Positive gravitational energy in arbitrary dimensions

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1 Introduction

The most elegant and convincing proof of the positive energy theorem is by using spinors, as did Witten in the case $n = 3$ inspired by heuristic works of Deser and Grisaru originating from supergravity. The aim of this Note is to present a streamlined, complete proof, valid in arbitrary space dimension n, and using only spinors on the oriented Riemannian space $(M^n; g)$, without invoking spacetime spinors.

2 Einsteinian spacetime.

An Einsteinian spacetime is a Lorentzian manifold $(\mathbf{M}^{n+1}, \mathbf{g})$ which satisfies the Einstein equations

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{R} = T_{\alpha\beta},$$

it satisfies the dominant energy condition if $u_\alpha T^{\alpha\beta}$ is non spacelike for every time like vector u.

On each spacelike slice M^n, x^0 =constant (we assume $\mathbf{M}^{n+1} = M^n \times R$), the induced metric g and extrinsic curvature K satisfy the constraints written here in a Cauchy adapted local frame

$$\mathbf{R}_{0j} \equiv D_h P^h{}_j = -T_{0j}, \quad P^{ih} := K^{ih} - g^{ih}\mathrm{tr}K \tag{2.1}$$

$$\mathbf{S}_{00} \equiv \frac{1}{2}\{R - |K|^2 + (\mathrm{tr}K)^2\} = T_{00}. \tag{2.2}$$

The mass m and linear momentum p, called ADM, associated to an asymptotically Euclidean end of (M^n, g) originate from an Hamiltonian formulation of the Einstein equations, they define a spacetime vector $\mathbf{E}$ with components

$$E^0 := m := \lim_{r\to\infty} \frac{1}{2}\int_{S_r^{n-1}} \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i}\right) n_i \mu_{\bar{g}}, \tag{2.3}$$

$$E^h := p^h := \lim_{r\to\infty} \int_{S_r^{n-1}} P^{ih} n_i \mu_{\bar{g}}.$$

The positive energy theorem says that

$$m \geq |p| \quad \text{if} \quad T_{00} \geq (g^{ij}T_{0i}T_{oj})^{\frac{1}{2}}$$

3 Asymptotically Euclidean space.

M^n is a smooth manifold union of a compact set W and a finite number N of sets Ω_I, diffeomorphic to the complement of a ball in R^n. One covers W by a finite number N' of open sets W_K each diffeomorphic to a ball in R^n. We denote by x^i local coordinates for a domain Ω_I or W_K. We set $r := \sum (x^i)^2\}^{\frac{1}{2}}$ and we take $r_0 > 0$ such that $\Omega_I := \{r > r_0\}$, $\Omega_I \cap W_K = \emptyset$ if $r < 2r_0$.

We consider a smooth partition of unity, f_I, $I = 1, ...N$, f_K, $K = 1, ...N'$, called a preparation of M^n; f_K with support in W_k, f_I support in Ω_I and $f_I = 1$ for $r > 2r_0$.

The Riemannian metric g is assumed continuous and uniformly bounded above and below in each Ω_I and W_K by constant positive definite quadratic forms.

A tensor field u on M^n is written

$$u \equiv \sum_{I=1,..N} u_I + \sum_{K=1,..N'} u_K \quad \text{with} \quad u_I := f_I u, \quad u_K := f_K u.$$

Norms on spaces of tensor fields are defined through their components in the Ω_I, W_K, each endowed with the Euclidean metric $e := \eta_{ij} dx^i dx^j \equiv \sum (dx^i)^2$, with pointwise norm $|.|$ and volume element μ_e. We use the Banach and Hilbert spaces C^k_β and $H_{s,\delta}$ with norms

$$||u||_{C^k_\beta} \equiv \sup_{I,K} \{ \sup_{\Omega_I} (r^{\beta+k} |\underline{D}^k u_I|), \sup_{W_K} |\underline{D}^k u_K| \}, \quad \underline{D}^k := \frac{\partial^k}{\partial x^{i_1} ... \partial x^{i_k}}. \tag{3.1}$$

$$||u||^2_{H_{s,\delta}} := \sum_{I=1,...N} \int_{\Omega_I} \sum_{0 \le k \le s} r^{2(k+\delta)} |\underline{D}^k u_I|^2 \mu_e + \sum_{K=1,...N'} \int_{W_K} \sum_{0 \le k \le s} |\underline{D}^k u_K|^2 \mu_e \tag{3.2}$$

