Global solutions of water waves equations in one dimension

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(joint work with Thomas Alazard)

1. Statement of the main result

Let $(t,x) \to (\eta,\psi)(t,x)$ be a function from $[T_0, T[\times\mathbb{R}$ to $\mathbb{R}^2$ solving the Craig-Sulem-Zakharov system:

$$\text{(CSZ)} \qquad \begin{aligned} \partial_t \eta &= G(\eta)\psi \\ \partial_t \psi &= -\eta - \frac{1}{2}(\partial_x \psi)^2 + \frac{[G(\eta)\psi + \partial_x\psi\partial_x\eta]^2}{2(1+(\partial_x\eta)^2)} \end{aligned}$$

where $G(\eta)$ is the Dirichlet-Neumann operator defined by $G(\eta)\psi = \sqrt{1+(\partial_x\eta)^2}\partial_n \Phi|_{y=\eta(t,x)}$, where

$$(\partial_x^2 + \partial_y^2)\Phi = 0 \text{ in } y < \eta(t,x), \; \Phi|_{y=\eta(t,x)} = \psi.$$

Known results: • Local existence in dimension 1 or 2 (Sijue Wu).
• Almost global existence with small data in dimension one: if $(\eta,\psi)|_{t=1} = \epsilon(\eta_0,\psi_0)$, the solution exists on an interval $[1, e^{c/\epsilon}[$ (Sijue Wu).
• Global existence in dimension 2 for $\epsilon \ll 1$ (Sijue Wu), global existence and scattering (Germain-Masmoudi-Shatah).

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Theorem

In dimension one, take $s \gg s_1 \gg 1$ integers. Let (η_0, ψ_0) be such that

$$(x\partial_x)^p \eta_0 \in H^{s-p}, (x\partial_x)^p |D|^{1/2}\psi_0 \in H^{s-p},\ p \leq s_1.$$

For $\epsilon \ll 1$, (CSZ) with data $(\eta, \psi)|_{t=1} = \epsilon(\eta_0, \psi_0)$ has a unique global smooth solution. Moreover, $u = |D|^{1/2}\psi + i\eta$ has the asymptotic expansion

$$u(t,x) = \frac{\epsilon}{\sqrt{t}} \alpha\Big(\frac{x}{t}\Big) \exp\left[\frac{it}{4|x/t|} + i\frac{\epsilon^2}{64}\frac{|\alpha(x/t)|^2}{|x/t|^5}\log t\right] + t^{-\frac{1}{2}-\kappa}\rho(t,x),$$

where $\kappa > 0$, $\alpha \in C_b^0(\mathbb{R})$, $\rho \in L^\infty$.

Reference: Ionescu-Pusateri obtained independently a similar result of global existence, with the following differences: Decay assumptions are weaker, and asymptotics are obtained for $\hat{u}(t,\xi)$ instead of $u(t,x)$.

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2. Strategy of proof

One uses the Klainerman vector field $Z = t\partial_t + 2x\partial_x$.

Model problem: $(D_t - P(D_x))u = \underbrace{N(u)}_{\text{cubic}}$, where $D_t = \frac{1}{i}\partial_t$, $P(\xi)$ real valued symbol (for instance $P(\xi) = |\xi|^{1/2}$).

Let Z be a vector field so that $[Z, D_t - P(D_x)] = 0$. One gets

$$(D_t - P(D_x))Z^k u = Z^k N(u) \sim u^2 Z^k u + \cdots$$

whence

$$\|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} + \int_1^t C\|u(\tau,\cdot)\|_{L^\infty}^2 \|Z^k u(\tau,\cdot)\|_{L^2}\, d\tau.$$

Consider the following two conditions (with $0 < \delta_k, \tilde{\delta}'_k \ll 1$):

(A) For t in some interval $[1, T[$, $\|u(t,\cdot)\|_{L^\infty} = O(\epsilon/\sqrt{t})$ and for $k = 0, \dots, s/2$, $\|Z^k u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}+\tilde{\delta}'_k})$,

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- Apparent loss of derivatives

Consider a quadratic para-differential approximation of the (CSZ) system $\partial_t U = T_A U$, with

$$U = \begin{bmatrix} \eta \\ |D|^{1/2}\psi \end{bmatrix}, \ T_A = \text{ paradiff. operator with symbol } A = A(U, x, \xi),$$

where A is a 2×2 matrix of symbols

$$A(U, x, \xi) = \underbrace{A_0(U, x, \xi)}_{\text{anti-self-adjoint}} + \underbrace{A_1(U, x, \xi)}_{\text{self-adjoint}}.$$

Because of A_1, the eigenvalues of A have non zero real part, so generate a loss of derivatives in energy inequalities.

