Brunn–Minkowski Theory in Minkowski space-times

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Convex bodies

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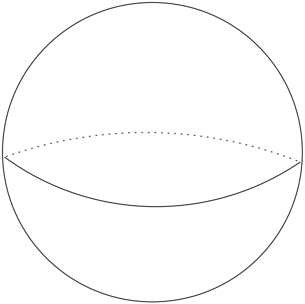

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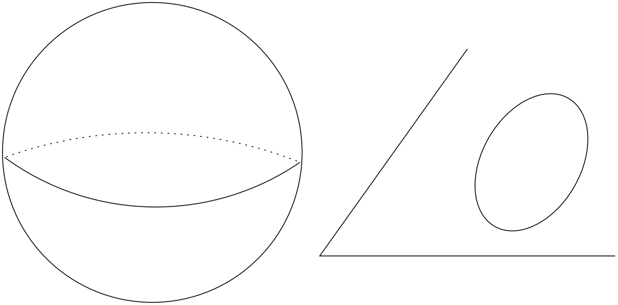

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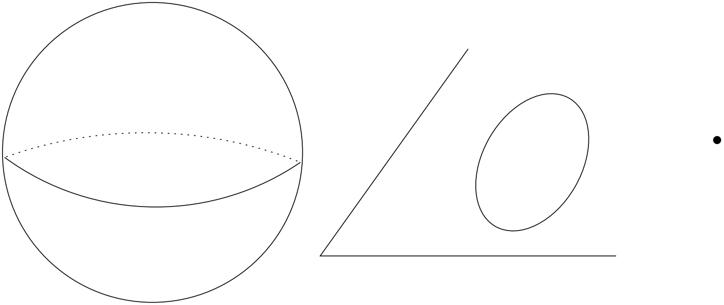

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The set of convex bodies of $\mathbb{R}^{d+1}$ is endowed with the volume functional V_E, and with an addition (Minkowski sum)

$$K + K' = \{x + x' | x \in K, x' \in K'\}.$$

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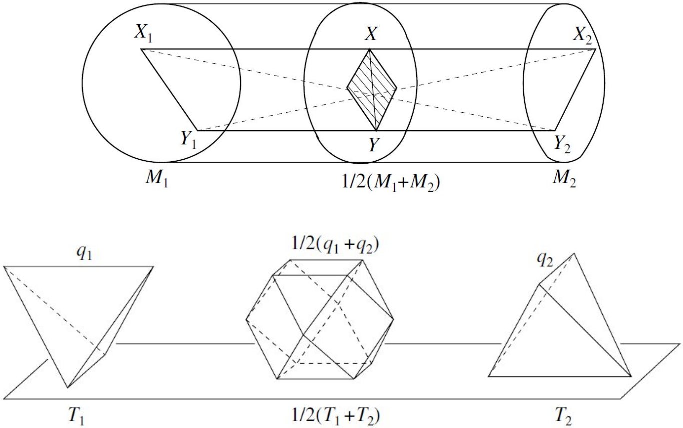

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Initiated by Minkowski in the beginning of the last century.
Developed by A.D. Alexandrov in the '30s.
Rolf Schneider *Convex Bodies: The Brunn–Minkowski Theory* 1993.

A fundamental result is

Brunn–Minkowski inequality

$$V_E((1-t)K + tK')^{\frac{1}{d+1}} \geq (1-t)V_E(K)^{\frac{1}{d+1}} + tV_E(K')^{\frac{1}{d+1}}$$

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i.e. $V_E^{\frac{1}{d+1}}$ is concave. This comes from the log-concavity

$$\ln(V_E((1-t)K + tK')) \geq (1-t)\ln V_E(K) + t\ln V_E(K')$$

$$\Leftrightarrow V_E((1-t)K + tK') \geq V_E(K)^{1-t}V_E(K')^t$$

and the homogeneity of the volume

$$V_E(tK) = t^{d+1}V_E(K).$$

Fuchsian convex bodies

The Minkowski space is $\mathbb{R}^{d+1}$ endowed with

$$\langle x, y \rangle_- = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1}$$

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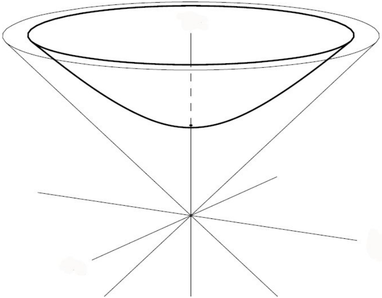
It is not one of the Minkowski spaces (normed space of finite volume) in which convex bodies are usually studied.

