Dynamics of spatially homogeneous cosmological models

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Main Goal

Classify the asymptotic dynamics of spatially homogeneous spacetimes with a general class of matter models.
We study the Einstein-matter equations under the assumption of spatial homogeneity using dynamical systems techniques. The matter model is not given explicitly but only through a set of physically motivated assumptions.

So far we have analyzed the following spatially homogeneous spacetimes:

- Bianchi type I spacetimes
- Bianchi class A spacetimes with Local Rotational Symmetry

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- We propose a class of anisotropic matter models that naturally generalize perfect fluids and which is defined through a set of physically motivated assumptions.
- There exist important explicit examples of matter models included in our class (Magnetic fields, Vlasov matter, Elastic matter).

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- Each orbit of the reduced dynamical system identifies in a unique way a family of solutions to the Einstein-matter equations. Orbits approaching the boundary of the state space correspond to solutions of the Einstein-matter equations approaching a singularity.

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- Each orbit of the reduced dynamical system identifies in a unique way a family of solutions to the Einstein-matter equations. Orbits approaching the boundary of the state space correspond to solutions of the Einstein-matter equations approaching a singularity.
- The α/ω-limit sets of the orbits to the reduce dynamical system uniquely characterizes the asymptotic behavior of the spacetime.

Spatially Homogeneous Spacetimes

We consider a spacetime $(M, \bar{g})$ with a 3-dimensional group G of isometries whose orbits are spacelike hypersurface that foliate M.

Let t denote the proper time of the geodesic congruence orthogonal to the orbits of G.

Let ω^i, $i = 1, 2, 3$ denote a time independent group invariant frame on the orbits of G. The metric can be written in the form

$$\bar{g} = -dt^2 + g_{ij}(t)\omega^i\omega^j$$

$g_{ij}(t_0)$ is the Riemannian metric induced on the orbit $t = t_0$ of the group G.

Class A Bianchi models

For spatially homogeneous models of Bianchi Class A, the dual forms ω^i can be chosen to satisfy

$$d\omega^1 = -n_1\,\omega^2 \wedge \omega^3\,, \quad d\omega^2 = -n_2\,\omega^3 \wedge \omega^1\,, \quad d\omega^3 = -n_3\,\omega^1 \wedge \omega^2$$

Bianchi	n_1	n_2	n_3	Simply Connected G
I	0	0	0	$\mathbb{R}^3$
II	1	0	0	$\mathrm{H}(3,\mathbb{R})$
VI_0	0	1	-1	$\mathrm{Iso}(\mathrm{M}^2)$
VII_0	0	1	1	$\mathrm{Iso}(\mathbb{R}^2)$
VIII	-1	1	1	$\mathrm{SL}(2,\mathbb{R})$
IX	1	1	1	$\mathrm{SO}(3,\mathbb{R})$

The Ricci curvature R_{ij} of the group orbits is a polynomial on g_{ij} whose coefficients depend on the structure constants (n_1, n_2, n_3).

Einstein equations

The Einstein equations (in units $8\pi G = c = 1$) split into evolution equations

$$\partial_t g_{ij} = -2k_{ij}, \quad \partial_t k^i{}_j = R^i{}_j + \left(g^{lm}k_{lm}\right) k^i{}_j - T^i{}_j + \tfrac{1}{2}\delta^i{}_j (g^{lm}T_{lm} - \rho).$$

and constraint equations

$$R - k^{ij}k_{ij} + (g^{ij}k_{ij})^2 = 2\rho\,, \qquad \nabla_i k^i{}_j - \nabla_j(g^{lm}k_{lm}) = -T_{0j}\,,$$

R_{ij} is the Ricci tensor of the Riemannian metric g_{ij}

$T_{\mu\nu}$ is the matter energy-momentum tensor, with $T_{00} = \rho$. To close the system we need to specify $T_{\mu\nu}$.