Way to overcome this difficulty: Use the "good unknown" of Alinhac. For the quadratic approximation above, this means introducing $\widetilde{U} = \begin{bmatrix} \eta \\ |D|^{1/2}\omega \end{bmatrix}$, with $\omega = \psi - T_{|D|\psi}\eta$. Then $\partial_t \widetilde{U} = T_{A_0}\widetilde{U}$. Since the eigenvalues of A_0 are purely imaginary, one gets Sobolev energy inequalities.

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• Quadratic character of the non-linearity

Because the non-linearity is quadratic, the energy inequality reads

$$\|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} + C\int_1^t \underbrace{\|u(\tau,\cdot)\|_{L^\infty}^2}_{O(\epsilon^2/\tau)} \|Z^k u(\tau,\cdot)\|_{L^2}\, d\tau,$$

which gives a $O(e^{\epsilon\sqrt{t}})$ estimate for the left hand side.

Way to overcome this difficulty: Use a Shatah normal form:
Look for a new unknown $W = \widetilde{U} + B(\widetilde{U},\widetilde{U})$, where B is a bounded quadratic map on H^s, chosen so that W solves an equation $\partial_t W = T_{A_0'} W + T_{A_0''} W$, where

- A_0' is sublinear in W and $T_{A_0'}$ is antiself-adjoint on H^s
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3. Klainerman-Sobolev estimates

We want to prove that $(A) + (B) \Rightarrow (A)$ (with a better constant).
We first show, with $u = |D|^{1/2}\psi + i\eta$,

- $\|u(t,\cdot)\|_{L^\infty} = O(\epsilon/\sqrt{t})$
- $\|Z^k u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}+\tilde{\delta}'_k}), k \le s/2$
- $\|Z^k u(t,\cdot)\|_{L^2} = O(\epsilon t^{\delta_k}), k \le s.$

$\Rightarrow \|Z^k u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}+\delta'_k})$ for $k \le s - 100$.

Define $v(t,x)$ by $u(t,x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t})$ and set $h = \frac{1}{t}$.

Notation: If a is a symbol, set

$$\mathrm{Op}_h(a)v = a(x, hD)v = \frac{1}{2\pi}\int e^{ix\xi} a(x, h\xi)\hat{v}(\xi)\, d\xi.$$

The (CSZ) system written on v is given by

$$(D_t - \mathrm{Op}_h(x\xi + |\xi|^{1/2}))v = \sqrt{h}Q_0(V) + h\left[C_0(V) - \frac{i}{2}v\right] + h^{1+\kappa}R(V),$$

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Set $\Lambda = \{(x,\xi); \xi \neq 0, 2x\xi + |\xi|^{1/2} = 0\} = \{(x,\xi); \xi = d\omega(x)\}$ with $\omega(x) = 1/(4|x|)$.

Proposition: Let γ_Λ be a smooth cut-off close to Λ, equal to one on a neighborhood of Λ, $\gamma_\Lambda^c = 1 - \gamma_\Lambda$, $v_\Lambda = \mathrm{Op}_h(\gamma_\Lambda)v$, $v_{\Lambda^c} = \mathrm{Op}_h(\gamma_\Lambda^c)v$. Then for $k \leq s - 100$,
(i) $\|Z^k v_{\Lambda^c}\|_{L^2} = O(\epsilon h^{\frac{1}{2}-\delta'_k})$.
(ii) $\|(hD_x - d\omega)Z^k v_\Lambda\|_{L^2} = O(\epsilon h^{1-\delta'_k})$.
(iii) $\|Z^k v\|_{L^\infty} = O(\epsilon h^{-\delta'_k})$.

Proof. (i) Use that on the support of γ_Λ^c, $2x\xi + |\xi|^{1/2}$ is elliptic. Then $\|Z^k v_{\Lambda^c}\|_{L^2} = O(\epsilon h^{\frac{1}{2}-\delta'_k})$.