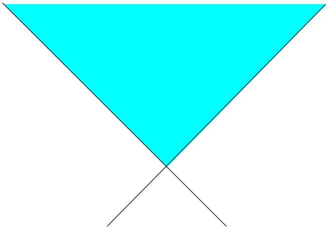
The hyperbolic space is a pseudo sphere in the Minkowski space

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} | \langle x, x \rangle_- = -1, x_{d+1} > 0\}$$



Fuchsian convex bodies

$\mathcal{F}$ will be the interior of the future cone



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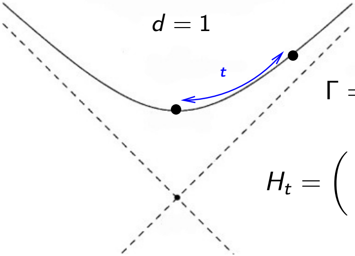

$$\Gamma = \langle H_t \rangle$$

$$H_t = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

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- strictly contained in $\mathcal{F}$ $\mathcal{F}$
- $\Gamma K = K$ (setwise)

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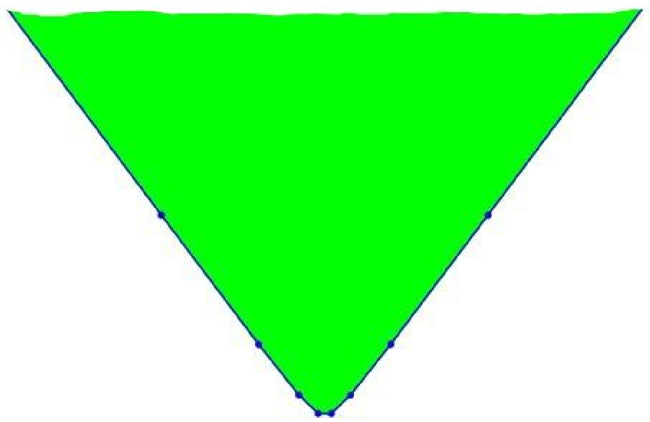
If Γ is a Fuchsian group and $x \in \mathbb{H}^d \subset \mathbb{R}^{d+1}$, the convex hull (in $\mathbb{R}^{d+1}$) of Γx is a Γ convex body ("convex hull construction"). This family of examples was introduced in the '80s, in papers of Robert Penner, with Bob Epstein and Marjatta Näätänen.

Fuchsian convex bodies



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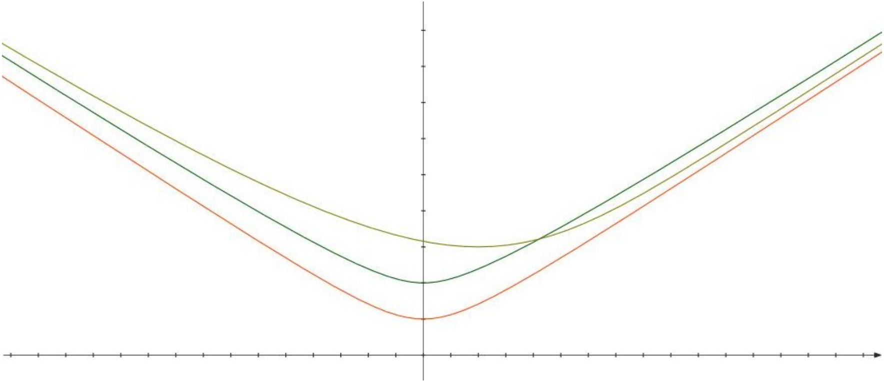

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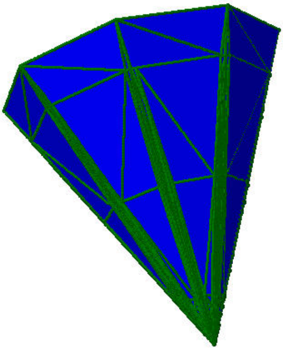

