Robustness of the conformal constraint equations in a scalar-field setting

Seminar on Mathematical General Relativity, Université Pierre et Marie Curie

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Scalar-field theory in General Relativity

Let $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, be a Lorentzian manifold, $\Psi \in C^\infty(\mathcal{M}^{n+1})$ a scalar-field and $V \in C^\infty(\mathbb{R})$ a potential. Ψ accounts for the distribution of matter and/or energy in the universe.

We will say that $(\mathcal{M}^{n+1}, h, \Psi)$ is a space-time if it satisfies the following Einstein equations:

$$\begin{cases} Ric(h)_{ij} - \frac{1}{2} R(h) h_{ij} = \nabla_i \Psi \nabla_j \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{ij}, \\ \Box_h \Psi = \frac{dV}{d\Psi}. \end{cases} \tag{E}$$

In a *space-time* satisfying (E) the gravitational field h only depends on Ψ.

Relevant physical cases:

- Vacuum case with no cosmological constant: $\Psi \equiv 0$, $V \equiv 0$.
- Vacuum case with positive cosmological constant: $\Psi \equiv 0$, $V \equiv \Lambda > 0$.
- Klein-Gordon fields: $V(\Psi) = \frac{1}{2} m \Psi^2$, $m > 0$.

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The Evolution Problem

Assume that the space-time is *globally hyperbolic*: $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$ with $(M^n, h_{|M^n})$ Riemannian. Ideal setting to express the Einstein equations as an evolution problem.

Initial data sets on M^n: Let $(M^n \times \mathbb{R}, h, \Psi)$. We let:

- $\tilde{g} = h_{|M^n}$ and $\tilde{\nabla}$ is the Levi-Civita connection for $\tilde{g}$ in M^n,
- $\tilde{K}$: second fundamental form of the embedding $M^n \subset M^n \times \mathbb{R}$,
- $\tilde{\psi} = \Psi_{|M^n}$ and $\tilde{\pi} = (N \cdot \Psi)_{|M^n}$. N is the future-directed unit normal to M^n.

Theorem (Choquet-Bruhat '52, Choquet-Bruhat-Geroch '69)

$(M^n \times \mathbb{R}, h, \Psi)$ solves (E) if and only if $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ solves in M^n the contraint system:

$$\begin{cases} R(\tilde{g}) + tr_{\tilde{g}} \tilde{K}^2 - \left|\left|\tilde{K}\right|\right|_{\tilde{g}}^2 = \tilde{\pi}^2 + |\tilde{\nabla}\tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}) , \\ \tilde{\nabla}(\mathrm{tr}_{\tilde{g}} \tilde{K}) - div_{\tilde{g}} K = -\tilde{\pi}\tilde{\nabla}\tilde{\psi} , \end{cases} \qquad (C)$$

In particular: any solution $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ of (C) evolves into a solution of the Einstein equations. A solution $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ is therefore called ***an initial data set****.*

System (C) has $n(n+1) + 2$ unknowns $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$: it is highly underdetermined.

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The conformal method (Lichnerowicz, Choquet-Bruhat, York)

The constraints system (C) determines the initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ for which the Einstein equations admit a well-posed formulation.

It is highly underdetermined. To overcome this, we restrict to a specific class of solutions, written as a parametrization from fixed background physics data. The goal is to reduce the resolution of (C) to the determination of the parameters.

Let (M^n, g) be a fixed Riemannian manifold. The (arbitrarily fixed) *physics data* are:

- V a potential,
- ψ, π scalar-field data,
- τ the mean curvature,
- σ, $(2,0)$-symmetric tensor field with $\mathrm{tr}_g \sigma = 0$ and $\mathrm{div}_g \sigma = 0$ ("TT tensor").