Different preparations of M^n give equivalent norms. A Riemannian manifold (M^n, g) is called asymptotically Euclidean (A.E) if

$$h_I := f_I(g - e) \in H_{s,\delta} \cap C^1_{n-2}, \quad f_K g \in H_s, \quad s > \frac{n}{2} + 1, \quad \frac{n}{2} - 2 > \delta > -\frac{n}{2}. \tag{3.3}$$

It can be proved, using the fact that $H_{s,\delta}$ is an algebra if $s > \frac{n}{2}, \delta > -\frac{n}{2}$ that an A.E (M^n, g) admits in each end Ω_I an orthonormal coframe

$$\theta^i := a^j_i dx^j \qquad a^j_i = \delta^j_i + \frac{1}{2}\lambda^j_i, \quad \lambda^j_i \in H_{s,\delta} \cap C^1_{n-2}. \tag{3.4}$$

Then, underlying components in the coordinates x^i of Ω_I, it holds that

$$\underline{g_{ij}} \equiv \sum_h a^h_i a^h_j \equiv \eta_{ij} + \underline{h_{ij}}, \quad \eta_{ij} := \delta^j_i \tag{3.5}$$

with

$$\underline{h_{ij}} \equiv \frac{1}{2}(\lambda^j_i + \lambda^i_j) + \frac{1}{4}\sum_h \lambda^h_j \lambda^h_i, \quad \lambda^h_j \lambda^h_i \in H_{s,2\delta+\frac{n}{2}} \cap C^1_{2n-4}. \tag{3.6}$$

The rotation coefficients c^h_{ij} of the coframe θ^h are, with (b^j_i) the matrix inverse of (a^i_j) and ∂_i the Pfaff derivative with respect to θ^i, $d\theta^h \equiv \frac{1}{2}c^h_{ij}\theta^i \wedge \theta^j$,

$$c^h_{ij} \equiv b^k_j \partial_i \lambda^h_k - b^k_i \partial_j \lambda^h_i \equiv \frac{1}{2}(\partial_i \lambda^h_j - \partial_j \lambda^h_i) + c'^h_{ij}, \quad c'^h_{ij} \in H_{s-1,2\delta+1+\frac{n}{2}}$$

The components of the Riemannian connection ω in the coframe θ^i are

$$\omega_{i,jh} \equiv \frac{1}{2}(-c^i_{jh} + c^h_{ij} - c^j_{ih})$$

hence by straightforward computation

$$\omega_{i,jh} \equiv \{\frac{1}{4}\{-\partial_j(\lambda^h_i + \lambda^i_h) + \partial_h(\lambda^i_j + \lambda^j_i)\} + \omega'_{i,hj}, \quad \omega'_{i,hj} \in H_{s-1,2\delta+1+\frac{n}{2}},$$

therefore, formula useful in the computation of the mass,

$$\omega_{i,hj} \equiv \frac{1}{2}\{\underline{\partial_h h_{ij}} - \frac{1}{2}\underline{\partial_j h_{ih}}\} + \zeta_{i,hj}, \quad \zeta_{i,hj} \in H_{s-1,2\delta+1+\frac{n}{2}}. \tag{3.7}$$

3.1 Mass m and linear momentum p of an end.

We assume the extrinsic curvature K of our A.E (M^n, g) slice to be like ω in $H_{s-1,\delta+1} \cap C^0_{n-1}$. We then say that (M^n, g, K) is A.E. The mass m and linear momentum p,

$$E^0 := m := \lim_{r\to\infty} \frac{1}{2} \int_{S_r^{n-1}} (\underline{\partial}_j \underline{h_{ij}} - \underline{\partial}_i \underline{h_{jj}}) n_i \mu_{\bar{g}}, \tag{3.8}$$

$$E^h := p^h := \lim_{r\to\infty} \int_{S_r^{n-1}} P^{ih} n_i \mu_{\bar{g}}$$

are then well defined because $\mu_{\bar{g}}$ is equivalent to $r^{n-1}\mu_{S_1^{n-1}}$ and n_i is uniformy bounded.