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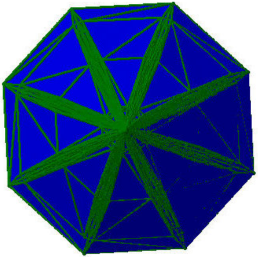

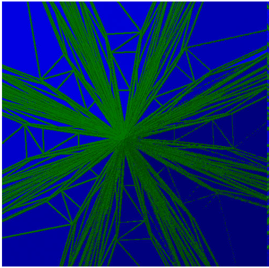

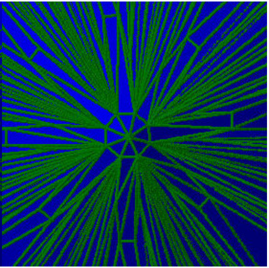

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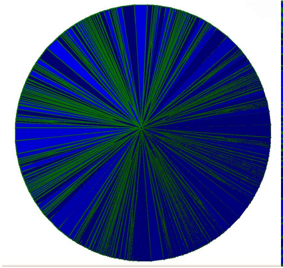

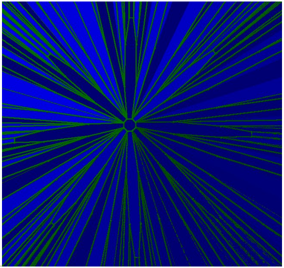

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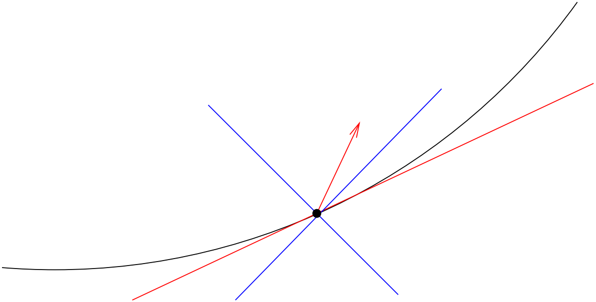
In the same way, the boundary of a Fuchsian convex body is homeomorphic to $\mathbb{H}^d$ by the projection along the half-lines from the origin. And $\partial K/\Gamma \simeq \mathbb{H}^d/\Gamma$.

Fuchsian convex bodies

Moreover Fuchsian convex bodies are lying in the future part of their support hyperplane, which are space-like.



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The translate of a Fuchsian convex body is not a Fuchsian convex body.

Fuchsian convex bodies

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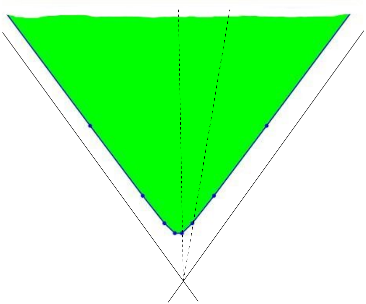
1. the sum of convex sets is a convex set,
2. the sum is a closed set in this case,
3. the sum of two future time like vectors is future time like,
4. the elements of Γ are linear.

The *volume* of a Γ convex body K is the volume of

$$(\mathcal{F} \setminus K)/\Gamma.$$



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that implies

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Fuchsian convex bodies

The convexity is the genuine analogue of the log-concavity in the following sense

$$f \text{ concave} \Longrightarrow f \text{ log-concave} \overset{f(tx)=t^n f(x)}{\Longrightarrow} f^{1/n} \text{ concave}$$
$$f \text{ log convex} \Longrightarrow f \text{ convex} \underset{f(tx)=t^n f(x)}{\Longrightarrow} f^{1/n} \text{ convex}$$

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The proofs are lying on the ones of the corresponding Euclidean results, because the boundary of a regular (resp. polyhedral) Fuchsian convex body is a Riemannian manifold (resp. a set of convex polytopes).