We will look for the unknown initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ under the following form:

$$\left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right) = \left(u^{\frac{4}{n-2}} g, \frac{\tau}{n} u^{\frac{4}{n-2}} g + u^{-2} \left(\sigma + \mathcal{L}_g W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \qquad (*)$$

where $u \in C^\infty(M)$, $u > 0$, $W \in T^*M$ and $\mathcal{L}_g W$ is the conformal Killing operator:

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Using the decomposition :

$$\tilde{K} = \frac{\tau}{n} u^{\frac{4}{n-2}} g + u^{-2} \left(\sigma + \mathcal{L}_g W \right),$$

and since for any $(2,0)$-tensor U there holds:

$$\mathrm{div}_{u^{\frac{4}{n-2}} g} U_i = u^{-\frac{4}{n-2}} \mathrm{div}_g U_i - \frac{2}{n-2} \frac{\partial_i u}{u} \mathrm{tr}_g U + 2 u^{-\frac{n+2}{n-2}} g^{jk} \partial_k u U_{ij}.$$

the vector constraint equation becomes:

$$\mathrm{div}_g \left(\mathcal{L}_g W \right) = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla \tau + \pi \nabla \psi.$$

Independently, there holds:

$$R(u^{\frac{4}{n-2}} g) = u^{-\frac{n+2}{n-2}} \left(8 \frac{4(n-1)}{n-2} \triangle_g u + R(g) u \right)$$

and, using the decomposition of $\tilde{K}$:

$$|\tilde{K}|_g^2 = u^{-\frac{4}{n-2}} \left| \sigma + \mathcal{L}_g W \right|^2 + \frac{\tau^2}{n} u^{\frac{8}{n-2}}.$$

The scalar constraint equation therefore becomes:

$$\frac{4(n-1)}{n-2} \triangle_g u + \left(R(g) - |\nabla \psi|_g^2 \right) u = \left(2V(\psi) - \frac{n-1}{n} \tau^2 \right) u^{\frac{n+2}{n-2}} + \left(\pi^2 + |\sigma + \mathcal{L}_g W|^2 \right) u^{-\frac{3n-2}{n-2}}$$

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The Conformal Constraint System:

In the end, $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ given by $(*)$ solve (C) if and only if (u, W) solve the following:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \quad (CC)$$

The problem is now determined. System (CC) is called *Conformal Constraints System* (or also Einstein-Lichnerowicz system) *of physics data* $(\psi, \pi, \tau, \sigma)$. The scalar equation is called *Einstein-Lichnerowicz equation*, the 1-form (or vector) equation is called the *Lamé equation*. In (CC) we have let:

$$h(g, \psi) = R(g) - |\nabla \psi|_g^2 \quad , \quad f(\tau, \psi, V) = 2V(\psi) - \frac{n-1}{n} \tau^2,$$

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By a solution (u, W) of (CC) we will always mean: $u > 0$.

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Choquet-Bruhat-Geroch-Lichnerowicz formalism (CBGL):

The conformal method sums up in the following 3-steps construction in (M^n, g):

Freely chosen physics data $(\psi, \pi, \tau, \sigma)$ and V

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Maximal development $(\mathcal{M}, h, \Psi)$

We question the relevance of the conformal method *via* the relevance of this construction:

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Continuous dependence of the solutions of the conformal method

We check the robustness step by step:

- **Step** 2: The initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ continuously depends on (u, W) *via* the parameterization $(*)$.
- **Step** 3: the globally hyperbolic space-time development $(\mathcal{M}, h, \Psi)$ continuously depends on the initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$, at least in finite time. It is the Choquet-Bruhat-Geroch well-posedness result + Cauchy stability.
- We are left to investigate the continuous dependence of Step 1. The question of the robustness of the CBGL formalism reformulates as follows:

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Setting considered

From now on: (M^n, g) **is closed**. Let $(\psi, \pi, \tau, \sigma)$ and V be given background physics data in (M^n, g). We investigate the system of physics data $(\psi, \pi, \tau, \sigma)$ and V:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\mathcal{L}_g W + \sigma|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi . \end{cases} \qquad (CC)$$

We will always work **in the focusing case:**

$$\text{focusing case: } f(\tau,\psi,V) > 0 \text{ in } M$$

(Remember: $\triangle_g$ is positive).

The focusing case arises when treating general nontrivial non-gravitational data. For instance:

- Positive cosmological constant setting, where $V \equiv \Lambda > 0$
- Klein-Gordon scalar-field, where $V(\psi) = \frac{1}{2} m \psi^2$, $m > 0$.

