Graceful exit from inflation for minimally coupled Bianchi A scalar field models

Florian Beyer

Reference: F.B. and Leon Escobar (2013), CQG, 30(19), p.195020.

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Background and motivation

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Graceful exit from inflation is the *transition*

Accelerated expansion (inflation) $\longrightarrow$ Decelerated expansion.

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However: Concerning *finite* phases of accelerated expansion there are several open questions:

- For which matter models is this possible?
- Even if inhomogeneities and anisotropies are small by the end of inflation, could they grow again in the decelerated epoch after inflation?

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Answers to these questions are highly relevant to justify the standard model of cosmology:

The universe is modeled as an expanding *exactly* homogeneous and isotropic solution of Einstein's equations with certain matter fields.

$\longrightarrow$ Friedmann-Robertson-Walker (FRW) spacetimes

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Intermediate step towards answering these questions:
Restrict to *spatially homogeneous*, but in general *anisotropic* models.

Relevant matter models

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2. Minimally coupled scalar field models. Focus on this for the rest of the talk.
3. . . .

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A solution (M, g, ϕ) of the following system of equation is called minimally coupled self gravitating scalar field model

1. Einstein's field equations

$$G_{\mu\nu} = T_{\mu\nu},$$

with energy momentum tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left(g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi / 2 + V(\phi) \right),$$

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2. Scalar wave equation

$$\nabla^\mu\nabla_\mu\phi - V'(\phi) = 0.$$

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- We could add further matter fields which satisfy the strong and dominant energy conditions. Most of the analytic results mentioned in the following allow this. For our purposes here, we consider pure self-gravitating scalar field models.
- The scalar wave equation reduces to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

in the spatially homogeneous case and for scalar fields which are constant on the surfaces of homogeneity.

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However: Not much is known about finite inflation and the *graceful exit problem*.

Our choice of potential

We have decided to study the following potential

$$V(\phi) = e^{-c_1\phi}(c_2 + \phi^2),$$

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$$F(\phi) := \sqrt{\frac{3}{2}}\frac{V'(\phi)}{V(\phi)} = \sqrt{\frac{3}{2}}\left(-c_1 + \frac{2\phi}{\phi^2 + c_2}\right).$$

$$\implies \quad \lim_{\phi\to\infty} F(\phi) = -\sqrt{3/2}\,c_1 \neq 0.$$

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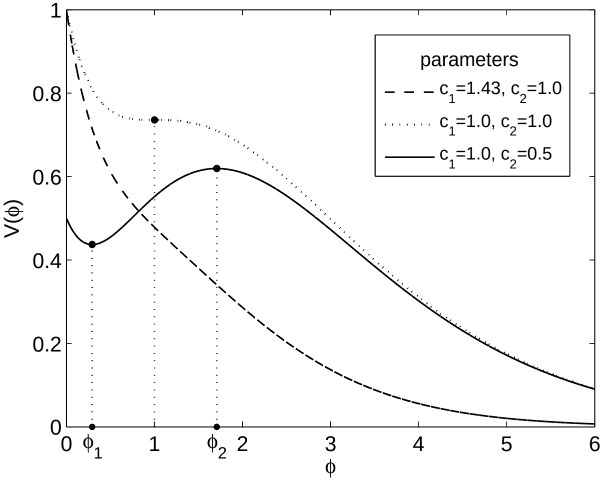
Notice: Studies of the graceful exit problem in the isotropic case for this potential were carried out by Parsons and Barrow [1995].



Assumption for today: Choose parameters of the potential c_1 and c_2 such that the potential is strictly monotonically decreasing.

Spatially homogeneous models

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Bianchi classification:

- Up to topological questions, such isometry groups are determined by their *3-dimensional real Lie algebras*.
- The Bianchi A case corresponds to the class of *unimodular real 3-dimensional Lie algebras*. For those, we can choose a basis $\{\xi_1, \xi_2, \xi_3\}$ such that

$$[\xi_b, \xi_c] = \sum_{a,e=1}^{3} \varepsilon_{bce} n^{ea} \xi_a,$$

for a symmetric constant matrix (n^{ea}).

