A study of some curvature operators near the Euclidian metric

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A geometric operator

Let $\kappa, \Lambda \in \mathbb{R}$ and let

$$Ein(g) = Ric(g) + \kappa R(g)\, g + \Lambda\, g \in S_2.$$

$$\Phi^*(Ein(g)) = Ein(\Phi^*(g))$$

- $\Lambda = \kappa = 0$: Ricci tensor

$$Ein(g) = Ric(g).$$

- $\kappa = -\frac{1}{2}$: Einstein tensor

$$Ein(g) = Ric(g) - \frac{1}{2} R(g) g + \Lambda g.$$

- $\Lambda = 0$, $\kappa = -\frac{1}{2(n-1)}$: Schouten tensor

$$Ein(g) = Ric(g) - \frac{1}{2(n-1)} R(g) g.$$

The geometric equation

Let E be a field of symmetric bilinear forms on $\mathbb{R}^n$, we want to find a metric g on $\mathbb{R}^n$ such that

$$Ein(g) = E.$$

This is a Quasi-linear system of PDE with $n(n+1)/2$ equations and $n(n+1)/2$ unknown.
For the talk we take $\kappa = 0$ first so

$$Ein(g) = Ric(g) + \Lambda g.$$

Preceding results ($\Lambda = 0$) :

- DeTurck 1981, near a point.
- Hamilton 1984, On S^n, near the canonical metric.
- D- 1999, On H^n, near the canonical metric.
- D- Herzlich 2001, On CH^n, near the canonical metric.
- D- 2002, On some AH manifold, near some Einstein metrics.
- Delanoë 2003, On some compact manifold, near some Einstein metrics.

Remarks about the equation

If the metric g is C^{k+2}, then $Ein(g)$ is C^k.
Let Φ be a C^{k+1} diffeomorphism and let

$$E := \Phi^* Ein(g).$$

Then E is C^k and

$$Ein(\Phi^* g) = \Phi^* Ein(g) = E,$$

but $\Phi^* g$ is only C^k !
In particular Ein can not be elliptic.

The result on $\mathbb{R}^n$

For $x \in \mathbb{R}^n$, let

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

For $s, t \in \mathbb{R}$, let us define the weighted Sobolev space :

$$H^{s,t} = \langle x \rangle^{-t} H^s = \{x \mapsto \langle x \rangle^{-t} u(x),\ u \in H^s\}.$$

Theorem

Let $s, t, \Lambda \in \mathbb{R}$ such that $s > \frac{n}{2}$, $t \geq 0$, $\Lambda > 0$. Then For all $e \in H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ close to zero, there exist a small h in $H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ such that

$$Ein(\delta + h) = Ein(\delta) + e.$$

Moreover, the map $e \mapsto h$ is smooth near zero between the corresponding Hilbert spaces.

Idea of proof step 1 : Linearised operators

$$D\ Ein(\delta)h = \frac{d}{dt}\left[Ein(\delta + th)\right]_{|_{t=0}} = \frac{1}{2}\Delta h + \Lambda h - \mathcal{L}B_\delta(h),$$

where

$$[B_g(h)]_j = -{}^g\nabla^k h_{kj} + \frac{1}{2}\partial_j(g^{kl}h_{kl}),$$

and

$$(\mathcal{L}\omega)_{ij} = \frac{1}{2}(\partial_i\omega_j + \partial_j\omega_i).$$

Recall the Bianchi identity

$$B_g(Ein(g)) = 0,$$

and note that

$$D[\mathcal{B}_{(.)}(Ein(\delta))](\delta)h = -\Lambda B_\delta(h)$$

Idea of proof step 2 : The modified equation

We thus define, for the moment formally, for h and e small

$$F(h, e) := Ein(\delta + h) - E - \frac{1}{\Lambda}\mathcal{L}_{\delta} B_{\delta+h}(E),$$

where $E = Ein(\delta) + e = \Lambda\delta + e$.
So if there is a solution to

$$Ein(\delta + h) = E$$

then

$$F(h, e) = 0.$$

And now we have

$$D_h F(0, 0) = \frac{1}{2}\Delta + \Lambda.$$

We want to prove, by the implicit function theorem is some Banach spaces, there is solution to $F(h, e) = 0$.

