

Existence of Dynamical Vacuum Black Holes

(joint with M. Dafermos and I. Rodnianski)

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The Black Hole Stability Problem

The Kerr family of solutions $(\mathcal{M}, g_{M,a})$ is believed to play the central role as the final state for vacuum gravitational collapse. The $(\mathcal{M}, g_{M,a})$ satisfy

$$R_{\mu\nu} = 0.$$

Do sufficiently small perturbations of Kerr initial data converge

- to a black hole solution?
- another Kerr solution (on the domain of outer communications)?

This question is still wide open. The only non-linear global stability result of this type (in the asymptotically flat context) is the celebrated “Stability of the Minkowski space” (Christodoulou-Klainerman, Bieri, Lindblad-Rodnianski).

Intense research over the past ten years recently culminated in a complete understanding of $\square_g \psi = 0$ for g a sub-extremal member of the Kerr family ($|a| < M$) by Dafermos and Rodnianski:

Solutions to the linear wave equation $\square_g \psi = 0$ for g a fixed sub-extremal Kerr background decay polynomially in time on the black hole exterior, including along the event horizon.

- Previous work: Andersson, Blue, Dafermos, Donniger, Luk, Metcalfe, Rodnianski, Schlag, Soffer, Tataru, Tohaneanu
- cosmological constant ($\Lambda > 0$: Bony-Haefner, Melrose, Dyatlov; $\Lambda < 0$: G.H., G.H.-Smulevici)
- extremal Kerr $|a| = M$: Aretakis

The non-linear problem is still quite far away. However, there is a simpler question which has not been answered satisfactorily:

Do there exist *any* non-trivial spacetimes converging in time to a member of the Kerr family?

There are examples arising from toy-problems in symmetry classes:

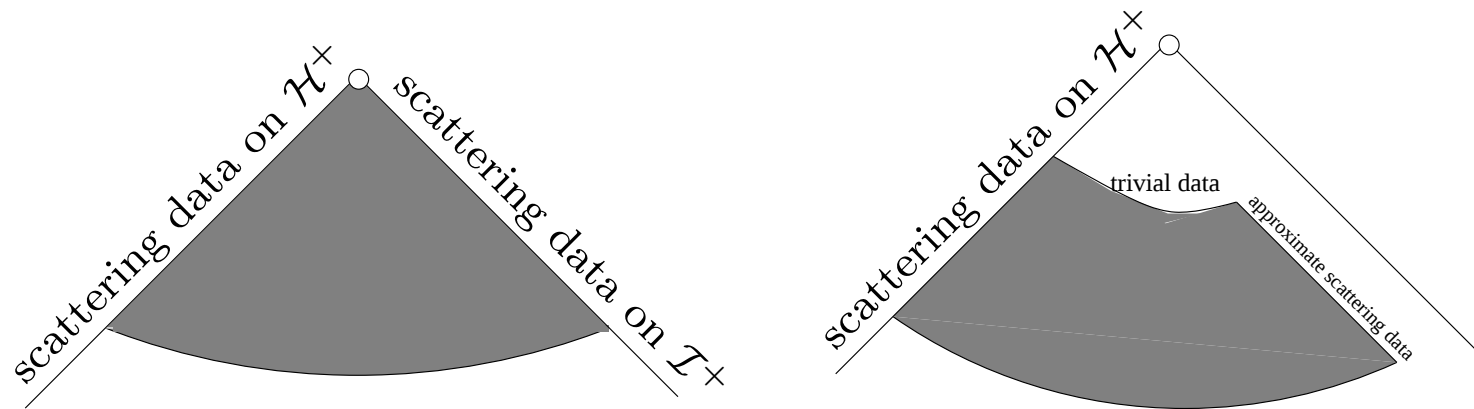
- spherically-symmetric self-gravitating scalar field in 4d
[Christodoulou, Dafermos-Rodnianski]; polynom. decay to Schwarzschild
- vacuum in 5d with bi-axial symmetry; polynomial decay to Schwarzschild [G.H]
- Robinson-Trautman metrics [Chrusciel]; exponential convergence to Schwarzschild

The content of this talk is the following

Theorem 1. *There exist smooth vacuum black hole spacetimes parametrized by “scattering data” with the full functional degree of freedom, which asymptote in time to a Kerr spacetime for any choice of parameters $|a| \leq M$.*