Assumptions on the matter model

The most commonly used energy-momentum tensor in cosmology is that of a (non-tilted) perfect fluid:

$$T_{\mu\nu} = \rho(dt \otimes dt)_{\mu\nu} + p[\bar{g}_{\mu\nu} + (dt \otimes dt)_{\mu\nu}] \, ,$$

where the pressure p and the energy density ρ are usually required to obey a linear equation of state,

$$p = w\rho \, , \qquad w = const.$$

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This energy-momentum tensor is spatially isotropic.
The Bianchi identites $\nabla_\mu T^{\mu\nu} = 0$ can be solved exactly to express ρ, and therefore the entire $T_{\mu\nu}$, in terms of the spatial metric:

$$\rho = \rho_0 n^{1+w} \, , \quad \rho_0 = \mathrm{constant} \, , \; n = (\det g)^{-\frac{1}{2}} \, .$$

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The components of the energy-momentum tensor are represented by smooth (at least $\mathcal{C}^1$) functions of the metric g_{ij}. We assume that $\rho = \rho(g_{ij})$ is positive (as long as g_{ij} is non-degenerate).

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Remark. In the previous assumption it is understood that also the matter fields which (possibly) enter the definition of the energy-momentum tensor can be written in terms of the metric g_{ij} by solving the Bianchi identities $\nabla_\mu T^{\mu\nu} = 0$ (this might be possible only under certain restrictions on the initial data for the matter fields).

Lemma

Under Assumption 1, the spatial component $T^i{}_j$ of the energy momentum tensor w.r.t. the frame ω^i satisfy

$$T^i{}_j = -2\frac{\partial \rho}{\partial g_{il}}\, g_{jl} - \delta^i{}_j\, \rho\,.$$

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The frame ω^i can be chosen so that g_{ij} and ${k^i}_j$ are diagonal for $t = 0$. We call this the initially orthogonal frame (IOF).

Assumption (diagonal models)

We assume that there exist initial data for the matter such that the energy-momentum tensor is diagonal in the IOF.

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By the Einstein equations it follows that $(g_{ij}, {k^i}_j, {T^i}_j)$ remain diagonal for all times.

We set

$$T^i{}_j = \mathrm{diag}(p_1, p_2, p_3)\,.$$

where p_i are the principal pressures and

$$p = \tfrac{1}{3}(p_1 + p_2 + p_3)\,.$$

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Assumption

We suppose that the isotropic pressure and the density are proportional, i.e., we assume $p = w\rho$, $w = \mathrm{const}$, *where*

$$w \in (-\tfrac{1}{3}, 1) \,.$$

Accordingly, the density and the isotropic pressure behave like those of a perfect fluid with a linear equation of state satisfying the strong and the dominant energy condition.

Lemma

For $i = 1,2,3$ (no sum over i) let $n = \sqrt{g^{11}g^{22}g^{33}}$ and

$$s_i = (g^{11} + g^{22} + g^{33})^{-1} g^{ii} \quad \Rightarrow s_1 + s_2 + s_3 = 1 \,.$$

There exists a function $\psi = \psi(s_1, s_2, s_3)$ such that the components of the energy-momentum tensor can be written as

$$\rho = n^{1+w}\psi(s_1, s_2, s_3) \,, \qquad p_i = w_i \rho \,,$$

where the normalized principal pressures w^i are given by

$$w_i(s_1, s_2, s_3) = w + 2\left(\frac{\partial \log \psi}{\partial \log s_i} - s_i \sum_l \frac{\partial \log \psi}{\partial \log s_l}\right).$$

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For $\psi =$constant we obtain the energy-momentum tensor of a perfect fluid with linear equation of state $p = w\rho$.

Example (John's Elastic Materials[1].)

The function ψ is given by

$$\psi(s_1, s_2, s_3) = 1 + \mu\,\mathfrak{s}$$

where $\mu =$const> 0 (modulus of rigidity) and

$$\mathfrak{s} = (s_1 s_2 s_3)^{-1/3} - 3 \quad \text{is the shear scalar}\,.$$

For this matter model we obtain

$$w_i(s_1, s_2, s_3) = w + 2\left(s_i - \frac{1}{3}\right)\frac{\mu}{1 + \mu\,\mathfrak{s}}\,.$$

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Remark. For elastic materials, the choice of the function ψ is equivalent to the choice of a constitutive equation.

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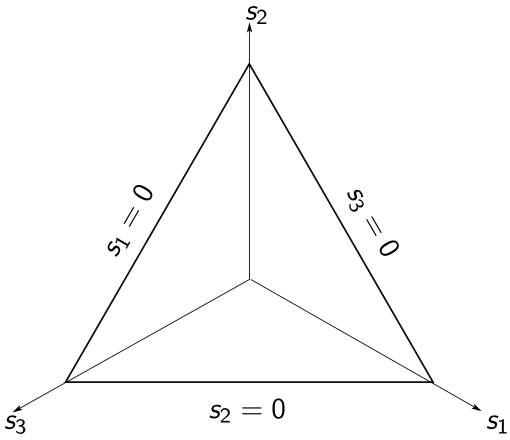
Since $s_1 + s_2 + s_3 = 1$, the functions $\psi(s_1, s_2, s_3)$ and $w^i(s_1, s_2, s_3)$ are defined in the interior of a triangle, which we call $\mathcal{T}$.
The boundary of the triangle $\mathcal{T}$ corresponds to a singularity of the metric.