Polyhedral case

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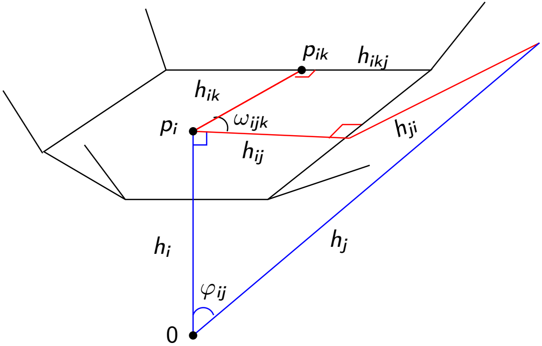
Those polyhedra are determined by vectors

$$h = (h_1, \dots, h_n)$$

whose entries are the “distances” between the origin and its orthogonal projections onto the hyperplanes containing the facets in a fundamental domain.

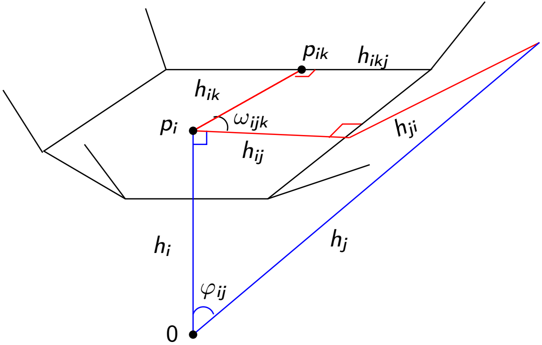
Polyhedral case

$$(h_i = \sqrt{-\langle p_i, p_i \rangle_-})$$



Polyhedral case

$$h_{ij} = -\frac{h_j - h_i \cosh \varphi_{ij}}{\sinh \varphi_{ij}} \qquad h_{ikj} = \frac{h_{ij} - h_{ik} \cos \omega_{ijk}}{\sin \omega_{ijk}}$$



Polyhedral case

The volume of P is

$$\frac{1}{d+1}\sum h_i A(F_i)$$

with

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- $A(F_i)$ is the area (the Euclidean d volume) of the facet supported by h_i.

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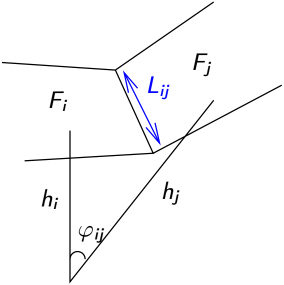
$$D_h^2 V(X, Y) = \langle X, D_h A(Y)\rangle$$

Polyhedral case

We can compute (in a simplified case)

$$\frac{\partial A(F_i)}{\partial h_j} = -\frac{L_{ij}}{\sinh \varphi_{ij}}$$

$$\frac{\partial A(F_i)}{\partial h_i} = \sum \cosh \varphi_{ik} \frac{L_{ik}}{\sinh \varphi_{ik}}$$



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As $\cosh \varphi_{ik} > 1$, we get

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In particular the volume is strictly convex.

$\square$

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Actually, one gets a more general result.

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If we perform a little parallel move to a facet of P, we get a simple Γ convex polyhedron, with the same combinatorics than P.

Moreover, the facets of P are simple convex polytopes.

Polyhedral case

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(volume of polyhedra is a polynomial in h_i whose coefficients depends on the combinatorics.)

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By identifying the hyperplanes containing parallel facets with $\mathbb{R}^d$, parallel facets are convex polytopes with the same property.
(volume of polyhedra is a polynomial in h_i whose coefficients depends on the combinatorics.)
In this case, it is well-known that there exists a unique d-linear form on $(\mathbb{R}^n)^d$, symmetric in each variable, the (Euclidean) *mixed volume*

$$V_E(\cdot, \dots, \cdot)$$

such that, if h is the vector made of the support numbers of a convex polytope P,

$$V_E(h, \dots, h) = V_E(P)$$

.

Polyhedral case

From the definition of the volume of the Γ convex polyhedra, we get a mixed volume ($d + 1$ linear form)

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By a deformation argument, we get that, for any $h^3, \dots, h^{d+1}$ support vectors of (suitable) polyhedra

$$V(\cdot, \cdot, h^3, \dots, h^{d+1})$$

is positive definite.

