Behaviour of vacuum Bianchi spacetimes close to their initial singularity

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Bianchi cosmological models : presentation

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Raisons d'être :

- natural **finite dimensional** class of spacetimes ;
- **BKL conjecture :** *generic spacetimes "behave like" spatially homogeneous spacetimes close to their initial singularity*.

Bianchi cosmological models : definitions

- A **Bianchi spacetime** is a spacetime (M, g) with

$$M \simeq I \times G \qquad\qquad g = -dt^2 + h_t$$

where $I = (t_-, t_+) \subset \mathbb{R}$,
G is 3-dimensional Lie group,
h_t is a left-invariant riemannian metric on G.

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- A **Bianchi spacetime** amounts to a one-parameter family of left-invariant metrics $(h_t)_{t \in I}$ on a 3-dimensional Lie group G.

Bianchi cosmological models : definitions

We will consider **vacuum type A** Bianchi spacetimes.

- **Type A :** G **is unimodular.**
- **Vacuum : Ric**$(g) = 0$.

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- **Type A :** G **is unimodular.**
- **Vacuum :** $\mathbf{Ric}(g) = 0$.

All the results/conjectures stated below *should* also hold when :

- G is **not unimodular** (type B Bianchi spacetimes).
- the matter content is a **non-tilted perfect fluid**.

Einstein equation

A Bianchi spacetime can be seen as a one-parameter family of left-invariant metrics $(h_t)_{t \in I}$ on a 3-dim Lie group G

$+$ The space of left-invariant metrics on G is finite-dimensional

$\Longrightarrow$ the Einstein equation $\mathrm{Ric}(g) = 0$ is a system of ODEs.

Einstein equation : coordinate choice

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- e_1, e_2, e_3 are tangent to $\{\cdot\} \times G$ and left-invariant ;
- $\nabla_{\frac{\partial}{\partial t}} e_i = 0$ for $i = 1, 2, 3$;
- (e_1, e_2, e_3) is orthonormal for h_t ;
- $[e_1, e_2] = n_3(t) e_3$;
  $[e_2, e_3] = n_1(t) e_1$;
  $[e_3, e_1] = n_2(t) e_2$;
- the second fundamental form of h_t is diagonal in (e_1, e_2, e_3).

Why ortho**normal** frames ?

- One studies the behavior of the **structure constants** n_1, n_2, n_3 instead of the behavior of **metric coefficients** $h_t(e_i, e_j)$;

- Easy to recover the coefficient of h_t in a constant frame : if (e'_1, e'_2, e'_3) is a constant frame, then $h_t(e'_i, e'_i) = \frac{1}{n_j n_k}$.

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- Key advantage: the various 3-dimensional Lie groups are treated altogether.

Variables

- The three structure constants $n_1(t)$, $n_2(t)$, $n_3(t)$;
- The three diagonal components $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$ of the traceless second fundamental form ;
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Actually, it is convenient to replace

- n_i and σ_i by $N_i = \dfrac{n_i}{\theta}$ and $\Sigma_i = \dfrac{\sigma_i}{\theta}$
- t by τ such that $\dfrac{d\tau}{dt} = -\dfrac{\theta}{3}$.

(Hubble renormalisation ; the equation for θ decouples).

The phase space

With these variables, the phase space $\mathcal{B}$ is a (non-compact) four dimensional submanifold in $\mathbb{R}^6$.

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$$\mathcal{B} = \left\{ (\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3) \in \mathbb{R}^6 \mid \Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \,,\ \Omega = 0 \right\}$$

where

$$\Omega = 6 - (\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \frac{1}{2}(N_1^2 + N_2^2 + N_3^2) - (N_1 N_2 + N_1 N_3 + N_2 N_3).$$

Wainwright-Hsu equations

$$\frac{d}{d\tau}\begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} (2-q)\Sigma_1 - R_1 \\ (2-q)\Sigma_2 - R_2 \\ (2-q)\Sigma_3 - R_3 \\ -(q+2\Sigma_1)N_1 \\ -(q+2\Sigma_2)N_2 \\ -(q+2\Sigma_3)N_3 \end{pmatrix}.$$

where

$$\begin{aligned} q &= \frac{1}{3}\left(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2\right) \\ R_i &= \frac{1}{3}\left(2N_i^2 - N_j^2 - N_k^2 + 2N_jN_k - N_iN_j - N_iN_k\right). \end{aligned}$$

Wainwright-Hsu equations

Let $X_{\mathcal{B}}$ be the vector field on $\mathcal{B}$ corresponding to this system of ODEs.

The vacuum type A Bianchi spacetimes can be seen as the orbits of $X_{\mathcal{B}}$.