Biance A classification continued

Result: Given any unimodular real 3-dimensional real Lie algebra, we can choose a basis $\{\xi_1, \xi_2, \xi_3\}$ such

$$(n^{ea}) = \mathrm{diag}(n_1, n_2, n_3),$$

where the real numbers n_1, n_2 and n_3 satisfy one of the following cases:

Bianchi I	$n_1 = n_2 = n_3 = 0$
Bianchi II	$n_2 = n_3 = 0,\ n_1 > 0$
Bianchi VI$_0$	$n_1 = 0,\ n_2 > 0,\ n_3 < 0$
Bianchi VII$_0$	$n_1 = 0,\ n_2 > 0,\ n_3 > 0$
Bianchi VIII	$n_1 < 0,\ n_2 > 0,\ n_3 > 0$
Bianchi IX	$n_1 > 0,\ n_2 > 0,\ n_3 > 0$

Formulate the field equations

1. Introduce a *symmetry invariant orthonormal frame* $\{e_0, e_1, e_2, e_3\}$ with e_0 perpendicular to the symmetry hypersurfaces. Let the symmetry hypersurfaces be labeled by a global time function t with unit gradient and choose coordinates (t, x^α) so that $e_0 = \partial_t$.

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2. Describe the models by

   - $H(t)$: Hubble scalar. Proportional to $\dot{vol}(t)/vol(t)$. Assume that $H(t) > 0$ during the whole evolution.
   - $\sigma_\pm(t)$: Anisotropy scalars.
   - $n_a(t)$: Eigenvalues of the matrix (n^{ea}) of the Lie algebra spanned by $\{e_1, e_2, e_3\}$. These determine the Bianchi type and the spatial curvature.
   - $\phi(t)$: Scalar field.

Formulate the field equations

3. Introduce Hubble time τ by $\frac{d\tau}{dt} = H$ and Hubble-normalized (dimensionless) quantities

$$(\Sigma_+, \Sigma_-, N_1, N_2, N_3) := (\sigma_+, \sigma_-, n_1, n_2, n_3)/H,$$

and

$$x := \dot{\phi}/(\sqrt{6}H), \quad y := \sqrt{V(\phi)}/(\sqrt{3}H),$$

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Result: The resulting state space is spanned by $(\Sigma_+, \Sigma_-, N_1, N_2, N_3, x, y, \phi) \in \mathbb{R}^8$.

Full set of equations – evolution equations

Evolution equations:

$$
\begin{aligned}
\Sigma'_\pm &= -(2-q)\Sigma_\pm - S_\pm, \\
N'_1 &= (q - 4\Sigma_+)N_1, \\
N'_2 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\
N'_3 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \\
x' &= x(q-2) - F(\phi)y^2, \\
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q :=& 2\Sigma_+^2 + 2\Sigma_-^2 + 2x^2 - y^2,\\
S_+ :=& \frac{1}{6}\big((N_2-N_3)^2\\
&- N_1(2N_1 - N_2 - N_3)\big),\\
S_- :=& \frac{1}{2\sqrt{3}}(N_3-N_2)(N_1-N_2-N_3),\\
F(\phi) =& \sqrt{\frac{3}{2}}\frac{V'(\phi)}{V(\phi)} = \sqrt{\frac{3}{2}}\left(\frac{2\phi}{\phi^2+c_2} - c_1\right).
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$$F(\phi) = \sqrt{\frac{3}{2}}\frac{V'(\phi)}{V(\phi)} = \sqrt{\frac{3}{2}}\left(\frac{2\phi}{\phi^2 + c_2} - c_1\right).$$

The quantity q is the deceleration scalar:

$q > 0$: Expansion is decelerated, $\ddot{vol} < 0$.

$q < 0$: Expansion is accelerated, $\ddot{vol} > 0$ (inflation).

Hence, a *sign change* $- \longrightarrow +$ *of* $q(\tau)$ *signals a graceful exit*.

Full set of equations – constraint

Hamiltonian constraint (generalized Friedmann equation):

$$1 = \Sigma_+^2 + \Sigma_-^2 + x^2 + y^2 + K,$$

with

$$K := -\frac{{}^3R}{6H^2} = \frac{1}{12}\Big(N_1^2 + N_2^2 + N_3^2 - 2(N_1 N_2 + N_2 N_3 + N_3 N_1)\Big),$$

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where ${}^3R$ is the spatial Ricci scalar.

Notice: $K \geq 0$ for all Bianchi A models, possibly except for Bianchi IX.

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Constraint implies: $R = \Sigma^2 + K$. Since $K \geq 0$ (except for Bianchi IX), it follows that both Σ^2 and K also decay during inflation.

