Groups of projective transformations of pseudo-Riemannian spaces

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The "Theorem"

Let (M, g) be a compact pseudo-Riemannian manifold. A diffeomorphism $f : M \to M$ is affine if it sends a geodesic to a geodesic. It is projective if it sends a non-parametrized (geometric) geodesic to a non-parametrized geodesic.

• *If g is Riemannian, then* $\mathrm{Proj(M, g)/Aff\,(M, g)}$ *is finite, unless M is a (finite) quotient of the standard sphere.*

•• *If g is Lorentzian, then* $\mathrm{Proj(M, g)/Aff\,(M, g)}$ *is finite.*

What is really new?

• In the Riemannian case, the theorem is known but for the identity components: $\mathrm{Proj}^0\mathrm{(M, g)} = \mathrm{Aff}^{\,0}\mathrm{(M, g)}$, unless M is covered by the sphere,
In other words
- **Hypothesis:** (M, g) admits a projective non-affine one parameter group of transformations
- Conclusion M is a quotient of the sphere

Here:

- **Hypothesis:** (M, g) has a projective transformation none of which powers is affine
Conclusion M is a quotient of the sphere

•• The Lorentz case is new: only partial (technical) results are know.

Introduction, motivations

Classical Problem

A Russian speciality

Current names: Bolsinov (UK), Topalov (Nantes), Matveev (Germany)...
Older: Solodovnokov, Sinjukov, Aminova,...

Italian:
Beltrami, Dini, Fubini, Levi-Civita....
Others,
Weyl, Eisenhart, Painlevé, Darboux, Lagrange, Cartan, Lie,...
Hall (Relativity)
Bryant,

Hierarchy of Groups

(M, g) pseudo-Riemannian
– Iso(M, g) the group of all isometries, i.e. diffeomorphisms such that $f^*g = g$

– Aff (M, g) the group of affine transformations, i.e. those preserving parametrized geodesics of (M, g). It contains

– Proj(M, g) which will interest us consists of transformations preserving non-parameterized geodesics of (M, g).

- Sim(M, g) the group of similarities (or homotheties) i.e. maps f such that $f^*g = ag$ for some constant.

Conf(M, g) the group of conformal transformations such that $f^*g = ag$, where a is a function on M.

Inclusions
Iso $\subset$ Sim $\subset$ Conf,

Focus here on:
Iso $\subset$ Sim $\subset$ Aff $\subset$ Proj.

They are all Lie groups with identity components $\mathrm{Iso}^0, \ldots, \mathrm{Aff}^0, \mathrm{Proj}^0$.

Lie group?

meaning: they are not necessarily connected...

There is a differentiable structure on G, such that the action $G \times M \to M$ is a smooth map.

e.g. $G = \mathbb{Z}$: data $\iff$ a diffeomorphism
???

"Beauty" of the sphere

Meaning: the inclusion chain is non-trivial!
• In conformal Geometry
Iso $\subset$ Sim $\subset$ Conf:

$\mathrm{Iso}(\mathbb{S}^n) = \mathrm{O}(n+1)$, $\mathrm{Conf}(\mathbb{S}^n) = \mathrm{O}(1, n+1)$

Better: Lichnerowicz conjecture (solved by Ferrand and Obatta):
$\mathbb{S}^n$ is the unique compact Riemannian manifold with Iso essentially different from Conf

The Projective case

$\mathrm{Proj}(\mathbb{S}^n) = \mathrm{PGL}_{n+1}(\mathbb{R})$
$(= \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^* = \mathrm{SL}_{n+1}(\mathbb{R})$ up to index 2)
$A = \mathrm{GL}_{n+1}(\mathbb{R})$, $A.x = \frac{Ax}{\|Ax\|}$
Alternatively,
PGL_{n+1} acts on $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$
$\mathrm{Aff}\,(\mathbb{S}^n) = \mathrm{Iso}(\mathbb{S}^n)$