The variables are choosen so that the "initial singularities of spacetimes" correspond to the "future asymptotic behaviours of the orbits of $X_{\mathcal{B}}$".

Dynamics of $X_{\mathcal{B}}$

Fundamental remark. — The classification of Lie algebras gives rise to an $X_{\mathcal{B}}$-invariant stratification of the phase space $\mathcal{B}$.

Bianchi classification

Name	N_1	N_2	N_3	$\mathfrak{g}$	dim. of the stratum
I	0	0	0	$\mathbb{R}^3$	1
II	$+$	0	0	heis_3	2
VI_0	$+$	$-$	0	$\mathrm{so}(1,1) \ltimes \mathbb{R}^2$	3
VII_0	$+$	$+$	0	$\mathrm{so}(2) \ltimes \mathbb{R}^2$	3
VIII	$+$	$+$	$-$	$\mathrm{sl}(2,\mathbb{R})$	4
IX	$+$	$+$	$+$	$\mathrm{so}(3,\mathbb{R})$	4

Type I models ($\mathfrak{g} = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- The subset of $\mathcal{B}$ corresponding to type I spacetimes is a euclidean circle : the *Kasner circle* $\mathcal{K}$.

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- Every point of $\mathcal{K}$ is a fixed point for the flow.

Type I spacetimes ($\mathfrak{g} = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)

- For every $p \in \mathcal{K}$, the derivative $DX_{\mathcal{B}}(p)$ has :
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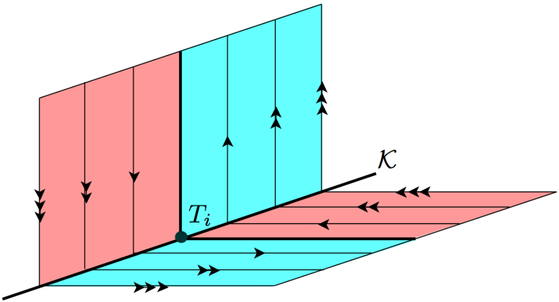
  - two distinct negative eignevalues,
  - a zero eigenvalue,
  - a positive eigenvalue.

- Except if p is one of the three special points T_1, T_1, T_3, in which case $DX_{\mathcal{B}}(p)$ has :
  - a negative eigenvalue,
  - a triple-zero eigenvalue.

Type I spacetimes ($\mathfrak{g} = \mathbb{R}^3$, $N_1 = N_2 = N_3 = 0$)



Type II spacetimes ($\mathfrak{g} = \mathrm{hein}_3$, one of the N_i's is non-zero)

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- The orbits on one ellipsoid "take off" from one third of $\mathcal{K}$, and "land on" the two other thirds.

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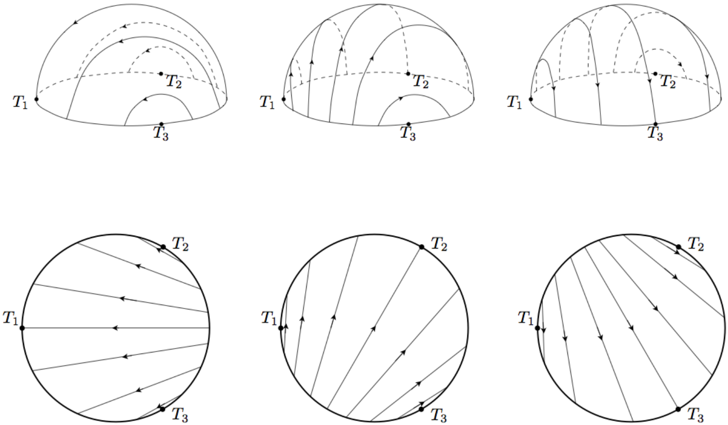

The Kasner map

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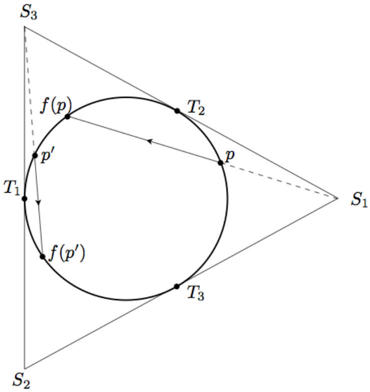
- We set $q = f(p)$. This defines a map $f : \mathcal{K} \longrightarrow \mathcal{K}$: the *Kasner map*.
- f maps the "asymptotic past state" of a type II spacetime to the "asymptotic future state of this spacetime.