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Open question: What happens after inflation, i.e., when q goes from negative to positive?

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Basic observations, which we discuss in detail now:

- We find $\lim_{\tau\to\infty} \phi(\tau) = \infty$ for generic initial data (monotonic potential). We replace ϕ by $\psi := 1/\phi$.
- If c_1 and c_2 satisfy certain conditions, then it turns out that there exist "future attractors" characterized by $\lim_{\tau\to\infty} q(\tau) > 0$. For inflationary initial data as above, a graceful exit then occurs naturally after some finite τ.

Bianchi I ($N_1 = N_2 = N_3 = 0$): Full set of equations

Evolution equations (dynamical system):

$$\begin{aligned}
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\psi' &= -\sqrt{6}x\psi^2.
\end{aligned}$$

with $q = 2\Sigma_+^2 + 2\Sigma_-^2 + 2x^2 - y^2$.

Constraint:

$$1 = \Sigma_+^2 + \Sigma_-^2 + x^2 + y^2.$$

Bianchi I ($N_1 = N_2 = N_3 = 0$): Fixed point

Numerical observation: Generic initially inflationary Bianchi I solutions approach the following *fixed point* of the dynamical system for $\tau \to \infty$

$$\Sigma_\pm = 0,\ N_1 = N_2 = N_3 = 0,\ x = c_1/\sqrt{6},\ y = \sqrt{1 - c_1^2/6},\ \psi = 0,$$

where $q = (c_1^2 - 2)/2$. This corresponds (in some sense) to a flat FRW perfect fluid solution.

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where $q = (c_1^2 - 2)/2$. This corresponds (in some sense) to a flat FRW perfect fluid solution.

Hence: This fixed point is in the decelerated regime if and only if

$$\sqrt{2} < c_1 < \sqrt{6}.$$

Bianchi I: Future non-linear stability of fixed point

Linear stability: Linearize Bianchi I evolution equations around fixed point. We get eigenvalues $\frac{1}{2}\left(c_1^2 - 6\right)$ (triple), 0 (single), $c_1^2 - 2$ (single). Hence for c_1 as above, we find:

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Linear stability: Linearize Bianchi I evolution equations around fixed point. We get eigenvalues $\frac{1}{2}\left(c_1^2-6\right)$ (triple), 0 (single), c_1^2-2 (single). Hence for c_1 as above, we find:

- 3-dimensional future stable subspace.
- 1-dimensional future unstable subspace: constraint violating mode (can be ignored if the constraints are satisfied).
- 1-dimensional center subspace.

Hence: Fixed point is not hyperbolic. Must apply center manifold theory.

Bianchi I: Dynamics on the center manifold and future non-linear stability

We find: Leading-order dynamics on the center manifold for $\eta = 1/\tau \to 0$ is

$$\Sigma_\pm(\eta) = 0,$$

$$\psi(\eta) = \eta/c_1 + \left(\phi_* - \frac{2\log\eta}{c_1^3}\right)\eta^2 + \dots,$$

$$x(\eta) = \frac{c_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}\frac{\eta}{c_1} - \sqrt{\frac{2}{3}}\eta^2\left(\phi_* + \frac{2}{c_1\left(6-c_1^2\right)} - \frac{2\log\eta}{c_1^3}\right) + \dots,$$

for a number ϕ_*.

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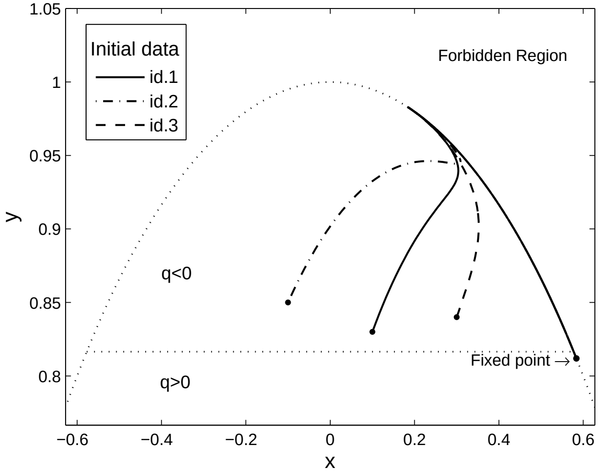
for a number ϕ_*.

In particular: Fixed point is future non-linearly stable within the Bianchi I class.