Some finite quotients of the sphere may have Proj non-compact,

Affine beauty of the torus

$\mathbb{T}^n = \mathbb{R}^n/\Lambda$, Λ lattice in $\mathbb{R}^n$, e.g. $\Lambda = \mathbb{Z}^n$
$\mathrm{Iso} = \mathbb{T}^n$, up to a finite index
$\mathrm{Aff} = \mathrm{GL}_n(\mathbb{Z})$ ($=\mathrm{SL}_n(\mathbb{Z})$ up to index 2),

But $\mathrm{Proj}(\mathbb{T}^n) = \mathrm{Aff}\,(\mathbb{T}^n)$

Projective Lichnerowicz Conjecture

Find all spaces (M, g) such that $\mathrm{Aff}\,(M, g) \subsetneq \mathrm{Proj(M, g)}$

- Variants:
- M compact
- (M, g) complete
- g Riemannian
- g pseudo-Riemannian

Weaker version with the hypothesis $Pro^0(M, g)$ non-contained in $\mathrm{Aff}\,(M, g)$

Precise question

Conjecture

Let (M, g) be a compact pseudo-Riemannian manifold. Assume (M, g) is not a quotient of the standard Riemannian sphere. Then, $\mathrm{Proj(M, g)}/\mathrm{Aff}\,(M, g)$ *is finite*

Weaker version:

Conjecture

(Killing fields) Let (M, g) be a compact pseudo-Riemannian manifold. Assume (M, g) is not a quotient of the standard Riemannian sphere. Then, $\mathrm{Proj^0(M, g)} = \mathrm{Aff}\,(\mathrm{M, g})$. *In other words, any projective Killing field is an affine Killing field.*

- The conjecture for Killing fields in the Riemannian case is proved by V. Matveev

- Here:
- we prove the full conjecture in the Riemannian case
- and in the Lorentz case....

Projective flatness

$f : (M, g) \to (M', g')$ projective diffeomorphism

(M, g) (locally) projectively flat if it projectively diffeomorphic to the Euclidean space.

Betrami: in this case (M, g) has constant sectional curvature

One defines projectively flat connection..., and projective structures,

Space of metrics

Restrict discussion to the Riemannian case $\rightarrow$ generalize...

$\mathcal{M}et(M)$ be the space of all Riemannian metrics on M.

The conformal equivalence relation relation on $\mathcal{M}et(M)$:
$g \sim^{conf} g'$ iff $g' = e^{\sigma} g$ for some function σ.
Conf(M, g) of diffeomorphisms preserving the conformal class of g

Affine and projective relations:
$g \sim^{aff} g'$ iff g and g' have the same parameterized geodesics.

$g \sim^{proj} g'$ if they have the same non-parameterized)

In opposite the the conformal relation, the affine and projective classes $\mathcal{M}et^{\mathrm{Aff}}(g)$ and $\mathcal{M}et^{\mathrm{Proj}}(g)$ of g are **finitely dimensional manifolds in $\mathcal{M}et(M)$.**
These contain in particular the multiples $\mathbb{R}g$,

Aff (M, g) and Proj(M, g) are stabilizers in Diff(M) of theses classes when acting on $\mathcal{M}et(M)$.

The action of Aff (M, g) and Proj(M, g) on $\mathcal{M}et^{\mathrm{Aff}}(g)$ and $\mathcal{M}et^{\mathrm{Proj}}(g)$, is a priori, neither trivial, nor transitive.
(it may happen that Aff (M, g) is trivial but not is $\mathcal{M}et^{\mathrm{Aff}}(g)$).

Philosophy

Let $f \in$ Diff(M) act naturally on $\mathcal{M}et(M)$

- The f- action has a fixed $\iff$ f is an isometry for some Riemannian metric on M.

Question What is the dynamical counterpart of the fact that the f-action preserves some (finite dimensional) manifold V in $\mathcal{M}et(M)$.
special case dim $V = 2$...
Teichmuller space...

Rank

Degree of (projective) mobility of (M, g)= dim $\mathcal{M}et^{\mathrm{Proj}}(g)$
$\mathcal{M}et^{\mathrm{Proj}}(g)$ contains the multiple $\mathbb{R}g$, hence mobility ≥ 1.