4 Spinor fields.

The gamma matrices associated with an orthonormal coframe θ^i of g at $x \in M^n$ are linear endomorphisms of a complex vector space S of dimension $p := 2^{[n/2]}$ which satisfy the identities

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij} I_p, \; i,j = 1, ...n, \;\; I_p \text{ identity matrix} \tag{4.1}$$

The γ_i are chosen hermitian, i.e. $\gamma_i = \tilde{\gamma}_i$, as is possible for an $O(n)$ group.

The spinor group $Spin(n)$, double homomorphism covering of $SO(n)$, can be realized by the group of invertible linear maps Λ of S which satisfy, with $O := (O_i^j)$ a $n \times n$ orthogonal matrix

$$\Lambda \gamma^i \Lambda^{-1} = O_j^i \gamma^j \quad \text{and} \quad \det \Lambda = 1. \tag{4.2}$$

In a subset Ω_I or W_K with a given field ρ_0 of orthonormal frames a spinor field ψ is represented by a mapping $(x^i) \mapsto \underline{\psi}(x^i) \in S$. Under an $O \in SO(n)$ change of frame, $\rho = O\rho_0$ the spinor ψ becomes represented by $\underline{\psi'} = \underline{\Lambda\psi}$ where some choice has been made of the correspondance between Λ and O. This can be made consistently on M^n if it admits a spin structure; that is, a homomorphism of a $Spin(n)$ principal bundle P_{Spin_n} onto the principal bundle of oriented orthonormal frames. It is a topological property of M^n, the vanishing of its second Stiefel-Whitney class, always true for orientable M^3. A spinor field on M^n is then a section of a vector bundle $\Psi_{Spin(n)}$ associated with $P_{Spin(n)}$, with base M^n and typical fiber S.

To a space of spinors corresponds a space of cospinors, replacing S by the adjoint (complex dual) vector space $\tilde{S}$ and the change of representation by $\underline{\phi'} = \underline{\phi}\Lambda^{-1}$. Using dual frames e_A of S and θ^A of $\tilde{S}$ we have $\psi \equiv \psi^A e_A$, $\phi = \theta^A \phi_A$, $\quad A = 1, ...p$, we denote the duality relation by

$$\phi\psi \equiv (\phi, \psi) := \phi_A \psi^A, \;\; \text{a frame independent scalar.}$$

By the definitions $\tilde{\psi}$ represented by $\tilde{\psi}_A := (\psi^A)^*$ is a cospinor if ψ is a spinor, $|\psi|^2 \equiv \tilde{\psi}\psi$ is positive definite.

A spin connection σ on (M^n, g) is deduced from an $O(n)$ connection ω by the isomorphism between the Lie algebras of $O(n)$ and $Spin(n)$ obtained by differentiation of the relation linking Λ and O, it is represented in each domain of the preparation by

$$\sigma_i \equiv \frac{1}{4}\gamma^h \gamma^k \omega_{i,hk}, \quad i = 1, ...n. \tag{4.3}$$

The covariant derivative of a spinor ψ is a covariant vector - spinor, the covariant derivative of a cospinor is a covariant vector - cospinor. Their components in the frames respectively $\theta^i \otimes e_A$ and $\theta^i \otimes \theta^A$ are

$$(D_i\psi)^A \equiv \partial_i \psi^A + (\sigma_i \psi)^A,$$

$$(D_i\phi)_A \equiv \partial_i \phi_A - (\phi\sigma_i)_A.$$

The hermiticity of γ_i shows that

$$\tilde{\sigma}_i = -\sigma_i, \qquad \text{hence} \qquad \widetilde{D_i\psi} \equiv D_i\tilde{\psi}.$$

The Riemannian connection together with the spin connection define a first order derivation operator mapping tensor-spinor-cospinor fields into tensor-spinor- cospinor fields with one more covariant tensorial index.