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(ii) Recall the equation

$$\mathrm{Op}_h(2x\xi + |\xi|^{1/2})v = -\sqrt{h}Q_0(V) + h\Big[\frac{i}{2}v - iZv - C_0(V)\Big] - h^{1+\kappa}R(V).$$

One may write $2x\xi + |\xi|^{1/2} = e(x,\xi)(\xi - d\omega(x))$, whence

$$\mathrm{Op}_h(\xi - d\omega)\mathrm{Op}_h(\gamma_\Lambda)v = -\sqrt{h}\mathrm{Op}_h(\gamma_\Lambda)\mathrm{Op}_h(\tilde{e})Q_0(V) + h(\cdots).$$

Since by (i), $v = v_\Lambda + O_{L^2}(h^{\frac{1}{2}-0})$,
$Q_0(v,\bar{v}) = Q_0(v_\Lambda, \bar{v}_\Lambda) + O_{L^2}(h^{\frac{1}{2}-0})$.
By localization, $\mathrm{Op}_h(\gamma_\Lambda)Q_0(v_\Lambda, \bar{v}_\Lambda) \sim 0$.

(*ii*) Recall the equation

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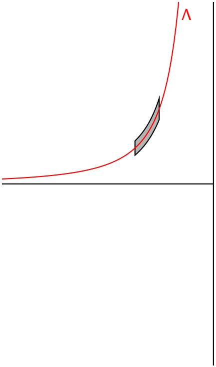

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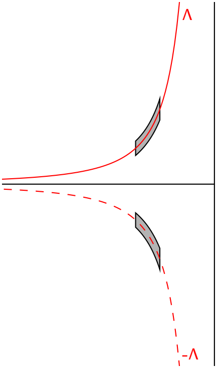

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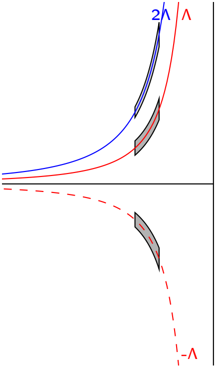

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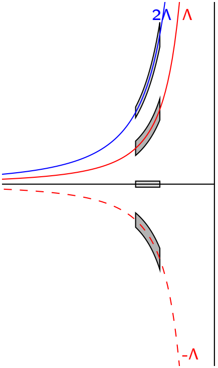

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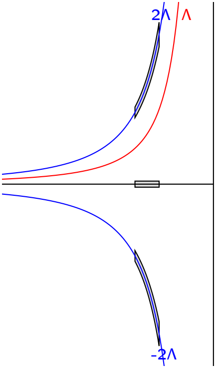

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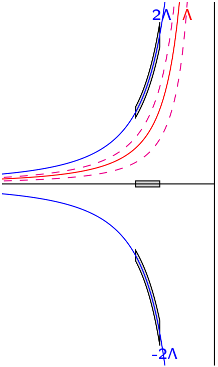

(iii) Let us prove that $\|Z^k v\|_{L^\infty} = O(\epsilon h^{-\delta'_k})$. One may reduce oneself to the case when v is microlocally supported in $h|\xi| \sim 1$. Then since by (i), $\|Z^k v_{\Lambda^c}\|_{L^2} = O(\epsilon h^{\frac{1}{2}-\delta'_k})$, by Sobolev injection, $\|Z^k v_{\Lambda^c}\|_{L^\infty} = O(\epsilon h^{-\delta'_k})$. On the other hand

$$\begin{aligned}
\|Z^k v_\Lambda\|_{L^\infty} &= \|e^{-i\omega/h} Z^k v_\Lambda\|_{L^\infty} \\
&\leq \|e^{-i\omega/h} Z^k v_\Lambda\|_{L^2}^{1/2} \|D(e^{-i\omega/h} Z^k v_\Lambda)\|_{L^2}^{1/2} \\
&\leq \|Z^k v_\Lambda\|_{L^2}^{1/2} \underbrace{\left\| \left(D - \frac{d\omega}{h} \right) Z^k v_\Lambda \right\|_{L^2}^{1/2}}_{= O(\sqrt{\epsilon} h^{-\delta'_k/2}) \text{ by } (ii)} = O(\epsilon h^{-\delta'_k}),
\end{aligned}$$

for $k \leq s - 100$.