The vacuum case is **defocusing:** $f(\tau,\psi,V) \leq 0$.

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The focusing case arises when treating general nontrivial non-gravitational data. For instance:

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Existence of solutions of (CC)

Theorem (P., *Comm. Math. Phys.*, '13)

Let (M^n, g) closed be such that $\overrightarrow{\triangle}_g$ has no kernel. Let V and $(\psi, \pi, \tau, \sigma)$ be ***focusing*** *physics data. There exists $\varepsilon(n, g) > 0$ such that if $(\pi, \sigma) \not\equiv (0, 0)$ and*

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Mathematical formulation of the stability problem

Let V be a potential and $\mathcal{D} = (\psi, \pi, \tau, \sigma)$ be given background physics data. Let $(V_\alpha)_\alpha$ and $(\mathcal{D}_\alpha)_\alpha$, $\mathcal{D}_\alpha = (\psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ sequences of potentials and background physics data converging to V and $\mathcal{D}$, in some suitable topology.

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The stability of the conformal constraint system (CC) wit respect to the choice of $(\psi, \pi, \tau, \sigma)$ and V reformulates as follows: does, up to a subsequence, $(u_\alpha, W_\alpha)_\alpha$ converge in some **strong** topology to some solution (u_∞, W_∞) of the limiting system (CC):

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- Critical and negative nonlinearities u^{2^*-1} and u^{-2^*-1}.
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Stability does not always hold for system (CC)!

Proposition (P., '14)

Let $(\mathbb{S}^3, h)$ be the standard sphere. There exist sequences $(\sigma_\alpha)_\alpha$ and $(Y_\alpha)_\alpha$, respectively of symmetric $(2,0)$-tensor fields and of vector fields, converging in $C^0(\mathbb{S}^3)$ respectively towards σ and Y, with $\sigma \not\equiv 0$, and there exists $(u_\alpha, W_\alpha)_\alpha$ with $u_\alpha > 0$ a sequence of solutions of the following system:

$$\begin{cases} \triangle_h u_\alpha + \frac{3}{4} u_\alpha = \frac{3}{4} u_\alpha^5 + \frac{|\sigma_\alpha + \mathcal{L}_h W_\alpha|_h^2}{u_\alpha^7}, \\ \overrightarrow{\triangle}_h W_\alpha = -\frac{2}{3} u_\alpha^6 \nabla \tau + Y_\alpha, \end{cases}$$

where $\tau \not\equiv 0$ is a smooth function in $\mathbb{S}^3$, such that $\max_M u_\alpha \to +\infty$ as $\alpha \to +\infty$.

Similar examples available in dimensions $n \geq 3$.

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Continuous dependence of the set of solutions of (CC) in the physics data

Theorem (Druet-P., Math. Ann. '14 – P., '14)

Let (M^n, g) be a closed locally conformally flat manifold. Let V a potential and $\mathcal{D} = (\psi, \pi, \tau, \sigma)$ be fixed background ***focusing*** *physics data. Assume further that:*

- $\pi \not\equiv 0$ *if* $3 \le n \le 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points in* M^n *if* $n \ge 6$.

Let $(V_\alpha)_\alpha$ and $(\mathcal{D}_\alpha)_\alpha$, $\mathcal{D}_\alpha = (\psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)$, be sequences of potentials and of physics data satisfying:

$$\|V_\alpha - V\|_{C^1} + \|\psi_\alpha - \psi\|_{C^1} + \|\tau_\alpha - \tau\|_{C^2} + \|\pi_\alpha - \pi\|_{C^0} + \|\sigma_\alpha - \sigma\|_{C^0} \underset{\alpha \to +\infty}{\longrightarrow} 0.$$

Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of (CC_α):

$$\begin{cases} \dfrac{4(n-1)}{n-2} \triangle_g u_\alpha + h(g, \psi_\alpha) u_\alpha = f(\tau_\alpha, \psi_\alpha, V_\alpha) u_\alpha^{2^*-1} + \dfrac{\pi_\alpha^2 + |\sigma_\alpha + \mathcal{L}_g W_\alpha|_g^2}{u_\alpha^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{n-1}{n} u_\alpha^{2^*} \nabla \tau_\alpha - \pi_\alpha \nabla \psi_\alpha \ . \end{cases} \qquad (CC_\alpha)$$

There exists (u_∞, W_∞) a solution of the limiting system (CC), with $u_\infty > 0$, such that, up to a subsequence and up to conformal Killing fields, $(u_\alpha, W_\alpha) \underset{\alpha \to \infty}{\longrightarrow} (u_\infty, W_\infty)$ in $C^{1,\theta}(M)$ for any $0 < \theta < 1$.