The gamma matrices are the components of a vector-spinor-cospinor which has covariant derivative zero.

The spin curvature ρ is a 2-tensor- spinor -cospinor, image by the mapping of Lie algebras of the curvature tensor of g. The Ricci identity for spinors reads

$$D_i D_j \psi - D_j D_i \psi \equiv \rho_{ij}\psi \quad \text{with} \quad \rho_{ij} := \frac{1}{4} R_{ij,hk}\gamma^h\gamma^k. \tag{4.4}$$

5 Dirac operator.

The Dirac operator on sections of the vector bundle $\Psi(n)$ reads locally

$$\mathcal{D}\psi \equiv \gamma^i D_i\psi \equiv \gamma^i(\partial_i\psi + \frac{1}{4}\omega_{i,hk}\gamma^h\gamma^k\psi), \quad \text{hence } \widetilde{\mathcal{D}\psi} \equiv D_i\tilde{\psi}\gamma^i. \tag{5.1}$$

The algebraic Bianchi identity together with definitions lead to the important well known formula

$$\mathcal{D}^2\psi \equiv \eta^{ij}D_iD_j\psi + \frac{1}{2}\gamma^i\gamma^j\rho_{ij}\psi \equiv \eta^{ij}D_iD_j\psi - \frac{1}{4}R\psi. \tag{5.2}$$

The Dirac operator is a first order linear operator with principal symbol $(\eta^{ij}\xi_i\xi_j)^{\frac{p}{2}}$, hence elliptic. Weighted Sobolev spaces for spinor fields on a prepared M^n are defined as for tensor fields after setting $\psi = \sum(\beta_I\psi + \beta_K\psi)$ and using representations $\underline{\psi}$. A known theorem (Christodoulou and C-B) gives:

Theorem 1 *On an A.E. (M^n, g) the Dirac operator is a Fredholm operator from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$, it is an isomorphism if injective. The same is true of $\mathcal{D}\psi + f\psi$ if f is a bounded linear map from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$.*

6 Gravitational mass.

We prove for arbitrary $n > 2$ the fundamental fact used by Witten for $n = 3$.

Theorem 2 *Let (M^n, g) be A.E. The mass m of an end Ω_I is equal to*

$$m = 2 \lim_{r \to \infty} \int_{S_r^{n-1}} \mathcal{U}_0^i n_i \mu_{\bar{g}}, \tag{6.1}$$

where

$$\mathcal{U}_0^i := \mathcal{R}_e\{\tilde{\psi}_0(\eta^{ij} - \gamma^i\gamma^j)\sigma_j\psi_0\},$$

with S_r^{n-1} the submanifold of the end Ω_I with equation $\{\sum(x^i)^2\}^{\frac{1}{2}} = r$, n_i its unit normal, $\mu_{\bar{g}}$ the volume element induced by g and ψ_0 a spinor constant in Ω_I (i.e. $\frac{\partial\psi_0}{\partial x^i} = 0$) and $|\psi_0| = 1$.

Proof. We first remark that, using $\gamma^i = \tilde{\gamma}^i$ and $\tilde{\sigma}_i = -\sigma_i$, one finds

$$\mathcal{R}_e(\tilde{\psi}_0\eta^{ij}\sigma_j\psi_0) \equiv \frac{1}{2}\tilde{\psi}_0\eta^{ij}(\sigma_j + \tilde{\sigma}_j)\psi_0 \equiv 0.$$

The definition of σ_j implies then ,

$$\mathcal{U}_0^i = \frac{1}{8}\sum_{j,h,k}\tilde{\psi}_0\omega_{j,hk}(\gamma^i\gamma^j\gamma^h\gamma^k - \gamma^h\gamma^k\gamma^j\gamma^i)\psi_0.$$

Recall that in an end of (M^n, g)

$$\omega_{j,hk} \equiv \frac{1}{2}\{\underline{\partial}_h\underline{h_{jk}} - \frac{1}{2}\underline{\partial}_k\underline{h_{jh}}\} + \zeta_{j,hk}, \quad \zeta_{j,hk} \in H_{s-1,2\delta+1+\frac{n}{2}}. \tag{6.2}$$

We denote by $\tilde{=}$ expressions which give the same limit for the integral when $\omega_{j,hk}$ is replaced by its first term, $\frac{1}{2}\{\underline{\partial}_h\underline{h_{jk}} - \frac{1}{2}\underline{\partial}_k\underline{h_{jh}}\}$.