Examples

Dini

On surfaces, near a generic point, g are $\bar{g}$ are projectively equivalent $\iff$ in some co-ordinate system:
$g = (X(x) - Y(y))(dx^2 + dy^2)$ and
$\bar{g} = (\frac{1}{Y(y)} - \frac{1}{X(x)})(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)})$
$X(x) > Y(y)$

REM: X and Y are (essentially) eigenfunctions of the tensor L, $\bar{g}(.,.) = g(L.,.)$.
At generic points L have simple eigenvalues.
The difficulty comes from accidents of eigenvalues

On $\mathbb{T}^2$

$$g = (f(x) - \frac{1}{f(y)})(\sqrt{f(x)}dx^2 + \frac{1}{\sqrt{f(y)}}dy^2)$$

$$\phi(x, y) = (y, x)$$

is projective, but not affine (except f very special)

Ellipsoid

(Topalov, Matveev, Tabachnikov)
Ellipsoid:

$$\Sigma_{i=1}^{i=n} \frac{(x_i)^2}{a_i} = 1$$

$$g = \text{ the metric induced from } \mathbb{R}^n$$

$$\bar{g} = \frac{1}{\Sigma(\frac{x_i}{a_i})^2}(\Sigma \frac{dx_i^2}{a_i})$$

Known results

Levi-Civita Theorem, Normal forms

(Dini in higher dimension, in the Riemannian case)
Let g and $\bar{g}$ projectively equivalent Riemannian metrics,
$\bar{g}(.,.) = g(T..)$: $T = g^{-1}\bar{g}$ ($\bar{g} = gT$)
T diagonalizable,
Assume multiplicities $(k_1, \ldots, k_m)$ constant,

There exists an orthogonal coordinate system:
$y = (y_1, \ldots, y_m)$, $y_i = (x_i^1, \ldots, x_i^{k_i})$
$M = \Pi_1^m M_i$, $\dim M_i = k_i$, y_i coordinates on M_i

There exist h_i $(= h_i(y_i)$) metric on M_i

$$g = \Sigma_1^m \pi_i(y) h_i(y_i), \quad \bar{g} = \Sigma_1^m \rho_i(y) \pi_i(y) h_i(y_i)$$

There exist functions λ_i such that:
$\lambda_i = (\lambda_i - \lambda_1) \ldots (\lambda_{i+1} - \lambda_i) \ldots (\lambda_m - \lambda_i)$
$\rho_i = \frac{1}{\lambda_1 \ldots \lambda_m} \frac{1}{\lambda_i}$

Furthermore:
$\lambda_i = \lambda_i(y_i)$ (i.e. λ_i function on M_i)
$\lambda_i = \textit{constant}$, if multiplicity $k_i > 1$.
$\lambda_1 < \ldots < \lambda_m$
(i.e. $\lambda_1(y_1) < \lambda_2(y_2) < \ldots \lambda_m(y_m)$, for any $y_1, \ldots, y_m$)

Conversely, two metrics like this are projectively equivalent.

REM: $\lambda_i = \frac{1}{\rho_i}(\rho_1 \ldots \rho_m)^{\frac{1}{m+1}}$

$\rho_1, \ldots, \rho_m$ are eigenvalues of T

(T is the transition endomorphism: $\bar{g} = gT$)

$\lambda_1, \ldots, \lambda_m$ are eigenvalues of L such that $T = \frac{L^{-1}}{\det L}$

Higher rank, Fubini

Assume (locally) $g, \bar{g}, g'$ projectively equivalent
- $g^{-1}\bar{g}$ has simple eigenvalues,
- g, $\bar{g}$, g' linearly independent

Then g has constant sectional curvature.

Theorem (Kiosak - Matveev)

Let (M, g) a compact pseudo-Riemannian that is not covered by the standard (Riemannian) sphere.
If the degree of projective mobility of (M, g) is ≥ 3, then any projectively equivalent metric to g is affinely equivalent to it.