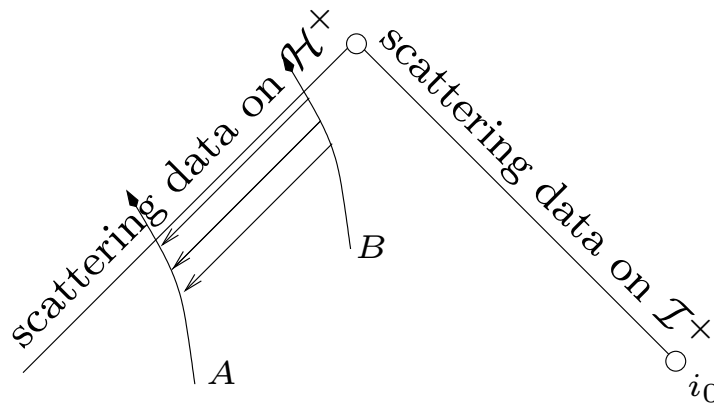
- no symmetry assumptions
- full functional degrees of freedom
- *exponential* convergence in time to Kerr (non-generic!)
- extremal case included

The construction proceeds by constructing appropriate “finite” problems and then show convergence:



Of course, establishing uniform estimates for the finite backwards problem is key.

The reason for the exponential decay can be traced back to the blue-shift effect:



At the level of the analysis, this can already be seen for the wave equation from the energy estimate arising from the divergence identity

$$\nabla^\mu (T_{\mu\nu} X^\nu) = T_{\mu\nu}^{(X)} \pi^{\mu\nu} .$$

Applying this with a vectorfield X everywhere timelike on the exterior [in order to produce non-degenerate energies!], one sees that the $T_{\mu\nu}^{(X)} \pi^{\mu\nu}$ has the wrong sign near the horizon for the backwards problem leading to an exponential forcing term.

The constant of the bad forcing term is related to the surface gravity of the horizon. From Gronwall's inequality, it is then easy to see that imposing sufficiently strong exponential decay can be propagated.

Remark

Note that we are dealing with *non-degenerate* energies here. For Schwarzschild, one could actually work with the degenerate energies arising from ∂_t and construct an isomorphism between scattering data on $\mathcal{H}^+$ and $\mathcal{I}^+$ and data on Σ_0 , cf. Dimock et al.

For Kerr or for a non-linear wave equation on Schwarzschild this breaks down which is why we work with non-degenerate energies from the beginning.

The analogy with the linear wave equation, explains the exponential decay.

- only the “naive” energy estimate is used – in particular, no symmetries/ approximate Killing properties
- the complicated geometry of the black hole exterior (superradiance, trapping) does not enter (no “Morawetz”-type estimate)
- Schwarzschild- and Kerr are “equally difficult”

Let us turn to the full problem (for Schwarzschild). Particularly important is an appropriate *formulation of the equations*, which involves the issue of *renormalization*. We formulate the vacuum Einstein equations as

$$\nabla^\alpha W_{\alpha\beta\gamma\delta} = 0 \quad \text{Bianchi equations}$$

$$\nabla\Gamma + \Gamma\Gamma = W \quad \text{(null)-structure equations}$$

These equations will be null-decomposed á la Christodoulou-Klainerman and then renormalized with respect to their Schwarzschild values.

One fixes the differentiable structure of the Schwarzschild manifold and wants to equip the manifold with metrics of the form

$$g = -4\Omega^2 dudv + g_{CD} (d\theta^C + b^C dv) (d\theta^D + b^D dv)$$

corresponding to a double-null foliation. [In Schwarzschild $\Omega^2 = 1 - \frac{2M}{r}$, $g = r^2(u, v) \gamma$, $b = 0$. We can write the metric arising from the mixed IVP locally in this way (Rendall, Christodoulou). We null-decompose with respect to the foliation and renormalize all quantities (Γ, ψ) with respect to their Schwarzschild values and finally obtain a system of hyperbolic and transport equations for *decaying* quantities:

$$\nabla_3 \psi = \mathcal{D}\tilde{\psi} + \Gamma\psi \quad , \quad \nabla_4 \tilde{\psi} = \mathcal{D}\psi + \Gamma\psi$$

and

$$\nabla_3 \Gamma = \Gamma\Gamma + \psi \quad , \quad \nabla_4 \Gamma = \Gamma\Gamma + \psi$$

The proof proceeds as follows.

- estimate the curvature in (r -weighted!) L^2 on spacelike and null-hypersurfaces via energy estimates from the null-Bianchi equations.
- estimate the Ricci-coefficients in L^2 on the spheres $S^2(u, v)$ from the transport equations using the curvature fluxes.
- Bootstrap appropriate exponential decay of these norms

The estimates for the Bianchi equations are done separately for each “Bianchi pair” (see “ultimately Schwarzschild spacetimes” [G.H]).