Computation of $\mathcal{U}_0^i$ is somewhat tedious, whatever path one uses. One can proceed as folows.

1. All indices are different, then anticommutation of the corresponding gamma matrices, the symmetry of $\underline{\partial}_k\underline{h_{hj}}g_{hj}$ in h and j and $\underline{\partial}_h\underline{h_{kj}}$ in k and j and change of names of indices lead to

$$\mathcal{U}_{0,diff}^i \tilde{=} \frac{1}{8}\sum_{j,h,k}\tilde{\psi}_0\underline{\partial}_h\underline{h_{kj}}(\gamma^i\gamma^j\gamma^h\gamma^k - \gamma^i\gamma^j\gamma^h\gamma^k)\psi_0 = 0,$$

2. Two indices coincide.
a. $h = k$: gives 0 since then $\omega_{j,hk} = 0$.
b. $j = i$: gives 0 since then

$$\mathcal{U}^i_{0,j=i} = \frac{1}{4}\tilde{\psi}_0\omega_{i,hk}(\gamma^h\gamma^k - \gamma^h\gamma^k)\psi_0 \equiv 0.$$

c. $h = i$, $h \neq k$ and $j \neq i$.

$$\mathcal{U}^i_{0,h=i} = \frac{1}{8}\sum_{j,h,k}\tilde{\psi}_0\omega_{j,hk}(\gamma^i\gamma^j\gamma^i\gamma^k - \gamma^i\gamma^k\gamma^j\gamma^i)\psi_0.$$

Using $i \neq k$ gives

$$\gamma^j\gamma^i\gamma^k - \gamma^k\gamma^j\gamma^i = -\gamma^j\gamma^k\gamma^i - \gamma^k\gamma^j\gamma^i = -2\eta^{jk}\gamma^i,$$

hence, using $\omega_{j,ij} \tilde{=} \frac{1}{2}\{\underline{\partial}_i \underline{h_{jj}} - \frac{1}{2}\underline{\partial}_j \underline{h_{ji}}\}$

$$\mathcal{U}^i_{0,h=i} \tilde{=} -\frac{1}{4}\sum_{j\neq i,h=i,k\neq i}\tilde{\psi}_0\gamma^i\omega_{j,ij}\psi_0 \tilde{=} \frac{1}{8}\{\underline{\partial}_j \underline{h_{ji}} - \underline{\partial}_i \underline{h_{jj}}\} \tilde{=} \mathcal{U}^i_{0,k=i}.$$

On the other hand, by the symmetries appearing in $\omega_{j,hk}$, $j \neq h$ and $h \neq i$, and no summation on i,

$$\mathcal{U}^i_{0,j=k} = \frac{1}{8}\sum_{j=k\neq i,k\neq h}\psi_0\omega_{j,hk}(\gamma^i\gamma^j\gamma^h\gamma^k - \gamma^h\gamma^k\gamma^j\gamma^i)\psi_0$$

hence using again symmetries

$$\mathcal{U}^i_{0,j=k} \tilde{=} \frac{1}{8}\tilde{\psi}_0\underline{\partial}_i \underline{h_{jj}}(\gamma^i\gamma^h + \gamma^h\gamma^i)\psi_0 = 0$$

The same argument gives $\mathcal{U}^i_{0,j=h} \tilde{=} 0$.
Hence finally

$$\mathcal{U}^i_0 \tilde{=} \frac{1}{4}|\psi_0|^2(\underline{\partial}_j \underline{h_{ji}} - \underline{\partial}_i \underline{h_{jj}}).$$

From which the theorem follows. ■

To study the positivity of the mass m one defines a vector $\mathcal{U}^i$ on $\mathbf{M}^{n+1}$ such that the integrals on S^{n-1}_r of $\mathcal{U}^i$ and $\mathcal{U}^i_0$ have the same limit when $r \to \infty$.