Assumption

We assume that the function ψ satisfies the following properties:

(i) *It is invariant under permutations of the axes, i.e.,*

$$\psi(s_1, s_2, s_3) = \psi(s_2, s_3, s_1) = \psi(s_3, s_1, s_2)\,.$$

(ii) *It has a smooth (at least $\mathcal{C}^1$) extension on the boundary of $\mathfrak{T}$.*

(iii) *It has only one critical point in the interior of $\mathfrak{T}$ and it is either a maximum or a minimum (or $\psi = const.$).*

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Remark. This assumption has the following physical interpretation:

(i) means that the energy density is invariant under permutation of the axes

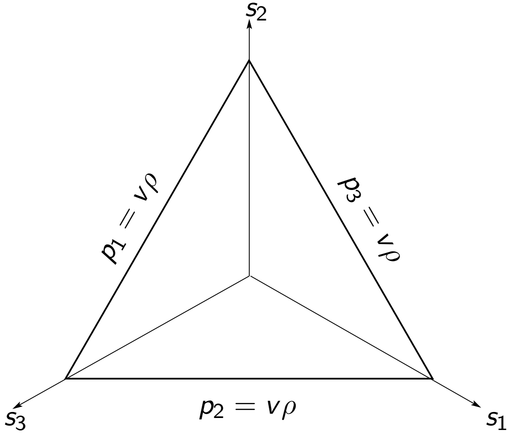
(ii) is a regularity condition on the variables s_i

(iii) is equivalent to require the existence of one and only one isotropic state of the matter ($w_i = w$, for all i) and that this state is either a maximum or a minimum state of energy.

The next assumption restricts the possible behavior of the matter source 'at the singularity' (to allow asymptotic self-similarity).

Assumption

We assume that there exists a constant v *such that* $w_i(s_1, s_2, s_3) = v$ *for* $s_i = 0$.



Definition

We define the anisotropy parameter β as

$$\beta = 2\frac{w - v}{1 - w}$$

and classify the matter models in different types according to the values of β as shown in the Table.

Type	β values	Energy cond.	Example
$\mathbf{D}_-$	$\beta \leq -2$	compatible	Electromagnetic fields
$\mathbf{C}_-$	$\beta \in (-2, -1)$	compatible	?
$\mathbf{B}_-$	$\beta = -1$	compatible	?
$\mathbf{A}_-$	$\beta \in (-1, 0)$	compatible	?
$\mathbf{A}_0$	$\beta = 0$	compatible	Elastic/perfect fluid matter
$\mathbf{A}_+$	$\beta \in (0, 1)$	compatible	Elastic matter
$\mathbf{B}_+$	$\beta = 1$	compatible	Elastic and Vlasov matter
$\mathbf{C}_+$	$\beta \in (1, 2)$	violated	Elastic matter
$\mathbf{D}_+$	$\beta \geq 2$	violated	Elastic matter

(some) Results for Bianchi type I

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- isotropize toward the future: $g^{ii} \sim t^{-\frac{4}{3(w+1)}}$ as $t \to +\infty$
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The larger is $|\beta|$, the more the asymptotic dynamics differs from the perfect fluid case (both in the past and the future direction).

Local Rotational Symmetry

For LRS models there exists a one-dimensional isotropy group that defines a plane of rotational symmetry. By using an adapted frame one can choose two of the components of the diagonal metric to be equal (these are the components of the metric induced on the plane of rotational symmetry). W.l.o.g. we make a choice of LRS frame such that

$$g_{22} = g_{33}$$

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In the following the qualitative behavior of LRS Bianchi type IX solutions will be analised.

Locally Rotationally Symmetric Solutions Bianchi type IX

In the perfect fluid case (with $p = w\rho$, $-\frac{1}{3} < w < 1$) the spacetime collapses in both time directions except for a zero measure set of initial data.