In fact, we first provide a systematic formulation of the equations.

$$\nabla_3 \psi_p = \not\partial \psi'_{p'} + E_3 [\psi_p] \quad (1)$$

$$\nabla_4 \psi'_{p'} + \gamma_4 (\psi'_{p'}) \operatorname{tr} \chi \psi'_{p'} = \not\partial \psi_p + E_4 [\psi'_{p'}] \quad (2)$$

with the index p indicating the radial decay at null-infinity.

- structure of equations (i.e. their p -decay) is *preserved* under commutation with the operators $\{\nabla_3, r\nabla_4, r\nabla\}$
- no symmetries are being used here!
- makes taking an additional derivative effortless
- control of non-linear errors reduces to counting decay + regularity

Systematic formulation of the null-structure equations:

$$\nabla_3^{(3)} \Gamma_p = \sum_{p_1+p_2 \geq p} (f_{p_1} + \Gamma_{p_1}) \Gamma_{p_2} + \psi_p$$

$$\nabla_4^{(4)} \Gamma_p = \boxed{\sum_{p_1+p_2=p+1} f_{p_1}^{(3)} \Gamma_{p_2}} + \sum_{p_1+p_2 \geq p+2} (f_{p_1} + \Gamma_{p_1}) \Gamma_{p_2} + \psi_{p+2}$$

Note the gain of two powers in the 4-direction except for the anomalous boxed term. The key observation is that whenever a boxed term appears, the Γ_p involved satisfies an equation in the 3 direction!

This structure is preserved under commutation with $\{\nabla_3, r\nabla_4, r\nabla\}$!

Think as follows

$$\begin{aligned} \int_{S^2(u,v)} r^{2p-2} |\Gamma_p|^2 \sqrt{g} d^2\theta \leq data + \int_u^{u_{hoz}} d\bar{u} \int_{S^2} r^{2p-2} |\psi_p|^2 \sqrt{g} d^2\theta \\ + \int_u^{u_{hoz}} d\bar{u} \int_{S^2} r^{2p-2} |\Gamma_p|^2 \sqrt{g} d^2\theta \end{aligned}$$

Insert bootstrap assumptions... No loss in r ! In the other direction,

$$\begin{aligned} \int_{S^2(u,v)} r^{2q-2} |\Gamma_p|^2 \sqrt{g} d^2\theta \leq data + \int_v^{v_\infty} d\bar{v} \int_{S^2} r^{2q-2} |\psi_{p+2}|^2 \sqrt{g} d^2\theta \\ + \int_v^{v_\infty} d\bar{v} \int_{S^2} \boxed{\frac{1}{r^2}} r^{2q-2} |\Gamma_p|^2 \sqrt{g} d^2\theta \end{aligned}$$

The $\frac{1}{r^2}$ is necessary for integrability near infinity! For the $\frac{1}{r}$ -term, $r^{2q-2} |\Gamma_p|^2$ will decay in r to ensure integrability and a smallness factor comes from the fact that the Γ_p involved has already been improved in the 3-direction.

This is, schematically, how the bootstrap assumptions can be improved. It essentially works because the bad linear terms (caused by the blueshift) can be estimated by choosing the exponential rate that is bootstrapped sufficiently large:

$$C \int_{t_1}^{t_2} dt \int_{\Sigma_t} |D\psi|^2 \leq C \frac{1}{A} e^{-At}$$

while all non-linear error-terms can be made small by choosing the t_0 of the bottom slice large. However, recall that understanding the radial decay was crucial!

This gives uniform control for every solution arising from $t_0 < t_f < \infty$. The final step is to obtain convergence. For this one needs to compare (i.e. identify) two spacetimes. The fixed differentiable structure provides a natural setting to do this.

One considers differences of null-structure and Bianchi equations and repeats the estimates. There is a slight simplification if one is willing to use elliptic estimates.

Final comments I

- We constructed a class of smooth dynamical black hole solutions without symmetry depending on the full scattering data.
(Previously: symmetry classes and Robinson-Trautman metrics [Chrusciel])
- Some of the estimates, as well as the formalism established, may be useful for the forward problem
- Generalization to de Sitter and Anti-de Sitter black holes [Friedrich, G.H.-Smulevici]

Final comments II

What about polynomial decay? The theorem is believed to be sharp in the following sense:

Conjecture 1. *Blue shift-conjecture: For generic, polynomially decaying scattering data there does **not** exist a spacetime (M, g) “bounded” by $\mathcal{H}^+$ and $\mathcal{I}^+$ and smooth up to $\mathcal{H}^+$.*