant on each component of the horizon, and achieve a local minimum there
- Use a version of the mountainpass lemma and we obtain a contradiction