Rank 2

Corollary: all what remains to consider is the challenging case when g has mobility 2:
$\dim \mathcal{M}et^{\mathrm{Proj}}(M, g) = 2$

Proj(M, g) acts on the surface $\mathcal{M}et^{\mathrm{Proj}}(M, g)$
Paradoxical situation!?
- this case is easier to handle since Proj acts on a smaller space $\mathcal{M}et$?
OR
- If $\mathcal{M}et^{\mathrm{Proj}}$ is small, then Proj itself is small...

Some differential projective geometry of pseudo-Riemannian metrics

$\mathcal{M}et^{\mathsf{Aff}}$

g and $\bar{g}$ affinely equivalent
$\iff$
$\nabla^g = \nabla^{\bar{g}}$ (Levi-Civita connections)
$\iff$
$\bar{g}$ is a parallel tensor with respect to g (i.e $\nabla^g \bar{g} = 0$): Any parallel transport between two points with respect to g, preserves $\bar{g}$.
$T = g^{-1}\bar{g}$
$(\bar{g}(u, v) = g(Tu, v))$

$T : TM \to TM$ endomorphism,
$(1, 1)$-tensor
De Rham Theorem solves the problem in the Riemannian case: the eigenspaces of T give rise to a (local) direct product structure....

More complicated history in the pseudo-Riemannian case!

Connections

Γ and $\bar{\Gamma}$ are projectively equivalent if they have the same geometric (unparametrized) geodesics

Geodesic equation
$\ddot{x}^k = \Gamma^k_{ij}(x)\dot{x}^i\dot{x}^j$

$A = \nabla - \bar{\nabla}$ tensor,
$A : TM \times TM \to TM$ is symmetric
Projective equivalence: $A(u, u) \wedge u = 0$
$\implies A(u, u) = 2l(u)u$, for some form l
$A(u, v) = l(u)v + l(v)u$

Say, a vector valued bilinear form $B : E \times E \to E$ is a pure trace if $B(u, v) \in \mathrm{Span}(u,v)$ for any u, v.

Projective equivalence $\iff$ $\nabla - \bar{\nabla}$ traceless

Linear equation on ∇...

Non-linear equation on g

$$\nabla_u \bar{g}(\xi, \eta) = \bar{g}(\xi, \eta) d\theta(u) + \frac{1}{2}\bar{g}(\xi, u) d\theta(\eta) + \frac{1}{2}\bar{g}(\eta, u) d\theta(\xi)$$

$$\theta = \ln\left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{1+n}}$$

Linearization!

$\bar{g} = \frac{1}{\det L} g L^{-1}$
i.e. $\bar{g}(u, v) = \frac{1}{\det L} g(L^{-1}u, v)$; $T = \frac{L^{-1}}{\det L}$
$L = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$

Proposition

g and $\bar{g}$ are projectively equivalent $\iff$ L satisfies the linear equation:
$g((\nabla_u L)v, w) = \frac{1}{2} g(v, u) d\mathrm{trace}(L)(w) + \frac{1}{2} g(w, u) d\mathrm{trace}(L)(v)$

Say that L is a $\mathcal{P}$-tensor, and $\mathcal{P}(M, g)$ their space

Parametrization of $\mathcal{M}et^{\mathrm{Proj}}(M, g)$

Let $L_0 = I$, $L_1, \dots L_k$ a basis of the space of $\mathcal{P}$-tensors
Let $\bar{g} \in \mathcal{M}et^{\mathrm{Proj}}(M, g)$
$\exists\ L = \Sigma a_i L_i$;
$\bar{g} = g_L = \frac{1}{\det L} g L^{-1}$

Conversely
$g_L \in \mathcal{M}et^{\mathrm{Proj}}(M, g)$, once g_L is non-degenerate

In particular, since $I \in \mathcal{P}(M, g)$,
$g^t = \frac{1}{\det(L - tI)} g.(L - tI)^{-1}$ is projectively equivalent to g (if t is not a spectral value of L)

Nijenhaus tensor

Some properties of $\mathcal{P}$-tensors
For L endomorphism
$N_L(u,v) = [Lu, Lv] - L[Lu, v] - L[u, Lv] - L^2[u, v]$