Bianchi I ($N_1 = N_2 = N_3 = 0$): Numerical studies

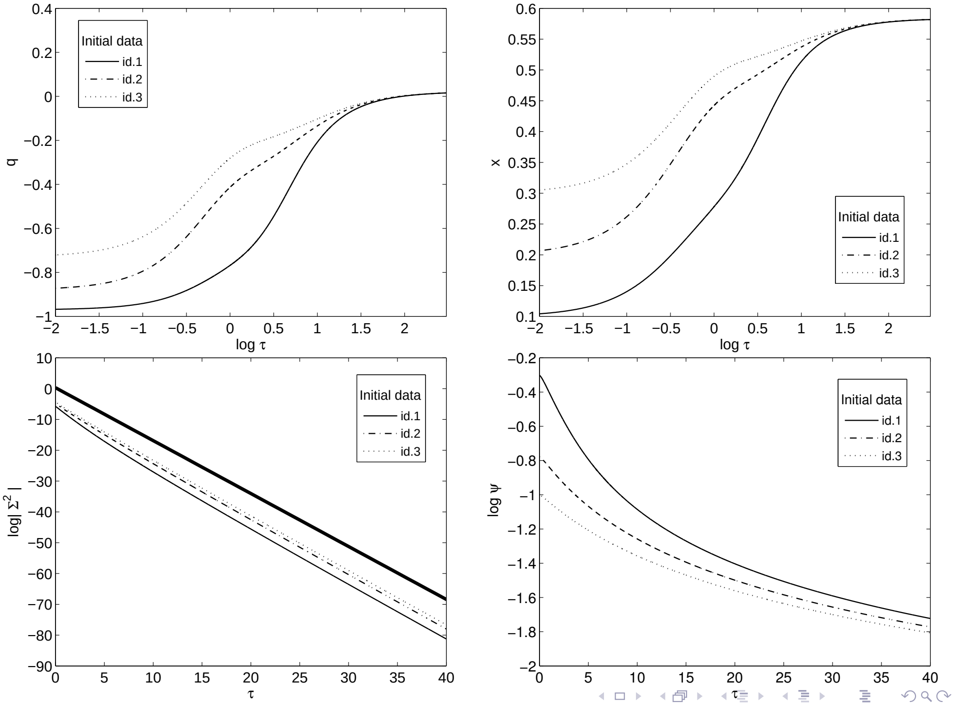
Numerical studies support the claim that the fixed point is indeed a future attractor if $\sqrt{2} < c_1 < \sqrt{6}$.

Bianchi I ($N_1 = N_2 = N_3 = 0$): Numerical studies



$c_1 = 1.43,$
$c_2 = 1.0.$

Notice: Constraint implies $x^2 + y^2 \leq 1$ and $q = 2 - 3y^2$. Recall that we have shown that $R = 1 - x^2 - y^2$ decays so long as $q < 0$.



Initial data
id.1
id.2
id.3
q
log τ
x
log τ
log| Σ² |
τ
log ψ
τ

Bianchi I ($N_1 = N_2 = N_3 = 0$): Results

We have found:

- If $\sqrt{2} < c_1 < \sqrt{6}$ and initial data are in the inflationary regime, then the corresponding Bianchi I solutions have graceful exits from inflation. We have not proven this, but our numerical studies suggest that the fixed point is a future attractor.

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- Fixed point is isotropic, so anisotropies continue to decay after inflation.
- Due to the unstable constraint violating mode, it is crucial to add constraint damping terms to the evolution equations to obtain reliable numerical results.

Bianchi II ($N_1 > 0$, $N_2 = N_3 = 0$): Full set of equations

Evolution equations:

$$\Sigma'_\pm = -(2-q)\Sigma_\pm - S_\pm,$$
$$N'_1 = (q - 4\Sigma_+)N_1,$$
$$x' = x(q-2) - F(\psi)y^2,$$
$$y' = F(\psi)xy + y(1+q),$$
$$\psi' = -\sqrt{6}x\psi^2.$$

with $q = 2\Sigma_+^2 + 2\Sigma_-^2 + 2x^2 - y^2$, $S_+ = -\frac{1}{3}N_1^2$ and $S_- = 0$.

Constraint:

$$1 = \Sigma_+^2 + \Sigma_-^2 + x^2 + y^2 + K,$$

with

$$K = \frac{1}{12}N_1^2.$$

Bianchi II ($N_1 > 0$, $N_2 = N_3 = 0$): Fixed point

Fact: Bianchi I fixed point before is unstable in Bianchi II.