The Kasner map



Type VIII or IX spacetimes ($\mathfrak{g} = \mathrm{sl}(2, \mathbb{R})$ or $\mathrm{so}(3, \mathbb{R})$, $N_1, N_2, N_3 \neq 0$)

From a naive viewpoint, iterating the Kasner map is a non-sense (it amounts to “concatenating” spacetimes).

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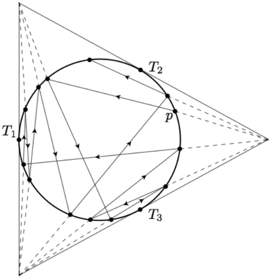
Nevertheless :

Vague conjecture (Belinskii, Khalatnikov, Lifschitz).
The dynamics of type VIII and IX spacetimes $\ll$ is driven $\gg$ by the dynamics of the Kanser map.

Dynamics of the Kasner map

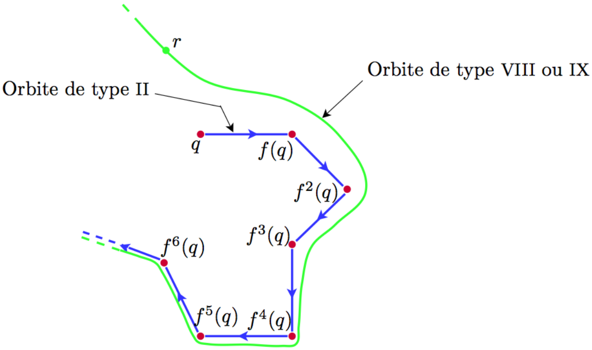


The dynamics of Kasner map f is chaotic but well-understood :

- f is topologically conjugated to $\theta \mapsto -2\theta$.
- it has a (unique) infinite invariant measure which is absolutely continuous with respect to the Lebesgue measure.
- The map induced on $\mathcal{K}/\mathfrak{S}_3$ is conjugated to the Gauss map.

Precise statements of the BKL conjecture

Let $q \in \mathcal{K}$. We seek for orbits of $X_{\mathcal{B}}$ which « shadow » the heteroclinic chain $q \to f(q) \to f^2(q) \to f^3(q) \to \ldots$.



These orbits are necessarily of type VIII or IX.

Precise statements of the BKL conjecture

Definition. The $X_{\mathcal{B}}$-orbit of a point r **shadows** the heteroclinic chain $q \to f(q) \to f^2(q) \to \dots$ if there exist $t_0 < t_1 < t_2 < \dots$ such that :

- $\mathrm{dist}(X_{\mathcal{B}}^{t_n}(r), f^n(q)) \underset{n\to\infty}{\longrightarrow} 0$;
- the Hausdorff distance between the orbit segment $\{X_{\mathcal{B}}^t(r)\ ;\ t_n \leq t \leq t_{n+1}\}$ and the type II orbit joining $f^n(q)$ to $f^{n+1}(q)$ goes to 0.

We denote by $W^s(q)$ the set of all such points r.

Results toward the BKL conjecture

Theorem (Béguin) There exists $k_0 \in \mathbb{N}$ with the following property. Consider $q \in \mathcal{K}$ such that the closure of the f-orbit of q does not contain any periodic orbit of period $\leq k_0$.

Then $W^s(q)$ is non-empty (it is a C^1 three-dimensional injectively immersed manifold).

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Proposition. The set of the points q satisfying the hypothesis of the theorem above is dense in $\mathcal{K}$, but has zero Lebesgue measure.

Results toward the BKL conjecture

Theorem (Georgi, Häterich, Liebscher, Webster). Consider a point $q \in \mathcal{K}$ which is periodic for f.

Then $W^s(q)$ is non-empty.

Results toward the BKL conjecture

The orbits of the Kasner map which satisfy the hypotheses of the previous theorems stay far from the Taub points T_1, T_2, T_3.

Results toward the BKL conjecture

Theorem (Reiterer, Trubowitz). There is a full Lebesgue measure subset E in $\mathcal{K}$ such that $W^s(q)$ is non-empty for $q \in E$.

Unfortunately, not much more precise than "non-empty".

Results toward the BKL conjecture

A point q is in E if and only if the "orbit of q does not approach to fast the Taub points T_1, T_2, T_3".

Precisely : the growth of the integers which appear in the continued fractions development of $u(q)$ at most polynomial.

Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity :

- For "many" vacuum Bianchi spacetimes, close to the initial singularity, the metric of each homogeneous slice is curved in only one direction.

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- But this direction oscillates in a complicated periodic or aperiodic way when approaching the singularity.