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Cauchy–Schwarz gives:

Reversed Alexandrov–Fenchel inequality

$V(P_1, P_2, P_3, \ldots, P_{d+1})^2 \leq$
$V(P_1, P_1, P_3, \ldots, P_{d+1}) V(P_2, P_2, P_3, \ldots, P_{d+1})$

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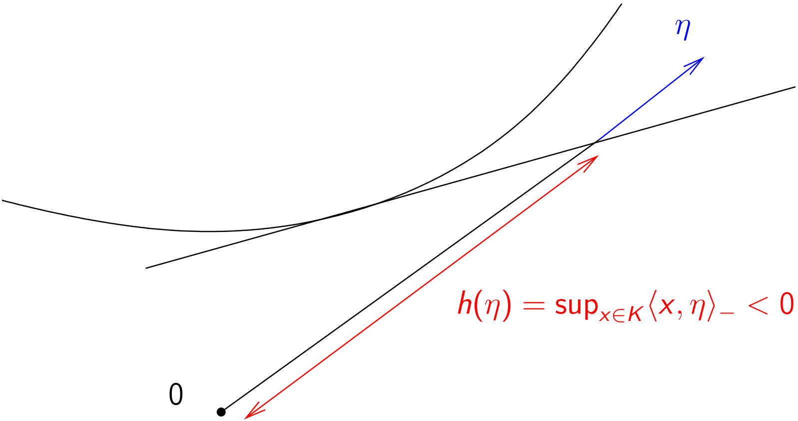
Euclidean case : mixed volume with signature $(+, -, \dots, -)$, reversed Cauchy–Schwarz $\Rightarrow$ Alexandrov–Fenchel
Minkowski case : mixed volume with signature $(+, \dots, +)$, Cauchy–Schwarz $\Rightarrow$ reversed Alexandrov–Fenchel

Regular case

We consider now strictly convex Γ convex bodies with C^2 boundary

Regular case

(As any Γ convex body) they are determined by their *support function* $h : \mathbb{H}^d/\Gamma \to \mathbb{R}_-$, which is C^2.



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$$V(h) = -\frac{1}{d+1}\int_{\mathbb{H}^d/\Gamma} h\kappa^{-1}(h)\mathrm{d}(\mathbb{H}^d/\Gamma) = -\frac{1}{d+1}\left(h, \kappa^{-1}(h)\right)$$

$(\cdot,\cdot)$ scalar product on $L^2(\mathbb{H}^d/\Gamma)$.

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$(\cdot,\cdot)$ scalar product on $L^2(\mathbb{H}^d/\Gamma)$.

Remark: a computation gives

$$\kappa^{-1}(h) = \det((\nabla^2 h)_{ij} - h\delta_{ij})$$

in an orthonormal frame on $\mathbb{H}^d/\Gamma$, and the determinant on the space of symmetric matrix has a polar form, that allows to define a mixed volume in the regular case.

Regular case

$$V(h) = -\frac{1}{d+1}\big(h, \kappa^{-1}(h)\big)$$

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$$D_h^2 V(X, Y) = \big(X, D_h \kappa^{-1}(Y)\big)$$

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$$D_h\kappa^{-1}(X) = \kappa^{-1}(h)\sum_{i=1}^d r_i^{-1}(h)D_h r_i(X)$$

as $r_i(h_0) = 1$,

$$D_{h_0}\kappa^{-1}(X) = \sum_{i=1}^d D_{h_0} r_i(X).$$

Regular case

On the other hand, in the Euclidean case we have

$$\triangle_{\mathbb{R}^{d+1}} H = \triangle_{\mathbb{S}^d} h + dh$$

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on the hyperbolic space.
In an adapted frame on $\mathbb{H}^d$, $\partial^2 H/\partial x_i^2 = r_i$ and $\partial^2 H/\partial \rho^2 = 0$ in the radial direction. Then

$$r_1 + \cdots + r_d = \triangle_{\mathbb{H}^d} h - dh$$

$$\sum_{i=1}^{d} D_{h_0} r_i^{-1}(X) = \triangle_{\mathbb{H}^d} X - dX$$

Regular case

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$$D_{h_0}\kappa^{-1}(X) = \triangle_{\mathbb{H}^d/\Gamma}X - dX$$

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$$D^2_{h_0}V(X,X) = -\big(X, D_{h_0}\kappa^{-1}(X)\big) = d\big(X,X\big) - \big(\triangle_{\mathbb{H}^d/\Gamma}X, X\big) > 0$$

$\square$

and

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Remark: In the Euclidean case we get

$$d\big(X, X\big) + \big(\triangle_{\mathbb{S}^d} X, X\big)$$

Regular case

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$$D^2_{h_0}V(X,X) = -\big(X, D_{h_0}\kappa^{-1}(X)\big) = d\big(X,X\big) - \big(\triangle_{\mathbb{H}^d/\Gamma}X, X\big) > 0$$

□

Remark: In the Euclidean case we get

$$d\big(X,X\big) + \big(\triangle_{\mathbb{S}^d}X, X\big)$$

one positive direction and a kernel with dimension $d+1$.