- In the past direction the solution blows-up like the vacuum Taub solution

$$\mathrm{T}: \quad g^{11} = t^2\,, \qquad g^{22} = g^{33} = 1\,.$$

- In the future direction the solution blows-up like the vacuum non-flat LRS Kasner solution

$$\mathrm{Q}: \quad g^{11} = t^{2/3}\,, \qquad g^{22} = g^{33} = t^{-4/3}\,.$$

For anisotropic matter models we have the following result:

Theorem

Suppose that the matter model satisfies the following inequalities:

$$-\frac{1}{3} < w < \frac{1-\sqrt{3}}{3} \approx -0.244 \quad \textit{and} \quad \beta_- < \beta < \beta_+$$

where

$$\beta_\pm = \frac{-1 \pm \sqrt{(1-3w)^2 - 3}}{3(1-w)}$$

(since $\beta_\pm \in (-\frac{1}{2}, 0)$ this is a special subcase of $\boldsymbol{A}_-$). Then

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(i) *The spacetime becomes singular in the past the and toward the singularity the solution blows-up like the vacuum Taub solution T*

(ii) *For an open set of initial data, the spacetime is forever expanding toward the future (no singularity!) and is asymptotic to a non-vacuum, anisotropic solution $\mathrm{R}_\flat$.*

Remark. The solution $\mathrm{R}_\flat$ of the Einstein equations is of Bianchi type I. The metric is given explicitly by

$$g^{11} = t^{-2\gamma_1}\,, \quad g^{22} = g^{33} = t^{-2\gamma_2}\,,$$

where

$$\gamma_1 = \frac{2(1+2\beta)}{3[\beta^2(1-w)+1+w]}\,, \qquad \gamma_2 = \frac{2(1-\beta)}{3[\beta^2(1-w)+1+w]}\,,$$

For $\beta \to 0$ this solution reduces to the isotropic perfect fluid solution:

$$g^{ii} \sim t^{-\frac{4}{3(w+1)}} \quad \forall i = 1,2,3\,.$$

In this limit, the set of initial data giving rise to this behavior has zero measure (cf. the statement for perfect fluids).

Proof of (ii)

The Einstein equations for the spatial metric g^{11}, $g^{22} = g^{33}$ are

$$\partial_t g^{11} = 2g^{11} k^1{}_1 \,, \quad \partial_t g^{22} = 2g^{22} k^2{}_2 \,, \quad m_1 = \frac{g^{22}}{\sqrt{g^{11}}} \,, \quad m_2 = \sqrt{g^{11}} \,,$$
$$\partial_t k^1{}_1 = \tfrac{1}{2} {m_1}^2 - 3Hk^1{}_1 - p_1 + \tfrac{1}{2}\left(p_1 + 2p_2 - \rho\right),$$
$$\partial_t k^2{}_2 = \tfrac{1}{2}\left[2m_1 m_2 - m_1^2\right] - 3Hk^2{}_2 - p_2 + \tfrac{1}{2}\left(p_1 + 2p_2 - \rho\right).$$

whereas the Hamiltonian constraint equation becomes

$$H^2 + \tfrac{1}{3} m_1 m_2 = \tfrac{1}{3}\rho + \tfrac{1}{12} m_1^2 + \sigma_+^2 \,,$$

where as usual we introduced the Hubble scalar and the shear variable

$$H = -\frac{\mathrm{tr}\, k}{3} = -\frac{k^1{}_1 + 2k^2{}_2}{3} \,, \qquad \sigma_+ = \frac{k^1{}_1 - k^2{}_2}{3} \,.$$

- For LRS solutions, the Einstein-matter equations form an autonomous dynamical system in the variables $(g^{11}, g^{22}, k^1{}_1, k^2{}_2) \in (0, \infty)^2 \times \mathbb{R}^2$ subject to a constraint.

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- We want to prove that under the assumptions of the Theorem, the solution is defined for $t \in (0, +\infty)$ and

$$g^{11} \sim t^{-2\gamma_1} \qquad g^{22} \sim t^{-2\gamma_2} \quad \text{as } t \to +\infty$$

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  (i.e., the solution is asymptotic to $\mathrm{R}_\flat$).
- **Dynamical system approach:** We replace $g^{11}, g^{22}, k^1{}_1, k^2{}_2$ with a new set of dynamical variables such that the relevant state space becomes 3-dimensional and bounded. In the new state space, the solution $\mathrm{R}_\flat$ is represented by a fixed point on the boundary.