Dynamics of type VIII or IX orbits

Informal interpretation of the results. Close to the initial singularity :

- For "many" vacuum Bianchi spacetimes, close to the initial singularity, the metric of each homogeneous slice is curved in only one direction.
- But this direction oscillates in a complicated periodic or aperiodic way when approaching the singularity.
- The way this direction oscillates is sensitive to initial conditions.

An important by-product : asymptotic silence

A Bianchi spacetime is called **asymptotically silent** if "two different particles cannot have exchanged information arbitrarily close to the initial singularity".

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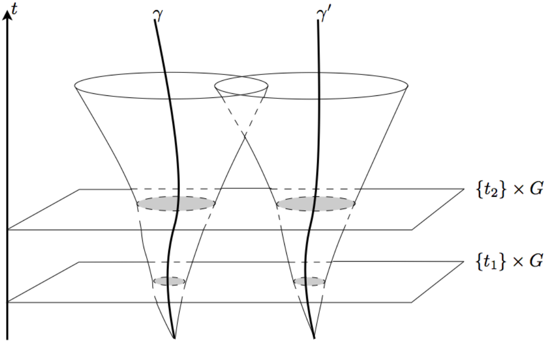
A Bianchi spacetime is called **asymptotically silent** if "two different particles cannot have exchanged information arbitrarily close to the initial singularity".

Formally : for every past inextendible timelike curve γ, the diameter of the set $J^+(\gamma) \cap (\{t\} \times G)$ goes to 0 as $t \to t_-$.



An important by-product : asymptotic silence

Theorem. For q as in one of the three preceding theorems, the orbits in $W^s(q)$ correspond to asymptotically silent spacetimes.

The proof uses the fact that :

- Type I and II spacetimes are asymptotically silent.
- Some estimates on the speed at which type VIII and IX spacetimes approach the set of type I and II spacetimes.

Further results on the BKL conjecture ?

Almost-theorem. There is a full Lebesgue measure subset E of $\mathcal{K}$ such that $W^s(q)$ is Lipschitz three-dimensional injectively immersed manifold for every $q \in E$.

Moreover, $E = \bigcup E_n$ with $\mathrm{Leb}(E_n) \to 1$, and $W^s(q)$ depends continuously on q when q ranges in E_n.

(**Loosely speaking :** for almost every orbit of the Kasner map, there are Bianchi spacetime with the "same asymptotic behaviour".)

Remark. $q \in E$ if and only if $a_n^2 \leq (a_1^3 + \cdots + a_{n-1}^3)$ for every n big enough, where $[a_1, a_2, \ldots]$ is the continued fraction development of $u(q)$.

Further results on the BKL conjecture ?

Conjecture. — The union of the manifolds $W^s(q)$ for $q \in E$ has positive Lebesgue measure in $\mathcal{B}$.

(**Loosely speaking :** many Bianchi spacetimes should behave asymptotically like an orbit of the Kasner map. Every asymptotic behaviour which is present in positive measure for the Kasner map is also present in positive measure for Bianchi spacetimes.)

To prove this conjecture, one needs to prove that $W^s(q)$ depends "more than continuously" on q (when q ranges in E_n). This is not true in general.

Further results on the BKL conjecture ?

Question. Does the union of the $W^s(q)$ has full Lebesgue measure ?

(**Loosely speaking :** for the Lebesgue measure viewpoint, do the typical asymptotic behaviours of Bianchi spacetimes correspond to the typical asymptotic behaviours of Kasner map orbits.)

To answer this question, one needs some completely different types of arguments. The only arguments I can imagine are in favour of a **negative answer**.

Further results on the BKL conjecture ?

Remark. With the current techniques, the set of Bianchi spacetimes we can control is *small from the topological viewpoint* (*i.e.* this is a meager set).

There is essentially no hope to control a set of Bianchi spacetimes which would be (locally) fat.

Further results on the BKL conjecture ?

Remark. Form the ergodic viewpoint, the system is rather pathological. In particular : for most points, *Birkhoff means should not converge.*

About the proof of the theorem.

The key is to understand what happens to type IX orbits when they pass *close to the Kasner circle*. Roughly speaking :

- close to the Kasner circle, there should be some "supra-linear contraction-dilatation phenomena" ;
- far from the Kasner circle, everything is "at most linear".

Control of a set of orbits with positive Lebesgue measure ?

- No linéarization results apply. One needs to prove "by brute force" some estimates of the contraction, the drift...

- One needs to control the size of the neighbourhood of p where the estimates hold. This size goes to zero exponentially fast as p approaches a Taub point.

- One needs to show that "many" orbits fall each time in the neighbourhoods where the estimates holds. Uses some results on the continued fraction development of almost every point.

- One needs to adapt Pesin theory.