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Bianchi II fixed point:

$$\Sigma_+ = \frac{2\left(c_1^2 - 2\right)}{c_1^2 + 16}, \qquad \Sigma_- = 0, \quad x = \frac{3\sqrt{6}c_1}{c_1^2+16}, \quad y = \frac{6\sqrt{8 - c_1^2}}{c_1^2+16},$$

$$N_1 = \frac{6\sqrt{-c_1^4 + 10c_1^2 - 16}}{c_1^2+16}, \quad \psi = 0.$$

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Hence: Since $q = 8(c_1^2 - 2)/(16 + c_1^2)$, the fixed point is in the decelerated regime if and only if

$$\sqrt{2} < c_1 < \sqrt{8}.$$

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Notice: This corresponds to the Collins-Stewart (II) perfect fluid solution (called $P_1^+(II)$ in Wainwright and Ellis [1997]).

Bianchi II ($N_1 > 0$, $N_2 = N_3 = 0$): Numerical studies

Numerical studies support the claim that the fixed point is a future attractor if $\sqrt{2} < c_1 < \sqrt{8}$.

Bianchi II ($N_1 > 0$, $N_2 = N_3 = 0$): Numerical studies

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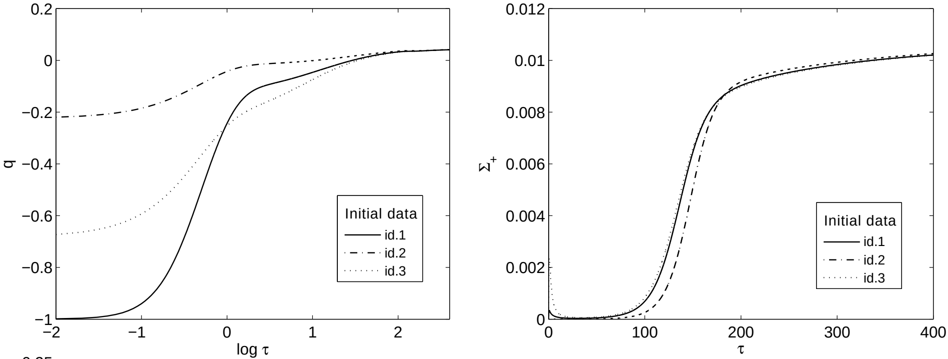
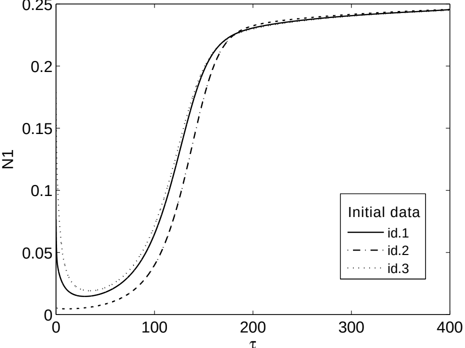
We find: This Bianchi II fixed point has similar stability properties within Bianchi II as the previous Bianchi I fixed point within Bianchi I.


q
log τ
Σ+
τ
Initial data
id.1
id.2
id.3


N1
τ
Initial data
id.1
id.2
id.3

$c_1 = 1.45,\ c_2 = 1.0$

Bianchi II ($N_1 > 0$, $N_2 = N_3 = 0$): Results

We have found:

- If $\sqrt{2} < c_1 < \sqrt{8}$ and initial data are in the inflationary regime, then the corresponding Bianchi II solutions have graceful exits from inflation.
- Fixed point is anisotropic and spatial curvature is not zero. Hence, anisotropies and spatial curvature do not stay small after inflation!

Bianchi VI_0 $(N_1 = 0, N_2 > 0, N_3 < 0)$

Bianchi VI_0 ($N_1 = 0$, $N_2 > 0, N_3 < 0$)

Skip ...

