Asymptotic behavior of convex Cauchy hypersurfaces

Belraouti Mehdi

Institut Fourier, UJF Grenoble

Novembre 2012

Table of contents

Table of contents

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors ($|v|^2 > 0$)

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors ($|v|^2 > 0$), timelike vectors ($|v|^2 < 0$)

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors $(|v|^2 > 0)$, timelike vectors $(|v|^2 < 0)$and lightlike vectors $(|v|^2 = 0)$.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors $(|v|^2 > 0)$, timelike vectors $(|v|^2 < 0)$and lightlike vectors $(|v|^2 = 0)$.
- A choice of a timelike vector fields $\longrightarrow$ chronological orientation.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors ($|v|^2 > 0$), timelike vectors ($|v|^2 < 0$)and lightlike vectors ($|v|^2 = 0$).
- A choice of a timelike vector fields $\longrightarrow$ chronological orientation.
- Orientation+chronological orientation $\longrightarrow$ a spacetime.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors $(|v|^2 > 0)$, timelike vectors $(|v|^2 < 0)$and lightlike vectors $(|v|^2 = 0)$.
- A choice of a timelike vector fields $\longrightarrow$ chronological orientation.
- Orientation+chronological orientation $\longrightarrow$ a spacetime.
- $T : M \to \mathbb{R}$ is strictly increasing along causal curves $\longrightarrow$ T is a time function.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors ($|v|^2 > 0$), timelike vectors ($|v|^2 < 0$)and lightlike vectors ($|v|^2 = 0$).
- A choice of a timelike vector fields $\longrightarrow$ chronological orientation.
- Orientation+chronological orientation $\longrightarrow$ a spacetime.
- $T : M \to \mathbb{R}$ is strictly increasing along causal curves $\longrightarrow$ T is a time function.
- The level sets of T are compact $\longrightarrow$ T Cauchy time function and the level sets of T are the Cauchy hypersurfaces.

Definition

A Lorentzian manifold (M, g) is a manifold M equipped with a pseudo-Riemannian metric g of signature $(- + ...+)$.

- Every tangent space is a Minkowski space $\longrightarrow$ spacelike vectors ($|v|^2 > 0$), timelike vectors ($|v|^2 < 0$)and lightlike vectors ($|v|^2 = 0$).
- A choice of a timelike vector fields $\longrightarrow$ chronological orientation.
- Orientation+chronological orientation $\longrightarrow$ a spacetime.
- $T : M \to \mathbb{R}$ is strictly increasing along causal curves $\longrightarrow$ T is a time function.
- The level sets of T are compact $\longrightarrow$ T Cauchy time function and the level sets of T are the Cauchy hypersurfaces.
- Existence of a Cauchy time function $\longrightarrow$ Globally hyperbolic Cauchy compact.

Table of contents

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Definition

If Ω is non empty and if Λ contains at least two elements, then Ω is called a flat regular domain.

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Definition

If Ω is non empty and if Λ contains at least two elements, then Ω is called a flat regular domain.

- Ω is a convex domain of $\mathbb{R}^{1,n}$.

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Definition

If Ω is non empty and if Λ contains at least two elements, then Ω is called a flat regular domain.

- Ω is a convex domain of $\mathbb{R}^{1,n}$.
- $\partial\Omega$ is the graph of a 1-Lipschitz convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$.

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Definition

If Ω is non empty and if Λ contains at least two elements, then Ω is called a flat regular domain.

- Ω is a convex domain of $\mathbb{R}^{1,n}$.
- $\partial\Omega$ is the graph of a 1-Lipschitz convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$.
- The Minkowski metric induce a pseudo-distance $d_{\partial\Omega}$ on $\partial\Omega$

Consider $\mathfrak{P}$ the space of lightlike hyperplanes. Let Λ be a closed subset of $\mathfrak{P}$. Define the open set: $\Omega = \bigcap_{P \in \Lambda} I^+(P)$.