Idea of proof step 2 : The modified equation

Proposition

The map

$$F : H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2) \times H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2) \longrightarrow H^{s,t}(\mathbb{R}^n, \mathcal{S}_2),$$

is well defined and smooth near $(0,0)$*.*

PROOF : First recall that

$$Ric(g)_{jk} = \partial_l \Gamma^l_{jk} - \partial_k \Gamma^l_{jl} + \Gamma^p_{jk}\Gamma^l_{pl} - \Gamma^p_{jl}\Gamma^l_{pk},$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks}(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}).$$

Idea of proof step 2 : The modified equation

Let us write

$$Ric(g) = \partial\Gamma + \Gamma\Gamma\ , \quad \Gamma = g^{-1}\partial g.$$

where $g = \delta + h$ and h is small in $H^{s+2,t}$, $s > \frac{n}{2}$, $t \geq 0$.
We can see that

$$g^{-1} = \delta^{-1} + \widetilde{h}\ , \ \ \widetilde{h} \in H^{s+2,t},$$

and

$$\|\widetilde{h}\|_{s+2,t} \leq \sum_{k\in} C^k_{s+2,t}\|h\|^{k+1}_{s+2,t} = \frac{\|h\|_{s+2,t}}{1 - C_{s+2,t}\|h\|_{s+2,t}}.$$

So if $\|h\|_{s+2,t} \leq \frac{1}{2C_{s+2,t}}$, we obtain

$$\|\widetilde{h}\|_{s+2,t} \leq 2\|h\|_{s+2,t}.$$

Idea of proof step 2 : The modified equation

We deduce

$$\Gamma = (\delta^{-1} + \widetilde{h})\partial h \in H^{s+1,t}\ , \ \ \|\Gamma\|_{s+1,t} \leq \|h\|_{s+2,t},$$

then

$$\partial\Gamma \in H^{s,t}\ , \ \ \|\partial\Gamma\|_{s,t} \leq \|\Gamma\|_{s+1,t} \leq \|h\|_{s+2,t},$$

thus,

$$Ric(g) \in H^{s,t}\ , \ \ \|Ric(g)\|_{s,t} \leq \|h\|_{s+2,t}.$$

Idea of proof step 2 : The modified equation

Let us study the Bianchi operator

$$B_g(E) = div_g E + \frac{1}{2} d \; Tr_g E.$$

We write abusively

$$\mathcal{B}_g(E) = g^{-1}(\partial E + \Gamma E) + \partial(g^{-1} E).$$

Recall that $E = \Lambda\delta + e$, then

$$\mathcal{B}_g(E) = (\delta^{-1} + \widetilde{h})[\partial e + \Gamma(\Lambda\delta + e)] + \partial[\delta^{-1} e + \widetilde{h}(\Lambda\delta + e)].$$

As before, we can deduce

$$\mathcal{B}_g(E) \in H^{s+1,t} \ , \ \ \|\mathcal{B}_g(E)\|_{s+1,t} \leq (\|h\|_{s+2,t} + \|e\|_{s+2,t}),$$

and

$$\mathcal{L}_\delta \mathcal{B}_g(E) \in H^{s,t} \ , \ \ \|\mathcal{L}_\delta \mathcal{B}_g(E)\|_{s,t} \leq \|\mathcal{B}_g(E)\|_{s+1,t} \leq (\|h\|_{s+2,t} + \|e\|_{s+2,t}).$$

■

Idea of proof step 2 : The modified equation

Proposition

The map

$$D_hF(0,0) = \frac{1}{2}\Delta + \Lambda : H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2) \longrightarrow H^{s,t}(\mathbb{R}^n, \mathcal{S}_2)$$

is an isomorphism.

PROOF : Use the Fourier transform. ■

Conclusion step 2 : from the implicit function theorem for e small in $H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ there exist a small h in $H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ such that $F(h, e) = 0$.

Idea of proof step 3 : Our metric is a solution

Recall we have a solution to

$$F(h,e) := Ein(\delta + h) - E - \frac{1}{\Lambda}\mathcal{L}_\delta B_{\delta+h}(E) = 0.$$

Apply $B_{\delta+h}$ to this equation, and define $\omega = \frac{1}{\Lambda}\mathcal{B}_{\delta+h}(E)$ then

$$P_{\delta+h}\omega := B_{\delta+h}\mathcal{L}_\delta\omega + \Lambda\omega = 0,$$

with $\omega \in H^{s+1,t}(\mathbb{R}^n,(\mathbb{R}^n)^*)$. But

$$P_\delta = \frac{1}{2}(\Delta + 2\Lambda)$$

is an isomorphism from $H^{s+1,t}(\mathbb{R}^n,(\mathbb{R}^n)^*)$ to $H^{s-1,t}(\mathbb{R}^n,(\mathbb{R}^n)^*)$, then $P_{\delta+h}$ stay injective if h is small enough. Finally

$$\omega = 0.$$

■

Image of the some Riemann-Christoffel type operators

Let $\mathcal{R}_3^1$ be the subspace of multi-linear forms on $(\mathbb{R}^n)^* \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with components satisfying

$$\tau^i_{ilm} = 0,\ \tau^i_{klm} = -\tau^i_{kml},\ \tau^i_{klm} + \tau^i_{mkl} + \tau^i_{lmk} = 0.$$