Evolution equations:

$$\Sigma'_\pm = -(2-q)\Sigma_\pm - S_\pm,$$
$$N'_2 = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \quad N'_3 = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3,$$
$$x' = x(q-2) - F(\psi)y^2, \quad y' = F(\psi)xy + y(1+q),$$
$$\psi' = -\sqrt{6}x\psi^2.$$

with

$$q = 2\Sigma_+^2 + 2\Sigma_-^2 + 2x^2 - y^2,$$
$$S_+ = \frac{1}{6}(N_2 - N_3)^2, \qquad S_- = \frac{1}{2\sqrt{3}}(N_2 - N_3)(N_2 + N_3).$$

Constraint:

$$1 = \Sigma_+^2 + \Sigma_-^2 + x^2 + y^2 + K,$$

with

$$K = \frac{1}{12}(N_2 - N_3)^2.$$

Bianchi VII_0: Heuristic analysis

Basic phenomenology: The constraint implies that while Σ_+, Σ_-, x, y are bounded, the variables N_2 and N_3 may become arbitrarily large!

Bianchi VII_0: Heuristic analysis

We find numerically: The quantities Σ_+, Σ_-, x, y and ψ approach stationary values 0, 0, x_*, y_* and 0 in the limit $\tau \to \infty$. Hence $q \to q_* = 2x_*^2 - y_*^2$.

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Based on this, the evolution equations imply (in consistency with the numerics) that

$$N_2 \to N_* e^{q_* \tau}, \quad N_3 \to N_* e^{q_* \tau},$$

for some $N_* > 0$ in a way such that

$$S_+ = (N_2 - N_3)^2/6 = 2K \to 0, \quad S_- = (N_2^2 - N_3^2)/2\sqrt{3} \to 0.$$

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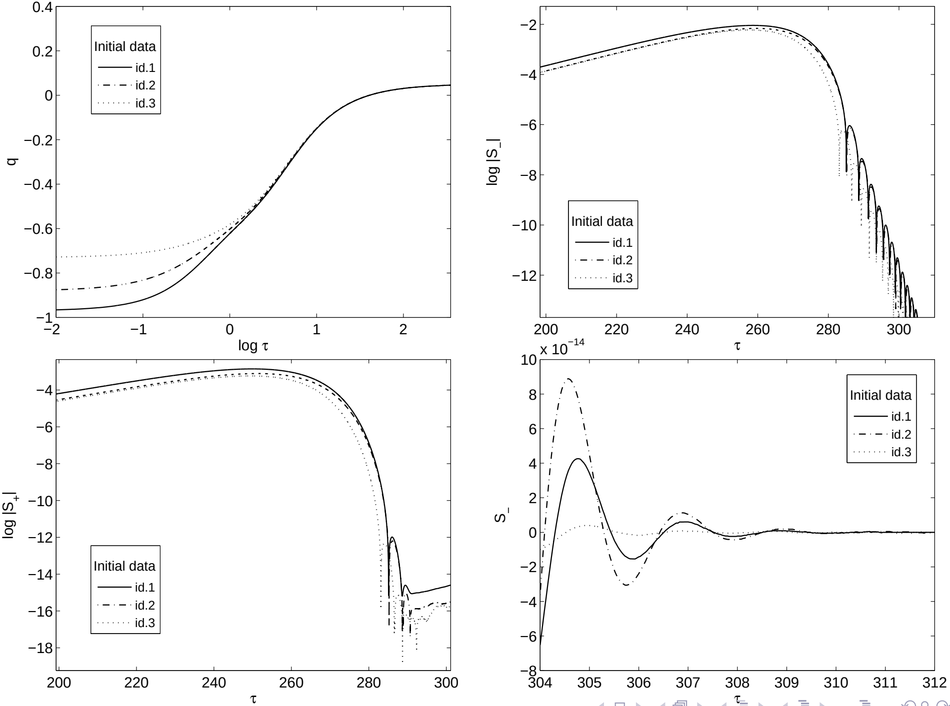
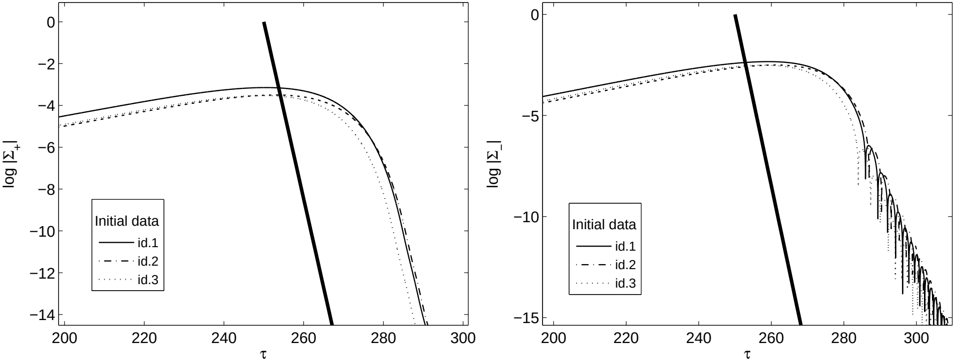
Then the equations imply: $x_* = c_1/\sqrt{6}, \quad y_* = \sqrt{1 - c_1^2/6}$, and hence

$$q_* = (c_1^2 - 2)/2 > 0 \quad \text{if and only if} \quad c_1 > \sqrt{2}.$$

Therefore graceful exits occur as before!