A $\mathcal{P}$-tensor has $N_L = 0$

Integrability

Almost complex structures, $N_L = 0 \iff$ integrability, i.e. complex structure

In general, if L diagonalizable,
- eigen- distributions are integrable
- An eigenfunction is constant along the leaves of the other distributions
- In particular eigen-functions with higher multiplicity (> 1) are constant

Levi-Cevita normal form

Levi-Civita normal form follows, IN THE RIEMANNIAN case, from these considerations,

General pseudo-Riemannian case:
- L is not diagonalizable
- Integrability of generalized eigen-distributions: OK
- Constancy of one along the leaves of the others: OK
- Constancy of eigenfunctions of of higher multiplicity: NO

Normal forms of Lorentz auto-adjoint (symmetric) endomorphisms

Let A be a self-adjoint endomorphism of a Lorentz space $(V, \langle,\rangle)$.
There exists G a timelike invariant subspace.
A is self-adjoint on the Euclidean $G^\perp$ (so diagonalizable)
On G, A has one of the normal forms:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} \lambda & \pm 1 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

The basis in the last two cases is lightlike (e_0, e_1, e_2): all product are 0, but $\langle e_0, e_2\rangle = \langle e_1, e_1\rangle = 1$

Real difficulty

There exist $\mathcal{P}$ tensor on Lorentz surfaces with ONE eigenfunction and L everywhere non-diagonalizable (over $\mathbb{R}$).

Examples

First Integrals

If g and $\bar{g}$ projectively equivalent, then
$h = H(g, \bar{g}) = (\frac{\det g}{\det \bar{g}})^{\frac{1}{n+1}} \bar{g}$
is a first integral of the geodesic flow of g
If $g = \frac{1}{\det L} g L^{-1}$,
then $h = (\det L) g L^{-1}$

Linear and quadratic integrals

$k : TM$ (or T^*M) $\to \mathbb{R}$, first integral of flow ϕ^t,
if $k \circ \phi^t = k$
Nother Theorem: a symmetry gives rise to a fiberwise linear integral

An orbital equivalence gives rise to a quadratic integral

The Hamiltonian flow (restricted to an energy level) possess 2 preserved contact forms.
The characteristic polynomial of the skew symmetric map relating the 2 forms,
is an integral of the flow.

Alternatively,
$L - tI$ is a $\mathcal{P}$-tensor,

$g_t = \frac{1}{\det(L-tI)} g(L - tI)^{-1}$ is projectively equivalent to g

$h_t(u) = \det(L - tI)g(L - tI)^{-1}$ is an integral of the geodesic flow of g (for any fixed t)

Geodesic cone fields

Fix t, $I = h_t$

I is a field of quadratic forms on M, with variable signature...

Cases:
- If t is not a spectral value of L_x, then I_x is non-degenerate
- If L_x diagonalizable and t spectral value of L_x with higher multiplicity, then $I_x = 0$
- Many intermediate cases....

Singularities

Assume g Lorentz
in nice case, I is Lorentz, except on singular set S, where $I_x = 0$

$C_x = I_x^{-1}(0)$ cone in T_xM
$x \to C_x$ a field of cones

This is a **geodesic cone field** (for g):
if $u \in C_x$, $\gamma_u(t)$ its geodesic
$\gamma'_u(t) \in C_{\gamma_u(t)}$ $\forall t$

Shape of S

$x \in M$
The visual cone of S
$C(x, S) = \{u \in T_xM$ such that the geodesic γ_u hits S.
Cone in T_xM

Find S, a subset of M such that $C(x, S)$ is a quadratic cone for any x (or say for generic x)

Example: $M = \mathbb{R}^3$ (Euclidean)
$S =$ a circle (of codimension 2)
Line
Parabola,
Hyperbola
Because of projective invariance

Elements of proof, Actions

Explicit Position of the problem

(M, g) with projective mobility 2: $\dim \mathcal{M}et^{\mathrm{Proj}}(M, g) = 2$
$\mathcal{P}(M, g)$ space of $\mathcal{P}$-tensors
$L \in \mathcal{P}(M, g)$, $L \notin \mathbb{R}I$
$\{I, L\}$ a basis of $\mathcal{P}(M, g)$