Definition

If Ω is non empty and if Λ contains at least two elements, then Ω is called a flat regular domain.

- Ω is a convex domain of $\mathbb{R}^{1,n}$.
- $\partial\Omega$ is the graph of a 1-Lipschitz convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$.
- The Minkowski metric induce a pseudo-distance $d_{\partial\Omega}$ on $\partial\Omega$

Definition

The initial singularity associated to Ω is the cleaning of $(\partial\Omega, d_{\partial\Omega})$

Theorem (Mess 1990 $2+1$ dimesion, Barbot 2003 $n+1 \geq 4$)

Let M be a maximal globally hyperbolic Cauchy compact (MGHC) flat space-time of dimension $n+1$. Then, reverting the time if necessary, M is the quotient of a flat regular domain Ω by a discrete subgroup $\Gamma \subset SO(1,n) \ltimes \mathbb{R}^{1,n}$ acting freely and properly discontinuously.

Theorem (Mess 1990 $2+1$ dimesion, Barbot 2003 $n+1 \geq 4$)

Let M be a maximal globally hyperbolic Cauchy compact (MGHC) flat space-time of dimension $n+1$. Then, reverting the time if necessary, M is the quotient of a flat regular domain Ω by a discrete subgroup $\Gamma \subset SO(1,n) \ltimes \mathbb{R}^{1,n}$ acting freely and properly discontinuously.

Theorem (Mess 1990)

Let S be a surface of genus ≥ 2. There is an one to one correspondence between the space of measured geodesic laminations on S and the space of flat maximal globally hyperbolic $2+1$ space-times admitting a Cauchy hypersurface homeomorphic to S.

Table of contents

Let $M = \Omega/\Gamma$ be a *MGHC* future complete flat space-time. Let $T : \Omega \to]0, \infty]$ be a Γ invariant Cauchy time function.

Let $M = \Omega/\Gamma$ be a *MGHC* future complete flat space-time. Let $T : \Omega \rightarrow]0, \infty]$ be a Γ invariant Cauchy time function.

- The restriction of Q to every $\tilde{S}_t^T \longrightarrow (\tilde{S}_t^T, g_t)$ Riemannian hypersurface.

Let $M = \Omega/\Gamma$ be a *MGHC* future complete flat space-time. Let $T : \Omega \to]0, \infty]$ be a Γ invariant Cauchy time function.

- The restriction of Q to every $\tilde{S}_t^T \longrightarrow (\tilde{S}_t^T, g_t)$ Riemannian hypersurface.
- $(\tilde{S}_t^T, g_t)$ Riemannian hypersurfaces $\longrightarrow$ one-parameter family $(\tilde{S}_t^T, d_t)$ of metric spaces .

Let $M = \Omega/\Gamma$ be a *MGHC* future complete flat space-time. Let $T : \Omega \to]0, \infty]$ be a Γ invariant Cauchy time function.

- The restriction of Q to every $\tilde{S}_t^T \longrightarrow (\tilde{S}_t^T, g_t)$ Riemannian hypersurface.
- $(\tilde{S}_t^T, g_t)$ Riemannian hypersurfaces $\longrightarrow$ one-parameter family $(\tilde{S}_t^T, d_t)$ of metric spaces .
- Γ acts by isometries on $(\tilde{S}_t^T, d_t)$.

Let $M = \Omega/\Gamma$ be a *MGHC* future complete flat space-time. Let $T : \Omega \to]0, \infty]$ be a Γ invariant Cauchy time function.

- The restriction of Q to every $\tilde{S}_t^T \longrightarrow (\tilde{S}_t^T, g_t)$ Riemannian hypersurface.
- $(\tilde{S}_t^T, g_t)$ Riemannian hypersurfaces $\longrightarrow$ one-parameter family $(\tilde{S}_t^T, d_t)$ of metric spaces .
- Γ acts by isometries on $(\tilde{S}_t^T, d_t)$.