4. Optimal L^∞ estimates

Proposition: One has an expression

$$v = v_\Lambda + \sqrt{h}(v_{2\Lambda} + v_{-2\Lambda}) + h(v_{3\Lambda} + v_{-\Lambda} + v_{-3\Lambda}) + h^{1+\kappa} g,$$

where $v_{\ell\Lambda}$ is microlocally supported close to $\ell\Lambda$ and is a semi-classical lagrangian distribution on $\ell\Lambda$ in the following sense

- $\|Z^k v_{\ell\Lambda}\|_{L^\infty} = O(\epsilon h^{-\delta'_k})$ and
- $\|\mathrm{Op}_h(e_\ell) Z^k v_{\ell\Lambda}\|_{L^\infty} = O(\epsilon h^{1-\delta'_k})$ for $\ell = -2, 1, 2$,
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where e_ℓ is a smooth symbol vanishing on $\ell\Lambda$. Moreover, $v_{\ell\Lambda}$ may be expressed as a polynomial in $v_\Lambda, \bar{v}_\Lambda$.

The fact that $v_{\ell\Lambda}$ are semi-classical Lagrangian distributions means that these functions have some oscillatory behavior along the phase $\ell\Lambda$. If, for instance, $w(x) = \alpha(x)\exp(i\omega(x)/h)$, then

$$\mathrm{Op}_h(\xi - d\omega)w = (hD_x - d\omega)w = O(h).$$

Idea of proof of the proposition: Argue as for the L^2 estimates, but using the a priori L^∞ bounds obtained above.

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Corollary: The solution v of (CSZ) satisfies an ODE

$$
\begin{aligned}
D_t v = &\frac{1}{2}(1-\chi)(xh^{-\beta})|d\omega|^{1/2} v \\
&- i\sqrt{h}(1-\chi)(xh^{-\beta})|d\omega|^{3/2}[\Phi_2(x)v^2 + \Phi_{-2}(x)\bar{v}^2] \\
&+ h(1-\chi)(xh^{-\beta})|d\omega|^{5/2}[\Phi_3(x)v^3 + \Phi_1(x)|v|^2 v \\
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&+ h^{1+\kappa} g
\end{aligned}
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where Φ_ℓ are real valued functions on $\mathbb{R}^*$, $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 0$ close to zero.

Corollary $\Rightarrow$ Uniform bound for v and asymptotics: Applying a normal form method to the ODE, one reduces to

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D_t f = \frac{1}{2}(1-\chi)(xh^{-\beta})|d\omega|^{1/2}\left[1 + \frac{|d\omega|^2}{t}|f|^2\right] f + O\Big(\frac{1}{t^{1+\kappa}}\Big).
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Proof: If $b(\xi)$ is a symbol, there is e with $e|_{\ell\Lambda} = 0$ such that

$$b(\xi) = b|_{\ell\Lambda} + e(x,\xi) = b(\ell d\omega) + e(x,\xi).$$

Then $\mathrm{Op}_h(b)v_{\ell\Lambda} = b(\ell d\omega(x))v_{\ell\Lambda} + \underbrace{\mathrm{Op}_h(e)v_{\ell\Lambda}}_{=O(h^{1-\delta_0'}) \text{ by the prop.}}$.

The quadratic form $Q_0(v,\bar{v})$ of the equation is made of terms of the form $\mathrm{Op}_h(b_0)[(\mathrm{Op}_h(b_1)v)(\mathrm{Op}_h(b_2)v)]$,... Then

$$\mathrm{Op}_h(b_0)[(\mathrm{Op}_h(b_1)v_\Lambda)(\mathrm{Op}_h(b_2)v_\Lambda)] = b_0(2d\omega)b_1(d\omega)b_2(d\omega)v_\Lambda^2 + O(h^{1-\delta_0'}).$$

Applying the same idea to the other non-local terms in

$$(D_t - \mathrm{Op}_h(x\xi + |\xi|^{1/2}))v = \sqrt{h}Q_0(V) + h\Big[C_0(V) - \frac{i}{2}v\Big] + h^{1+\kappa}R(V),$$

one gets the wanted ODE.