New dynamical variables. Define

$$s = \frac{g^{22}}{g^{11} + 2g^{22}} \in (0, \tfrac{1}{2})$$

In LRS symmetry the variable s is equivalent to the variables (s_1, s_2, s_3) defined before, since $s_2, s_3 = s$, $s_1 = 1 - 2s$.

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Define the dominant variable D as

$$D = \sqrt{H^2 + \frac{m_1 m_2}{3}} \quad (m_1 = \frac{g^{22}}{\sqrt{g^{11}}}\, , m_2 = \sqrt{g^{11}})$$

The Hamiltonian constraint

$$H^2 + \tfrac{1}{3} m_1 m_2 = \sigma_+^2 + \tfrac{1}{12}\, m_1^2 + \tfrac{1}{3}\rho\, ,$$

guarantees that the square root is real.

We employ D to introduce normalized variables according to

$$H_D = \frac{H}{D}\,, \qquad \Sigma_+ = \frac{\sigma_+}{D}\,, \qquad M_1 = \frac{m_1}{D} > 0\,, \qquad M_2 = \frac{m_2}{D} > 0\,.$$

In addition we define a normalized energy density Ω by

$$\Omega = \frac{\rho}{3D^2} \geq 0\,,$$

and we replace the cosmological time t by a rescaled time variable τ via

$$\frac{d}{d\tau} = \frac{1}{D}\,\frac{d}{dt}\,.$$

Henceforth, a prime denotes differentiation w.r.t. τ.

In the new variables the Einstein eqs split into a decoupled equation for D and a system of equations for $(H_D, \Sigma_+, M_1, M_2)$

$$
\begin{aligned}
H_D' &= -(1 - H_D^2)(q - H_D \Sigma_+) \\
\Sigma_+' &= (q-2) H_D \Sigma_+ + (1 - H_D^2)\Sigma_+^2 + \tfrac{1}{3} M_1^2 + 3\Omega\big(w_2(s) - w\big) \\
M_1' &= M_1\big(q H_D - 4\Sigma_+ + (1 - H_D^2)\Sigma_+\big) \\
M_2' &= M_2\big(q H_D + 2\Sigma_+ + (1 - H_D^2)\Sigma_+\big)
\end{aligned}
$$

where $q = 2\Sigma_+^2 + \frac{1}{2}(1+3w)\Omega$. The constraint eq. implies

$$
s = \left(2 + \frac{3(H_D^2 - 1)}{M_1^2}\right)^{-1}, \qquad \Omega = 1 - \Sigma_+^2 - \tfrac{1}{12}\, M_1^2 \, .
$$

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$$
\begin{aligned}
H_D' &= -(1-H_D^2)(q - H_D\Sigma_+)\\
\Sigma_+' &= (q-2)H_D\Sigma_+ + (1-H_D^2)\Sigma_+^2 + \tfrac{1}{3}M_1^2 + 3\Omega\big(w_2(s) - w\big)\\
M_1' &= M_1\big(qH_D - 4\Sigma_+ + (1-H_D^2)\Sigma_+\big)\\
M_2' &= M_2\big(qH_D + 2\Sigma_+ + (1-H_D^2)\Sigma_+\big)
\end{aligned}
$$

where $q = 2\Sigma_+^2 + \frac{1}{2}(1+3w)\Omega$. The constraint eq. implies

$$
s = \left(2 + \frac{3(H_D^2 - 1)}{M_1^2}\right)^{-1}, \qquad \Omega = 1 - \Sigma_+^2 - \tfrac{1}{12}\,M_1^2\,.
$$

The equation on M_2 decouples$\rightarrow$ 3-dimensional state space.

The reduced dynamical system consists of the equations on H_D, Σ_+, M_1 and is defined over the state space

$$\mathcal{X}_{\mathrm{IX}} = \left\{ (H_D, \Sigma_+, M_1) \,\middle|\, H_D \in (-1, 1)\,,\ M_1 > 0\,,\ \Sigma_+^2 + \tfrac{1}{12} M_1^2 < 1 \right\}$$

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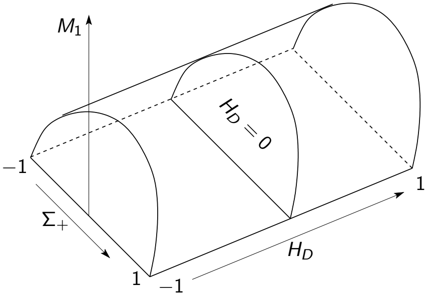

Remark.