Consequence: isoperimetric inequality

From

Reversed Brunn–Minkowski inequality

$$V((1-t)K + tK')^{\frac{1}{d+1}} \leq (1-t)V(K)^{\frac{1}{d+1}} + tV(K')^{\frac{1}{d+1}}$$

we can obtain

Reversed Minkowski inequality

$$V(K_1, K_2, \ldots, K_2)^{d+1} \leq V(K_2)^d V(K_1)$$

Consequence: isoperimetric inequality

Taking for K_2 the convex side B of $\mathbb{H}^d$ and reordering the terms, we get

Isoperimetric inequality

$\left(\frac{S(K)}{S(B)}\right)^{d+1} \leq \left(\frac{V(K)}{V(B)}\right)^{d}$

with

$$S(K) = \lim_{\epsilon \to 0^+} \frac{V(K + \epsilon B) - V(K)}{\epsilon} = (d+1)V(B, K, \ldots, K)$$

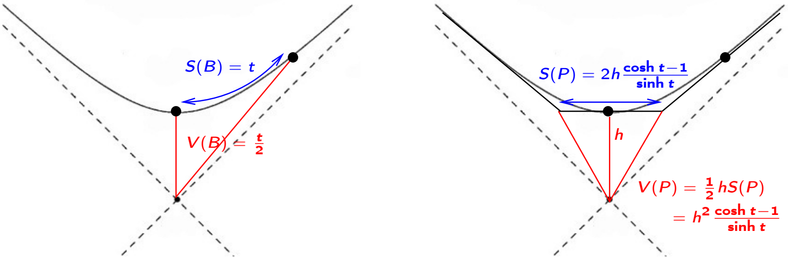
the *area* (of the boundary) of K.

Consequence: isoperimetric inequality

Isoperimetric inequality

$$\left(\frac{S(K)}{S(B)}\right)^{d+1} \leq \left(\frac{V(K)}{V(B)}\right)^{d}$$

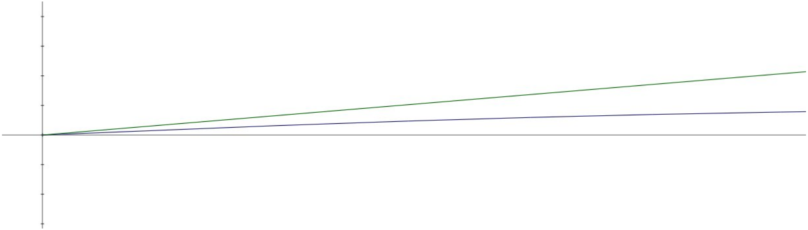
Example in $d = 1$



Consequence: isoperimetric inequality

The isoperimetric inequality becomes

$$2\frac{\cosh t - 1}{\sinh t} \leq t$$



Fuchsian geometry

Convex Fuchsian hypersurfaces are an extension of the compact convex hypersurfaces of the Euclidean space as natural as the study of compact hyperbolic manifolds with respect to the sphere.

Fuchsian geometry

Convex Fuchsian hypersurfaces are an extension of the compact convex hypersurfaces of the Euclidean space as natural as the study of compact hyperbolic manifolds with respect to the sphere.

In this way, already existing results can be put into the perspective of a “classical Fuchsian differential geometry”.

Vladimir Oliker and Udo Simon, 1983.

Minkowski Theorem (regular)

Let Γ be a Fuchsian group and $f : \mathbb{H}^d \to \mathbb{R}_+$ be a C^∞ positive Γ-invariant function. Then there exists a unique regular Γ-convex body with Gauss curvature f.

Polyhedral analogue, F. 2011.

François Labourie and Jean-Marc Schlenker, 2000.