The Stokes formula applied to the integral of the divergence of $\mathcal{U}^i$ will give information on this limit.

We set

$$\mathcal{U}^i := \mathcal{R}_e\{\tilde{\psi}(\eta^{ij}D_j\psi - \gamma^i\gamma^j D_j\psi)\}. \tag{6.3}$$

Lemma 3 *On an A.E manifold (M^n, g) it holds that*

1.

$$D_i\mathcal{U}^i \geq 0 \quad \textit{if} \;\; R \geq 0 \qquad \textit{and} \quad \mathcal{D}\psi = 0.$$

2. If $\psi = \psi_0 + \psi_1$ with $\underline{\partial_i}\psi_0 = 0$ in Ω_I and $\psi_1 \in H_{s,\delta}$ then in Ω_I

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{U}_0^i n_i \mu_{\bar{g}} = \lim_{r\to\infty}\int_{S_r^n} \mathcal{U}^i n_i \mu_{\bar{g}}. \tag{6.4}$$

Proof. 1. By elementary computation, using $D_i\tilde{\psi} = \widetilde{D_i\psi}$ and the Lichnerowicz identity one finds

$$D_i\mathcal{U}^i \equiv |D\psi|^2 - |\mathcal{D}\psi|^2 + \frac{1}{4}R|\psi|^2 \tag{6.5}$$

Therefore $D_i\mathcal{U}^i \geq 0$ if $R \geq 0$ and ψ satisfies the equation $\mathcal{D}\psi = 0$.

2. To study the limit of the integral on S_r^{n-1} of $\mathcal{U}^i$ when $\psi = \psi_0 + \psi_1$ we write

$$\mathcal{U}^i \equiv \mathcal{U}_0^i + \mathcal{V}^i, \tag{6.6}$$

$$\mathcal{V}^i \equiv \frac{1}{2}\mathcal{R}_e\{\tilde{\psi}_0[\gamma^j, \gamma^i]D_j\psi_1 + \tilde{\psi}_1[\gamma^j, \gamma^i]D_j\psi)\} \tag{6.7}$$

Then

$$D_i\mathcal{U}^i = D_i\mathcal{U}_0^i + D_i\mathcal{V}^i$$

Embedding and multiplication properties of Sobolev spaces give

$$\tilde{\psi}_1[\gamma^j, \gamma^i]D_j\psi) \in H_{s,\delta} \times \{C_{n-1}^1 \cap H_{s-1,\delta+1}\} \subset H_{s-1,2\delta+1+\frac{n}{2}} \subset C_\beta^0$$

$$\beta < 2\delta + 1 + \frac{n}{2} + \frac{n}{2} < 2n - 3, \quad \text{hence} \quad \beta > n-1 \quad \text{if} \quad n > 2.$$

To estimate the other term one remarks that

$$D_i\{\tilde{\psi}_0[\gamma^j,\gamma^i]D_j\psi_1\} \equiv D_iD_j\{\tilde{\psi}_0[\gamma^j,\gamma^i]\psi_1\} - D_i\{D_j\tilde{\psi}_0[\gamma^j,\gamma^i]\psi_1\} \tag{6.8}$$

the first parenthesis is an antisymmetric 2-tensor hence its double divergence D_iD_j is identically zero. The second parenthesis is

$$D_j\tilde{\psi}_0[\gamma^j,\gamma^i]\psi_1 \in C^1_{n-1} \times H_{s,\delta} \subset C^1_\beta.$$

One deduces from the Stokes formula that

$$\int_{M^n_r} D_i\mathcal{U}^i\mu_g = \int_{S^{n-1}_r} \mathcal{U}^i n_i \mu_{\bar{g}} \equiv \int_{S^{n-1}_r} (\mathcal{U}^i_0 + \mathcal{V}^i) n_i \mu_{\bar{g}}$$

The fall off properties found for $\mathcal{V}^i$ imply

$$\lim_{r\to\infty} \int_{S^{n-1}_r} \mathcal{U}^i n_i \mu_{\bar{g}} = \lim_{r\to\infty} \int_{S^{n-1}_r} \mathcal{U}^i_0 n_i \mu_{\bar{g}}. \tag{6.9}$$