Question

The asymptotic behavior of $(\Gamma, \tilde{S}_t^T, d_t)$ *when* $t \to \infty$*?*

Question

The asymptotic behavior of $(\Gamma, \tilde{S}_t^T, d_t)$ *when* $t \to \infty$*? when* $t \to 0$*?*

Question

The asymptotic behavior of $(\Gamma, \tilde{S}_t^T, d_t)$ *when* $t \to \infty$*? when* $t \to 0$*? For which topology?*

Question

The asymptotic behavior of $(\Gamma, \tilde{S}_t^T, d_t)$ *when* $t \to \infty$*? when* $t \to 0$*? For which topology?For which time?*

Question

The asymptotic behavior of $(\Gamma, \tilde{S}_t^T, d_t)$ *when* $t \to \infty$*? when* $t \to 0$*? For which topology?For which time?*

Definition

A Cauchy time function $T : \Omega \to \mathbb{R}$ *is quasi-concave if the level sets* $S_t = T^{-1}(t)$ *are convex, i.e. if the future of* S_t *is convex.*

Definition

A Cauchy time function $T : \Omega \to \mathbb{R}$ is quasi-concave if the level sets $S_t = T^{-1}(t)$ are convex, i.e. if the future of S_t is convex.

- If S is C^2 then it is equivalent to the semi-positivity of the second fondamental form.

Definition

A Cauchy time function $T : \Omega \to \mathbb{R}$ *is quasi-concave if the level sets* $S_t = T^{-1}(t)$ *are convex, i.e. if the future of* S_t *is convex.*

- If S is C^2 then it is equivalent to the semi-positivity of the second fondamental form.
- It is also equivalent to the **Hubble law**, i.e. the metrics on the level sets increase along the gradient lines.

Definition

A Cauchy time function $T : \Omega \to \mathbb{R}$ *is quasi-concave if the level sets* $S_t = T^{-1}(t)$ *are convex, i.e. if the future of* S_t *is convex.*

- If S is C^2 then it is equivalent to the semi-positivity of the second fondamental form.
- It is also equivalent to the **Hubble law**, i.e. the metrics on the level sets increase along the gradient lines.

When $t \to 0$?

When $t \to 0$?

Theorem (Bonsante 2005)

Let T be the cosmological time of Ω. Then $(\Gamma, \tilde{S}_t^T, d_t)$ converge on the Gromov equivariant topology to the cleaning of $(\Gamma, \partial\Omega, d_{\partial\Omega})$. Moreover $lim_{t\to 0} l_t(\gamma) = l_\Sigma(\gamma)$

When $t \to 0$?

Theorem (Bonsante 2005)

Let T be the cosmological time of Ω. Then $(\Gamma, \tilde{S}_t^T, d_t)$ converge on the Gromov equivariant topology to the cleaning of $(\Gamma, \partial\Omega, d_{\partial\Omega})$. Moreover $lim_{t\to 0} l_t(\gamma) = l_\Sigma(\gamma)$

Theorem (Bonsante and Benedetti 2006)

Let M be a MGHC future complete $2+1$ space-time and consider the associated measured geodesic lamination (S, λ). Then the cleaning of $(\partial\Omega, d_{\partial\Omega})$, equipped with the isometric action of $\pi_1(S)$, is equivariantly isometric to the real tree dual to the lamination λ.

Suppose that $n = 2$. Let T be the CMC time function defined on Ω.

Suppose that $n = 2$. Let T be the CMC time function defined on Ω.

Question (Benedetti and Guadagnini 2001)

$lim_{t\to 0} l_t(\gamma) = l_{\Sigma(\gamma)}$

Suppose that $n = 2$. Let T be the CMC time function defined on Ω.

Question (Benedetti and Guadagnini 2001)

$lim_{t\rightarrow\ 0} l_t(\gamma) = l_{\Sigma(\gamma)}$

Theorem (Andersson 2003)

If the measured geodesic lamination λ associated to Ω is locally finite. Then the CMC levels converge in the Hausdorff Gromov equivariant topology to the real tree dual to λ. Moreover $lim_{t\rightarrow 0} l_t(\gamma) = l_\Sigma(\gamma)$.