- The plane $H_D = 0$ acts as a semipermeable membrane for the flow of the reduced dynamical system, since $H'_{D\,|H_D=0} < 0$. This makes $H_D > 0$ a subset that is past invariant under the flow, while $H_D < 0$ is future-invariant.

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- The physical interpretation is the following: A type IX model that is expanding, i.e., $H > 0$ ($\Leftrightarrow H_D > 0$), at $t = t_0$ must have been expanding up to that time, i.e., for $0 < t < t_0$; conversely, a model that is contracting (i.e., $H < 0$) at $t = t_1$ must continue to contract $\forall t > t_1$ (which eventually leads to a big crunch); finally, a model that satisfies $H = 0$ at some time, is a model that starts from an initial singularity, expands to a state of maximum expansion ($H = 0$), from which it then recontracts to a final singularity.

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$$H_D \to -H_D\,,\ \Sigma_+ \to -\Sigma_+\,,\ \tau \to -\tau\,;$$

the flow in $H_D < 0$ is the image of the flow in $H_D > 0$ under this discrete symmetry. In particular, it suffices to analyze the asymptotic behavior of solutions in the half $H_D > 0$; the asymptotic behavior of solutions in $H_D < 0$, follows by applying the discrete symmetry.

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In particular we can restrict ourselves to study the flow on the half $H_D > 0$.

Asymptotic behavior of orbits

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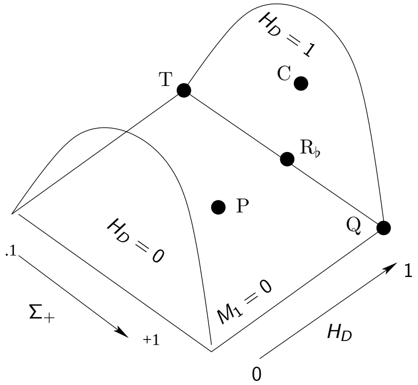
- Since the state space is bounded, the $\alpha/\omega-$limit sets of interior orbits are non-empty
- There exists a monotone function in the interior of the state space $\Rightarrow$ the $\alpha/\omega-$limit sets are located at the boundary of the state space.
- The only problem left is to study the flow induced on the boundary of $\mathcal{X}_{\mathrm{IX}} \cap \{H_D > 0\}$.
- Under the inequalities on (w, β) in the theorem, there exist 5 fixed points on the boundary, each of which corresponds to a self-similar solution of the Einstein equations.



If we prove that the fixed point $R_\flat$ is a local sink of interior orbits we are done!

However there is a problem: While the reduced dynamical system admits a regular extension on each of the boundary components $H_D = 1$ and $M_1 = 0$ separately, it does not extend regularly on the line

$$\mathcal{L} = \{H_D = 1\} \cap \{M_1 = 0\}$$

The problem lies in the equation

$$\Sigma_+' = (q-2)H_D\Sigma_+ + (1-H_D^2)\Sigma_+^2 + \tfrac{1}{3}\big(M_1^2 - M_1M_2\big) + 3\Omega\big(w_2(s) - w\big)$$

In fact, the variable

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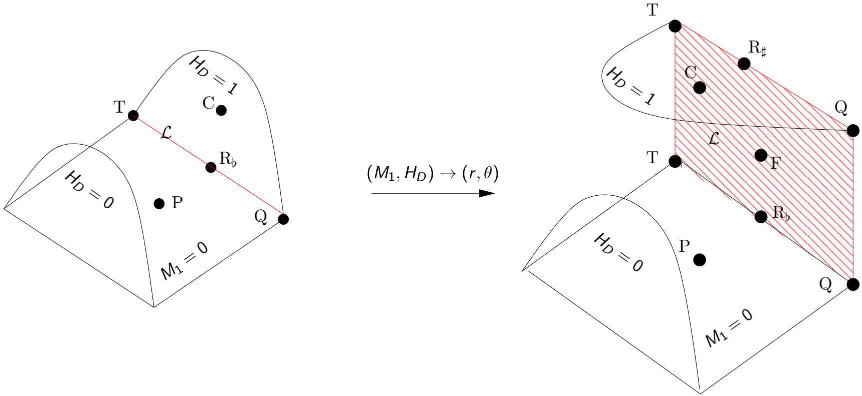
does not have a well defined limit when $H_D \to 1$ and $M_1 \to 0$ simultaneously. We need new coordinates.