■

Lemma 4 *If (M^n, g) is A.E. $R \geq 0$ and ψ_0 is a smooth spinor constant in Ω_I and zero in the other ends there exists on M^n a spinor $\psi \equiv \psi_0 + \psi_1$, such that $\mathcal{D}\psi = 0$, $\psi_1 \in H_{s,\delta}$.*

Proof. The hypotheses made on ψ_0 show that $\mathcal{D}\psi_0 \in H_{s-1,\delta+1}$. Theorem 1 implies the existence of ψ_1. ■

The lemmas imply that $m \geq 0$ if $R \geq 0$, that is if (M^n, g) is a maximal submanifold of $(\mathbf{M}^{n+1}, \mathbf{g})$. We will now lift this restriction, proving moreover that $m \geq |p|$.

7 Positive energy.

We assume the extrinsic curvature K of our A.E (M^n, g) slice to be in $H_{s,\delta} \cap C^0_{n-1}$. We then say that (M^n, g, K) is A.E. Inspired by 2.4 we define a real vector $\mathcal{P}$ on (M^n, g) by[1]

$$\mathcal{P}^i := \frac{1}{2}\tilde{\psi}\gamma_h P^{ih}\psi \equiv \frac{1}{2}\tilde{\psi}(\gamma_h K^{ih} - \gamma^i\gamma^j\gamma^h K_{jh})\psi, \quad P^{ih} = K^{ih} - \delta^{ih}\mathrm{trK}$$

If ψ_0 is as before a smooth spinor constant in one end of M^n and zero in the other ends and ψ is a spinor on M^n such that $\psi - \psi_0 \in C^0_\beta$, $\beta > 0$, then

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{P}^i n_i \mu_{\bar{g}} = \frac{1}{2}\tilde{\psi}_0\gamma_h p^h\psi_0, \quad p^h := \lim_{r\to\infty}\int_{S_r^{n-1}} P^{ih} n_i \mu_{\bar{g}}. \tag{7.1}$$

It is elementary to check using the properties of the $\gamma's$ that $\gamma_h p^h$ is an hermitian matrix with eigenvalues $\pm|p|$. If we choose for ψ_0 an eigenvector of the eigenvalue $-|p|$ we have

$$\tilde{\psi}_0\gamma_h p^h\psi_0 = -|\psi_0|^2\ |p|. \tag{7.2}$$

To estimate the limit 7.1. we use again the Stokes formula, with

$$D_i\mathcal{P}^i \equiv \frac{1}{2}D_i(\tilde{\psi}\gamma_h P^{ih}\psi) \equiv \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)P^{ih} + \frac{1}{2}\tilde{\psi}\gamma_h\psi D_i P^{ih}. \tag{7.3}$$

The momentum constraint gives

$$D_i P^{ih} = -T^h_0.$$

On the other hand, the Lichnerowicz identity together with the Hamiltonian constraint implies that

$$D_i\mathcal{U}^i \equiv |D\psi|^2 - |\mathcal{D}\psi|^2 + (\frac{1}{2}T_{00} + \frac{1}{4}|K|^2 - \frac{1}{4}|\mathrm{tr}K|^2)|\psi|^2. \tag{7.4}$$

One introduces the notations

$$\nabla_i\psi := D_i\psi + \frac{1}{2}\gamma^h K_{ih}\psi, \quad \not\nabla = \gamma^i\nabla_i \tag{7.5}$$

[1] Remark that we do not introduce a matrix γ_0

Elementary computation using $D_i\eta^{hj} \equiv 0$, $D_i\gamma^h \equiv 0$ gives

$$|\nabla\psi|^2 := \eta^{ij}\widetilde{\nabla_i\psi}\nabla_j\psi \equiv |D\psi|^2 + \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)K^{ih} + \frac{1}{4}|K|^2|\psi|^2 \qquad (7.6)$$