For $g \in C^\infty$ a riemannian metric on $\mathbb{R}^n$, let

$$\mathcal{E}in(g) = Riem(g) + \frac{1}{2(n-1)} \Lambda g^{-1}(g \owedge g) \in C^\infty(\mathbb{R}^n, \mathcal{R}_3^1).$$

We thus have

$$\mathcal{E}in(g)^l_{ilj} = Ein(g)_{ij}.$$

Let us define the smooth maps

$$\rho(.) \equiv \mathcal{E}in(\delta + .) - \mathcal{E}in(\delta) : C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2) \longrightarrow C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1),$$

and

$$Tr : C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1) \longrightarrow C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2)$$

defined by $\tau^i_{klm} \mapsto \tau^i_{kim}$.

Let

$$e(.) \equiv Ein(\delta + .) - Ein(\delta) : C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2) \longrightarrow C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2),$$

and let

$$h(.) : C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2) \longrightarrow C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2),$$

our solution of $e(h(e)) = e$. We have

$$Tr \circ \rho \equiv e(.),$$

and

$$h \circ e \equiv e \circ h \equiv Id_{C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2)}.$$

Proposition

For all $t \geq 0$,

$$C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1) = ImD\rho(0) \oplus KerTr.$$

PROOF :

$$Tr \circ D\rho(0) = De(0).$$

$$Dh(0) \circ De(0) \equiv De(0) \circ Dh(0) \equiv Id_{C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2)}.$$

The following diagram is then commutative :

$$C^{\infty,t}(\mathbb{R}^n, \mathcal{S}_2) \xrightarrow{D\rho(0)} C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1) \xrightarrow{Tr} C^{\infty}(\mathbb{R}^n, \mathcal{S}_2)$$

$$De(0)$$

$$Dh(0)$$

Theorem

$Im\rho$ is a smooth submanifold in $C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1)$, graph from $ImD\rho(0)$ to $KerTr$ given by

$$\Psi \longrightarrow (\rho \circ h \circ Tr)(\Psi) - \Psi.$$

PROOF

We also have the (non-linear) commutative diagram

$$C^{\infty,}(\mathbb{R}^n,\mathcal{S}_2) \xrightarrow{\rho} C^{\infty,t}(\mathbb{R}^n,\mathcal{R}_3^1) \xrightarrow{Tr} C^{\infty,t}(\mathbb{R}^n,\mathcal{S}_2), \quad C^{\infty,}(\mathbb{R}^n,\mathcal{S}_2) \xrightarrow{e} C^{\infty,t}(\mathbb{R}^n,\mathcal{S}_2), \quad C^{\infty,t}(\mathbb{R}^n,\mathcal{S}_2) \xrightarrow{h} C^{\infty,}(\mathbb{R}^n,\mathcal{S}_2).$$

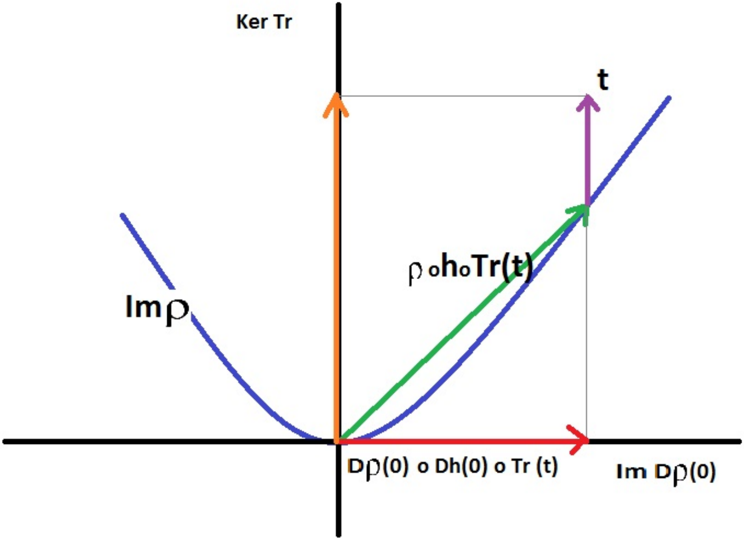
For $\tau \in C^{\infty,t}(\mathbb{R}^n,\mathcal{R}_3^1)$ close to zero :

$$\tau = \underbrace{[D\rho(0)\circ Dh(0)\circ Tr](\tau)}_{\in Im D\rho(0)} + \underbrace{\tau - [D\rho(0)\circ Dh(0)\circ Tr](\tau)}_{\in Ker Tr}$$

and

$$\tau = \underbrace{(\rho\circ h\circ Tr)(\tau)}_{\in Im\rho} + \underbrace{\tau - (\rho\circ h\circ Tr)(\tau)}_{\in Ker Tr}.$$