$f \in \mathrm{Proj(M, g)}$, $K = K_f$ its g-distortion (stress) tensor
$f^* g = \frac{1}{\det K} g K^{-1}$
$K \in \mathcal{P}(M, g)$, more generally,

$T_n(x) = (D_x f^n)^* D_x f^n$
$L_n(x)$ such that $T_n(x) = \frac{1}{\det L_n(x)} (L_n(x)^{-1}$
$L_n \in \mathcal{P}(M, g)$
$L_1 = K$

Dimension 2 Hypothesis: $\forall n$, $\exists$ a_n, b_n such that $L_n = a_n I + b_n L_1$
- a_n and b_n do not depend of x!!!

If x is fixed, $f(x) = x$
A corresponding linear problem!

Action

Proj(M, g) acts on $\mathcal{M}et^{\mathrm{Proj}}(M, g)$
$(f, g) \in \mathrm{Proj(M, g)} \times \mathcal{M}\mathrm{et}^{\mathrm{Proj}}(\mathrm{M, g}) \to \mathrm{f^*g}$

Transported action on $\mathcal{P}(M, g)$ via the map
$L \to g_L = \frac{1}{\det L} g L^{-1}$

$$(f, L) \in \mathrm{Proj(M, g)} \times \mathcal{P}(\mathrm{M, g}) \to \mathrm{f^*L.K_f} \in \mathcal{P}(\mathrm{M, g})$$

The action is linear !

Introduction
History
Projective geometry
Actions

Representation

$\rho : \mathrm{Proj(M, g)} \to \mathrm{GL}\,(\mathcal{P}(\mathrm{M, g})) = \mathrm{GL}\,_2(\mathbb{R})$, 2-dimensional linear representation

$\rho(f) = 1 \iff K_f = I \iff f$ isometry

$\rho(f)$ homothety $\iff K_f = aI$, f is a similarity, impossible if M compact (unless $a = \pm 1$)

So consider $\rho : \mathrm{Proj(M, g)} \to \mathrm{SL}\,_2(\mathbb{R})$

Zeghib Projective groups

Introduction
History
Projective geometry
Actions

Homography

Fix f, $K = K_f$
$\{I, K\}$ a basis of $\mathcal{P}(M, g)$
Let $L \in \mathcal{P}(M, g)$,
$f^*L.K = aI + bK$
Take $L = K$

$$f^*K = \frac{aI + bK}{K}$$

Zeghib Projective groups

Introduction
History
Projective geometry
Actions

$A = A_f = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$
Homographic action on $\mathbb{C}$: $A\,.\,z = \frac{az+b}{z}$
The action is defined for any $A \in \mathrm{SL}\,_2$

The action is defined for endomorphisms of TM...

Consequences:
$f^{n*}K = A^n\,.\,K$

Zeghib Projective groups

$x \to Sp(x)$ spectrum of $K(x)$
$Sp(x) \subset \mathbb{C} \times \ldots \mathbb{C}$...

$Sp(f^n x) = A^n . Sp(x)$

Up to ordering:
If $\lambda : M \to \mathbb{R}$ is eigenfunction,

$$\lambda(f^n) = A^n . \lambda(x)$$

λ semi-conjugates the two dynamical systems $(M, f) \to (\mathbb{C}, A)$

Classification of elements of SL_2

elliptic
parabolic
hyperbolic

Case $\rho(f)$ elliptic

Assume λ real
$= \lambda(M) \subset \mathbb{R}$ is a compact A-invariant interval
If a rotation A has an invariant interval, then then $A^2 = 1$

Parabolic is impossible

Hyperbolic case

A has two fixed points λ_- and λ_+
South-North dynamics between them,

- Prove that there is at most one non-constant eigenfunction
- The possible other eigenfunctions are the constants λ_-, λ_+
- Consider the (C^0, i.e non related to a measure) Lyapunov spaces
- Prove the Weyl projective tensor vanishes...
- Curvature is positive

Lorentz case!