The identity 7.5. can therefore be written after simplification

$$D_i\mathcal{U}^i \equiv |\nabla\psi|^2 - |\mathcal{D}\psi|^2 + (\frac{1}{2}T_{00} - \frac{1}{4}|\mathrm{tr}K|^2)|\psi|^2 - \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)K^{ih}). \qquad (7.7)$$

We deduce from the definition

$$|\not\nabla\psi|^2 \equiv |\mathcal{D}\psi|^2 + \frac{1}{2}D_i(\widetilde{\psi}\gamma^i\psi)\mathrm{tr}K + \frac{1}{4}|\mathrm{tr}K|^2|\psi|^2$$

which gives

$$D_i\mathcal{U}^i \equiv |\nabla\psi|^2 - |\not\nabla\psi|^2 + \frac{1}{2}T_{00}|\psi|^2 - \frac{1}{2}D_i(\tilde{\psi}\gamma^h\psi)P_{ih}. \qquad (7.8)$$

The identities 7.4 and 7.6. give the simplified identity

$$D_i(\mathcal{U}^i + \mathcal{P}^i) \equiv |\nabla\psi|^2 - |\not\nabla\psi|^2 + \mathcal{T} \qquad (7.9)$$

with

$$\mathcal{T} := \frac{1}{2}(T_{00}|\psi|^2 - \tilde{\psi}\gamma^h\psi T_{0h}) \qquad (7.10)$$

with $\mathcal{T} \geq 0$ under the dominant energy condition, because

$$|\tilde{\psi}\gamma^h\psi T_{0h}| \equiv |\psi|^2(\eta^{ih}T_{0i}T_{0h})^{\frac{1}{2}} \leq T_{00}|\psi|^2|.$$

We see that $D_i(\mathcal{U}^i + \mathcal{P}^i) \geq 0$ if $\not\nabla\psi = 0$.

Lemma 5 *Under the hypotheses made on* (M^n, g, K) *and* ψ_0, *the equation* $\not\nabla\psi = 0$ *has a solution* $\psi \equiv \psi_0 + \psi_1$, $\psi_1 \in H_{s,\delta}$.

Proof. The operator $\not\nabla$ has the same principal part as $\mathcal{D}$, therefore is also elliptic. It maps $H_{s,\delta}$ into $H_{s-1,\delta+1}$. The equation $\not\nabla\psi_1 = -\not\nabla\psi_0 \in H_{s-1,\delta+1}$ has one and only one solution if $\not\nabla$ is injective on $H_{s,\delta}$. To show injectivity we remark that the identity 7.7 was established without restriction on ψ, starting from the definitions of $\mathcal{U}^i$ and $\mathcal{P}^i$. We make $\psi = \psi_1$ in 7.7 and integrate it on M^n, the fall off of ψ_1 implies that the divergence gives no contribution, the equation $|\nabla\psi_1|^2 = 0$ implies therefore that on M^n, if $\mathcal{T} \geq 0$

$$|\nabla\psi_1|^2 = 0, \quad \text{i.e.} \quad D_i\psi_1 + \frac{1}{2}\gamma^h K_{ih}\psi_1 = 0, \quad \text{with} \quad \gamma^h K_{ih} \in H_{s-1,\delta+1}$$

The Poincaré inequality in weighted Hilbert spaces leads to $\psi_1 = 0$ if $\psi_1 \in H_{s,\delta}$, $s > \frac{n}{2} + 1$ and $-2 + \frac{n}{2} > \delta > -\frac{n}{2}$. ■

We arrive at the final theorem

Theorem 6 *If an Einsteinian spacetime satisfies the dominant energy condition, the energy momentum vector $E^0 = m$, $E^i = p^i$ of an A.E. slice (M^n, g, K) satisfies the inequality*

$$m \geq |p|.$$

Proof. We have proved the result by considering one end. It holds in the case of several ends by adding their enegy momentum vectors. ■

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I thank Piotr Chrusciel for communicating to me a text of his 2010 lectures in Krakow. A long list of references can be found there. For references prior to 1983 one can consult my survey on positive energy theorems for les Houches 1983 school reproduced in Y.Choquet-Bruhat "General Relativity and the Einstein equations", Oxford University press 2009.