Decompositions of $C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1)$



Let us define the map Φ from $C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1)$ to $C^{\infty,t}(\mathbb{R}^n, \mathcal{R}_3^1)$ given by

$$\Phi(\tau) = \tau' = [D\rho(0) \circ Dh(0) \circ Tr](\tau) + \tau - (\rho \circ h \circ Tr)(\tau)$$

(note that $Tr(\tau') = Tr(\tau)$).
Then

$$\Phi^{-1}(\tau') = \tau = \tau' - [D\rho(0) \circ Dh(0) \circ Tr](\tau') + (\rho \circ h \circ Tr)(\tau').$$

And we can see that

$$\Phi(Im\rho) = ImD\rho(0)$$

near zero . ■

When $\kappa \neq 0$?

The trace of the equation

$$Ein(g) = Ric(g) + \kappa R(g)g + \Lambda g = E,$$

gives

$$Tr_g Ein(g) = (1 + n\kappa)R(g) + n\Lambda = Tr_g E.$$

Thus the equation is equivalent to

$$Ric(g) = E - \frac{\kappa \; Tr_g E + \Lambda}{1 + n\kappa} g.$$

We define the Bianchi type operator :

$$\mathcal{B}_g(E) = div_g E + \frac{2\kappa + 1}{2(1 + \kappa n)} d \; Tr_g E = B_g(E) - \frac{(n-2)\kappa}{2(1 + \kappa n)} d \; Tr_g E,$$

When $\kappa \neq 0$

Let us define

$$\mathcal{F}(h,e) := Ric(\delta + h) - E + \frac{\kappa \; Tr_{\delta+h}E + \Lambda}{1+\kappa n}(\delta + h) - \frac{1}{\Lambda}\mathcal{L}_{\delta}\mathcal{B}_{\delta+h}(E),$$

where

$$E = Ein(\delta) + e = \Lambda\delta + e.$$

But here we have

$$D_h\mathcal{F}(0,0)h = \frac{1}{2}\Delta h + \Lambda h - \frac{\kappa\Lambda}{1+\kappa n} \; Tr_{\delta} h \, \delta - \frac{(n-2)\kappa}{2(1+\kappa n)}\partial\partial \; Tr_{\delta} h.$$

Note also that when $\kappa \neq 0$, $D_h\mathcal{F}(0,0)$ does not preserve the splitting

$$\mathcal{G} \oplus \mathring{S}_2.$$

When $\kappa \neq 0$

- In the conformal direction, where $h = u\delta$ we find

$$D_h\mathcal{F}(0,0)(u\delta) =$$

$$\frac{1}{2(1+\kappa n)}[(1+2(n-1)\kappa)\Delta u + 2\Lambda u]\delta - \frac{(n-2)n\kappa}{2(1+\kappa n)}\mathring{H}ess\ u,$$

where $\mathring{H}ess\ u$ is the trace free part of the Hessian of u.

- In the trace free direction $h = \mathring{h}$ we have

$$D_h\mathcal{F}(0,0)\mathring{h} = \frac{1}{2}(\Delta + 2\Lambda)\mathring{h}.$$

When $\kappa \neq 0$

But the operator

$$D_h\mathcal{F}(0,0) : S_2 = \mathcal{G} \oplus \mathring{S}_2 \to \mathcal{G} \oplus \mathring{S}_2$$

is "triangular" :

$$D_hF(0,0) = \frac{1}{2}\begin{pmatrix} (1+2(n-1)\kappa)\Delta + 2\Lambda & 0 \\ -\frac{(n-2)n\kappa}{1+\kappa n}\,\mathring{Hess} & \Delta + 2\Lambda \end{pmatrix}$$

If $\kappa > -1/2(n-1)$ and $\Lambda > 0$ it is an isomorphism.

When $\kappa \neq 0$

Theorem

Let $s, t, \kappa, \Lambda \in \mathbb{R}$ such that $s > \frac{n}{2}$, $t \geq 0$, $\kappa > -\frac{1}{2(n-1)}$, $\Lambda > 0$. Then For all $e \in H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ close to zero, there exist a small h in $H^{s+2,t}(\mathbb{R}^n, \mathcal{S}_2)$ such that

$$Ein(\delta + h) = Ein(\delta) + e.$$

Moreover, the map $e \mapsto h$ is smooth near zero between the corresponding Hilbert spaces.