Suppose that $n = 2$. Let T be the *CMC* time function defined on Ω.

Question (Benedetti and Guadagnini 2001)

$lim_{t\rightarrow\ 0} l_t(\gamma) = l_{\Sigma(\gamma)}$

Theorem (Andersson 2003)

If the measured geodesic lamination λ associated to Ω is locally finite. Then the CMC levels converge in the Hausdorff Gromov equivariant topology to the real tree dual to λ. Moreover $lim_{t\rightarrow 0} l_t(\gamma) = l_\Sigma(\gamma)$.

Theorem (B 2011)

Let M be a flat maximal globally hyperbolic Cauchy compact $2+1$ space-time and let λ be the associated measured lamination. Consider $T : \tilde{M} \to R$ quasi-concave Cauchy time function Γ invariant. Then the action of Γ on level sets of T converge on the Gromov equivariant topology to the real tree dual to λ. Moreover $lim_{t \to 0} l_t(\gamma) = l_{\Sigma(\gamma)}$.

- Positive answer to the question of Benedetti and Guadagnini 2001
- Application to the k time

Theorem (B 2011)

Let M be a flat maximal globally hyperbolic Cauchy compact $2+1$ space-time and let λ be the associated measured lamination. Consider $T : \tilde{M} \rightarrow R$ quasi-concave Cauchy time function Γ invariant. Then the action of Γ on level sets of T converge on the Gromov equivariant topology to the real tree dual to λ. Moreover $lim_{t \rightarrow 0} l_t(\gamma) = l_{\Sigma(\gamma)}$.

- Positive answer to the question of Benedetti and Guadagnini 2001
- Application to the k time
- The same results for de Sitter and anti de Sitter cases.

Theorem (B 2012)

Let M be a flat maximal globally hyperbolic Cauchy compact $n+1$ space-time. Consider $T : \tilde{M} \to R$ quasi-concave Cauchy time function Γ invariant. Then the action of Γ on level sets of T converge in the Gromov equivariant topology a $CAT(0)$ space. The limit is the same for all quasi-concave time functions. Moreover $lim_{t \to 0} l_t(\gamma) = l_{\Sigma(\gamma)}$.

When $t \to +\infty$

Theorem (Bonsante 2005)

Let T be the cosmologic time. Then $(\Gamma, X, t^{-1}d_t)$ converge in the Gromov equivariant topology to $(\Gamma, \mathbb{H}^n, d_{\mathbb{H}^n})$ when t goes to ∞.

When $t \to +\infty$

Theorem (Bonsante 2005)

Let T be the cosmologic time. Then $(\Gamma, X, t^{-1}d_t)$ converge in the Gromov equivariant topology to $(\Gamma, \mathbb{H}^n, d_{\mathbb{H}^n})$ when t goes to ∞. Moreover $\lim_{t\to\infty} t^{-1} l_t(\rho(\gamma)) = l_{\mathbb{H}^n}(\gamma)$.

When $t \to +\infty$

Theorem (Bonsante 2005)

Let T be the cosmologic time. Then $(\Gamma, X, t^{-1}d_t)$ converge in the Gromov equivariant topology to $(\Gamma, \mathbb{H}^n, d_{\mathbb{H}^n})$ when t goes to ∞. Moreover $lim_{t\to\infty} t^{-1} l_t(\rho(\gamma)) = l_{\mathbb{H}^n}(\gamma)$.

Theorem (B 2012)

There is a constant K depending only on the space-time M such that: for every quasi-concave Γ-invariant Cauchy time function $T : \Omega \to]0, +\infty[$, the renormalized level sets $(\Gamma, \tilde{S}_t^T, \frac{1}{\sup_{\tilde{S}_t^T} \tau} d_t)$ converge on the Gromov equivariant topology to a K-bilipschitz space of $(\mathbb{H}^n, d_{\mathbb{H}^n})$.