q
log τ
Initial data
id.1
id.2
id.3
log |S_|
τ
log |S+|
τ
x 10^−14
S_
τ



Recall: $\Sigma'_\pm = -(2-q)\Sigma_\pm - S_\pm$.

Bianchi VII$_0$ ($N_1 = 0$, $N_2, N_3 > 0$): Results

We have found:

- If $\sqrt{2} < c_1$ and initial data are in the inflationary regime, then the corresponding Bianchi VII$_0$ solutions have graceful exits from inflation.

Bianchi VII_0 ($N_1 = 0$, $N_2, N_3 > 0$): Results

We have found:

- If $\sqrt{2} < c_1$ and initial data are in the inflationary regime, then the corresponding Bianchi VII_0 solutions have graceful exits from inflation.
- The solutions continue to isotropize and spatial curvature continues to decay even after inflation is over.

Bianchi VII_0 ($N_1 = 0$, $N_2, N_3 > 0$): Results

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- If $\sqrt{2} < c_1$ and initial data are in the inflationary regime, then the corresponding Bianchi VII_0 solutions have graceful exits from inflation.
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- The solutions continue to isotropize and spatial curvature continues to decay even after inflation is over.
- Future asymptotic behavior is much more complicated than in previous Bianchi cases. The N-variables are unbounded and certain other variables are oscillatory.
- In the pure vacuum case, a result by Ringström [2001] implies that for generic Bianchi VII_0 initial data, the N-variables are bounded and that the solutions generically approach the flat spacetime.

Bianchi VIII ($N_1 < 0, N_2, N_3 > 0$)

Basic phenomenology: The constraint implies that while Σ_+, Σ_-, x, y and N_1 are bounded, the variables N_2 and N_3 may become arbitrarily large!

Bianchi VIII: Heuristic analysis

A similar heuristic discussion as before yields the following picture:

$$\Sigma_+ \to \Sigma_{+*} = \frac{c_1^2 - 2}{2\left(c_1^2 + 1\right)}, \quad \Sigma_- \to 0,$$

$$x \to \sqrt{\frac{3}{2}}\,\frac{c_1}{c_1^2 + 1}, \quad y \to \sqrt{\frac{3}{2}}\,\frac{\sqrt{c_1^2 + 2}}{c_1^2 + 1}, \quad \psi \to 0.$$

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This implies that

$$q \to q_* = \frac{c_1^2 - 2}{2(c_1^2+1)} > 0 \quad \text{if and only if} \quad c_1 > \sqrt{2}.$$

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This implies that

$$q \to q_* = \frac{c_1^2-2}{2(c_1^2+1)} > 0 \quad \text{if and only if} \quad c_1 > \sqrt{2}.$$

Moreover,

$$N_1 \to N_{1*} e^{(q_* - 4\Sigma_{+*})\tau}, \quad N_2 \to N_* e^{(q_* + 2\Sigma_{+*})\tau}, \quad N_3 \to N_* e^{(q_* + 2\Sigma_{+*})\tau},$$

such that

$$N_2 - N_3 \to 0, \quad N_2^2 - N_3^2 \to 0, \quad \frac{1}{6}|N_1|(N_2+N_3) \to (2-q_*)\Sigma_{+*}.$$

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Bianchi VIII: Results

- Generic initially inflating Bianchi VIII solutions have a graceful exit if $c_1 > \sqrt{2}$.
- Similar to Bianchi VII_0, S_- approaches zero in an oscillatory manner and hence Σ_-. However, now S_+, K and Σ_+ approach non-zero finite values.
- Asymptotics for $\tau \to \infty$ is neither flat nor isotropic.

Summary and outlook

- We have studied the graceful exit problem for minimally-coupled scalar field Bianchi A models with a particular scalar field potential (except for Bianchi IX).

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- Do our results also apply to larger classes of potentials? Work in progress.

Further reading I

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