Weyl problem ($d = 2$)

Let g be a (Riemannian) metric with negative curvature on a compact surface of genus > 1. Then there exists a unique (up to congruences) Fuchsian convex surface (S, Γ) such that S/Γ is isometric to g.

Polyhedral analogue (Alexandrov theorem), F. 2011.

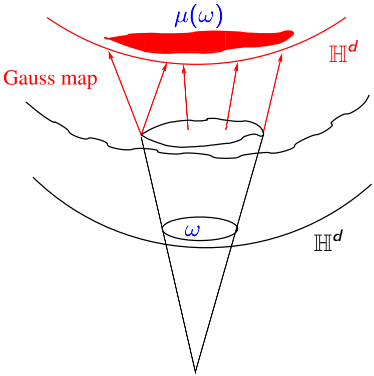
Polyhedral infinitesimal rigidity (Dehn Theorem), Jean-Marc Schlenker, 2007.

Fuchsian geometry

Jérôme Bertrand, 2011. (Polyhedral case $d = 2$ Igor Iskhakov 2000.)

Alexandrov prescribed curvature theorem (general case)

Let μ be a Borelian probability measure on a compact hyperbolic manifold $\mathbb{H}^d/\Gamma$. Then there exists a unique (up to homotheties) Γ convex body with μ as "Gauss curvature".



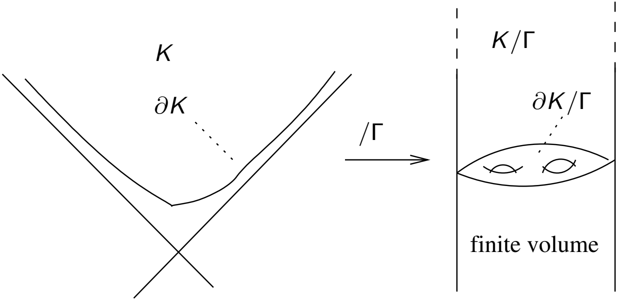
F.–Giona Veronelli, in preparation.

Christoffel Problem

Find necessary and sufficient conditions on a given function defined on a compact hyperbolic manifold to be the mean curvature radius (=sum of the radii of curvature) of a unique Fuchsian convex body.

Similar question for any symmetric function of the radii of curvature ("Christoffel–Minkowski problem").

The quotient of the interior of the future cone by Γ is a flat Lorentzian manifold, and the boundaries of Γ convex bodies give Cauchy surfaces.



It is the simplest example of flat "Globally hyperbolic Maximal Compact" spacetimes:

- they have a *Cauchy surface*: space-like hypersurface which meets exactly once any inextensible causal curve (globally hyperbolic), and which is compact (spatially compact)
- any isometric embedding into an analogue manifold is surjective (maximal).

Another way of saying this, is that we now consider isometries of the Minkowski space with a non trivial translation part.

Thierry Barbot, François Béguin and Abdelghani Zeghib, 2011.

Minkowski Theorem $d = 2$

Let Γ be a Fuchsian group, $f : \mathbb{H}^2 \to \mathbb{R}_+$ smooth and Γ invariant. Then there exists a space-like surface in the Minkowski space, strictly convex, such that f is its Gauss curvature.
Moreover, if $\overline{\Gamma}$ is a group of isometries of the Minkowski space which projects bijectively onto Γ, then there is a unique $\overline{\Gamma}$ invariant such surface.

Flat Lorentzian manifolds

The classical theory of convex bodies fits perfectly well to the Γ-invariant convex bodies case.
Study the Γ-invariant convex bodies with the same tools?

Two remarks:

1) K, K' $\overline{\Gamma}$-invariant, $K + K'$ is not $\overline{\Gamma}$-invariant.

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But for any $\lambda \in [0, 1]$, $\lambda K + (1 - \lambda)K'$ is $\overline{\Gamma}$-invariant.
So the question of the convexity of the volume has a meaning.

2) As *any* closed convex set, K is determined by its support function $H_K(\eta) = \sup_{x \in K} \langle x, \eta \rangle_-$. It is not Γ-invariant .

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The closure of the domain of dependance is also a closed convex set, it is determined by its support function H.
And $H_K - H$ is Γ-invariante. Give a function on $\mathbb{H}^d/\Gamma$.