We introduce polar coordinates (r, θ) centered on the line $\mathcal{L}$. This gives rise to a blow-up of the line $\mathcal{L}$ into a rectangle, in which the flow is equivalent to the Bianchi type I flow.

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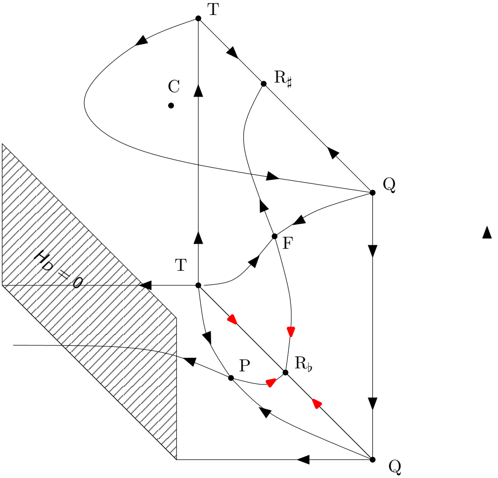
The analysis of the flow in the boundary reveals that $\mathrm{R}_\flat$ is a sink:



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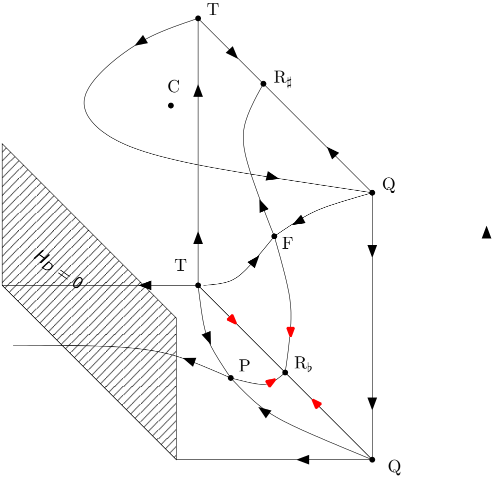
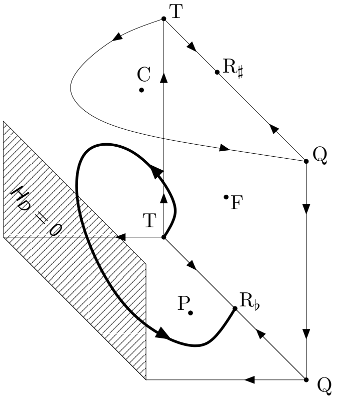


The proof is complete!

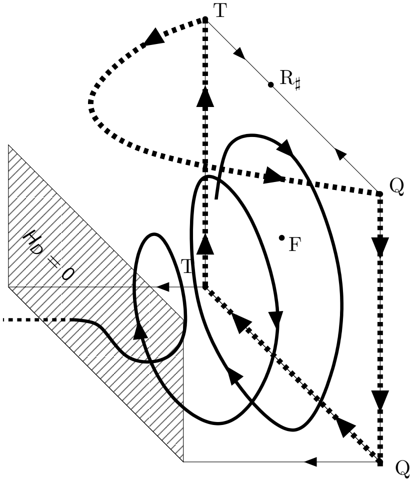
The structure of the flow on the boundary and an application of the monotonicity principle reveal that the $\alpha-$limit set of the non-recollapsing orbits is the fixed point T (i.e., the Taub solution), as claimed in the Theorem.



This is the only case in which the solution does not cross the barries $H_D = 0$, i.e., it does not recollapse. A two parameters family of interior orbits have the fixed point P as the ω-limit set.

Oscillations toward the singularity in the $\mathbf{B}_+$ case

For materials of type $\mathbf{B}_+$ ($\beta = 1$, e.g. Vlasov matter) the approach toward the singularities of LRS Bianchi type IX models is oscillatory.



Conclusions and Open Questions

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- The assumption of Local Rotational Symmetry is undesirable, but hard to relaxe. Partial results for Vlasov Matter in Bianchi IX have been obtained (work in progress with Håkan Andréasson and Mark Heinzle)
- An interesting open question is whether there is any relation between the classification of the matter models and the geometric structure of the singularity (Curvature vs Coordinates sing.)