Singularity formation in solutions of the Einstein-Vlasov system

(based on joint work with Juan Velázquez, arXiv:1009.2596))

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Outline of the talk

- Background: gravitational collapse and collisionless matter
- Self-similarity and reduction to an ODE problem
- Analysis of the ODE system: a shooting argument
- Prospects for further progress

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Gravitational collapse

- Dynamics of solutions of the Einstein-matter equations
- Initial data localized in space
- What is the nature of the corresponding time evolution?
- Are there singularities?
- What are their qualitative properties?
- Naked singularities, (weak) cosmic censorship?
- Spherical symmetry is difficult enough

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Collisionless matter 1

- A common matter model is dust (fluid with zero pressure)
- Develops singularities in the absence of gravity
- A more regular alternative is collisionless matter
- Described by Vlasov equation
- Basic unknown is a real-valued function $f(t, x^a, v^b)$
- Density of particles with given position and velocity at time t
- Dust corresponds to a distributional solution
- $f(t, x^a, v^b) = (1 + \delta_{cd} u^c u^d)^{1/2} \rho(t, x^a) \delta(v^b - u^b(t, x^a))$

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Collisionless matter 2

- $T^{\alpha\beta}(t, x^a) = \int f(t, x^a, v^b) v^\alpha v^\beta (v^0)^{-1} dv$
- $v^0 = \sqrt{1 + \delta_{ab} v^a v^b}$
- For comparison $T^{\alpha\beta} = \rho u^\alpha u^\beta$ in case of dust
- Dust often forms naked singularities
- Smooth solutions of Einstein-Vlasov not known to form naked singularities
- Do they ever do so? Open question
- Major efforts have produced only limited results

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Collionless matter 3: interpolation

- Try to interpolate between dust and smooth Vlasov
- For dust support of f has codimension 3
- Einstein clusters (Einstein (1939)) have codimension 2, static
- Dynamical generalization (Datta (1970), Bondi (1971))
- As bad as dust
- We construct solutions with codimension 1

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Reduction procedure

- Go to massless particles
- Assume self-similarity
- Support is an appropriate hypersurface
- Everything can be reduced to an ODE problem
- Cf. work of Martín García and Gundlach (2002) (partly numerical)

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The ODE problem 1

- System of four ODE for unknowns (Q_1, Q_2, G, Z)
- Two parameters $(y_0, \theta) \in (0, \infty)$
- Point P_0 determined by the application
- Stationary point P_1 depending on (y_0, θ)
- Main result: for $y_0 > 0$ sufficiently small there exists θ such that for the given values of the parameters the solution starting at P_0 tends to P_1 as the independent variable tends to infinity

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The ODE problem 2

- P_1 is hyperbolic for any values of the parameters
- One positive and three negative eigenvalues
- Three-dimensional stable manifold $\mathcal{M}_\theta$
- Aim is to find θ so that $P_0 \in \mathcal{M}_\theta$
- Intuitive picture

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The limiting case $y_0 = 0$

- The system extends analytically to $y_0 = 0$
- For $y_0 = 0$ dependence on Q_i via $Q_1^2 + Q_2^2$
- Let $Q = \sqrt{\frac{Q_1^2+Q_2^2}{2}}$
- The limiting system is

$$\frac{dQ}{d\chi} = -2GQZ$$
$$\frac{dG}{d\chi} = 2G\left[Z(1-G) - \theta(1+Z^2)^{\frac{3}{2}}Q^2\right]$$
$$\frac{dZ}{d\chi} = \left(3G - 1 - \theta(1+Z^2)^{\frac{1}{2}}ZQ^2\right)(Z^2+1)$$

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Scaling invariance in the limit

- Invariant under simultaneous scaling of Q and θ. Let $q = \sqrt{\theta} Q$
- Scaling moves point P_0 with coordinates $(1, 1, 0)$
- Find value of q_0 such that the solution starting at $(q_0, 1, 0)$ tends to $(0, \frac{1}{3}, 0)$.
- Then perturb this
- Application of implicit function theorem
- Hypothesis follows from scaling property of limiting system

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The shooting argument 1

- Shooting arguments are common in numerical work
- Here this idea is used for a proof
- Cf. proof of Bizoń and Wasserman for critical solution
- For q_0 small $\lim_{\chi\to\chi^*} Z(\chi) = -\infty$
- For q_0 large $\lim_{\chi\to\chi^*} Z(\chi) = +\infty$
- Let U_1 be the set of q_0 leading to the first possibility
- Let U_2 be the set of q_0 leading to the second possibility
- Both are non-empty

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The shooting argument 2

- U_1 and U_2 are open
- By connectedness of the interval $(0, \infty)$ there is some q_0^* in the complement of $U_1 \cup U_2$
- The solution correponding to q_0^* has Z positive and bounded
- This solution has the required asymptotics
- $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ blows up as the origin of scaling is approached

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- The solution correponding to q_0^* has Z positive and bounded
- This solution has the required asymptotics
- $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ blows up as the origin of scaling is approached

Proof of the shooting argument 1

- If Z becomes zero at some time then $G \leq \frac{1}{3}$
- If Z becomes zero at some time it becomes negative
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Proof of the shooting argument 2

- Suppose that $Z \to \infty$
- Step 1: $\lim_{\chi \to \chi^*} q(\chi) = 0$
- Step 2: $\lim_{\chi \to \chi^*} (Zq)(\chi) = 0$
- The system can be written in the non-autonomous form

$$\frac{dG}{d\chi} = 2G[Z(1-G) - (Z+1)\delta_2(\chi)]$$
$$\frac{dZ}{d\chi} = (3G - 1 - \delta_1(\chi))(Z^2 + 1)$$

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- If a solution $(\bar{q}, \bar{G}, \bar{Z})$ starts close enough to (q, G, Z) then at late times the following set of inequalities is preserved for some $\delta > 0$:
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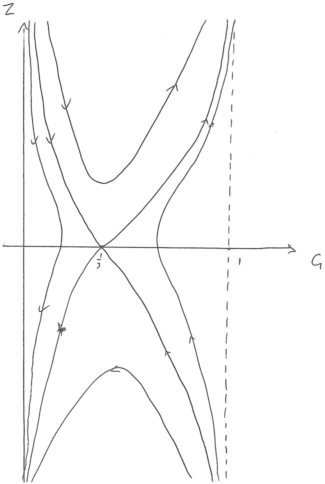
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Conclusions and open questions

- It has been shown that non-smooth self-similar solutions of the Einstein-Vlasov system with massless particles exist
- Can these solutions be smoothed?
- Can they be cut off so as to become asymptotically flat?
- Can the massless particles be replaced by massive ones?
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- 3. stream of infalling particles in Schwarzschild
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