The Theorem does not require any smallness assumption on the physics data. And it is **sharp**: blowing-up sequences can be constructed when $\pi \equiv 0$ and/or ψ and τ have a common critical point.

The Beig-Chruściel-Schoen result shows that the convergence generically holds for the W_α's.

This is in particular a compactness result. For the physical dimension $n = 3$ we have the following corollary:

Corollary (Druet - P., '14)

In dimension 3, the CBGL formalism is robust with respect to the choice of focusing physics data $(\psi, \pi, \tau, \sigma)$ and V, on a locally conformally flat manifold.

The Theorem covers the setting of a positive cosmological constant where $V \equiv \Lambda > 0$, and the one of Klein-Gordon fields, where $V(\psi) = \frac{1}{2} m \psi^2$, $m > 0$. The vacuum case is not covered here (it is a defocusing case).

For the conformal constraint system, matter creates stability!

Remark: Compactness results for critical elliptic equations are known. For instance for the Yamabe problem (Khuri-Marques-Schoen, Brendle-Marques, Li, Zhu, Druet, Zhang, Brendle, Marques). For the Einstein-Lichnerowicz equation, see Druet-Hebey and Hebey-Veronelli.

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Steps of the proof:

Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the following system:

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where $(h_\alpha, f_\alpha, b_\alpha, X_\alpha, Y_\alpha, \sigma_\alpha) \underset{\alpha \to +\infty}{\longrightarrow} (h, f, b, X, Y, \sigma)$ in a suitable topology.

Easy case: If $\sup_M u_\alpha \le M$ for all α. By standard elliptic theory on the vector equations + an *ad hoc* Harnack inequality on the scalar equations the sequence (u_α, W_α) converges (up to a subsequence and up to conformal Killing vector fields).

General case: Proof by contradiction. Assume that $\sup_M u_\alpha \to +\infty$: (u_α, W_α) therefore develops concentration points in M.

The proof goes through three uneven steps:

1. We locate the regions in M where loss of compactness for u_α and $|\mathcal{L}_g W_\alpha|_g$ occurs.
2. *A priori* analysis: we obtain **sharp asymptotic pointwise estimates** on u_α *and* $|\mathcal{L}_g W_\alpha|_g$ in the regions where they blow-up.
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Step 1: Construction of the concentration points

We locate the concentration points $x_{i,\alpha}$ by the property that we have a weak estimate on u_α and $\mathcal{L}_g W_\alpha$ around them:

Proposition

There exists a sequence $(N_\alpha)_\alpha$ of integers, $N_\alpha \geq 2$ (possibly going to $+\infty$), and sequences $(x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha})_\alpha$ of concentration points of M, that is satisfying for any α:

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,
- $d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \geq 1$ *for* $i, j \in \{1, \ldots, N_\alpha\}, i \neq j$
- *for x close to $x_{i,\alpha}$:*

$$u_\alpha(x) \leq \frac{1}{d_g(x_{i,\alpha}, x)^{\frac{n-2}{2}}} \quad \text{and} \quad |\mathcal{L}_g W_\alpha|_g \leq \frac{1}{d_g(x_{j,\alpha}, x)^n}.$$

We hope to exhaust in this way the regions of M where loss of compactness is likely to occur.

Step 2: Sharp pointwise asymptotics around concentration points

Concentration point: around one of the points x_α identified in Step 1, we let $\rho_\alpha > 0$ be the radius on which the weak estimate holds:

$$u_\alpha^{2^*}(x) + |\mathcal{L}_g W_\alpha|_g(x) \leq \frac{C}{d_g(x_\alpha, x)^n} \text{ in } B_{x_\alpha}(8\rho_\alpha)$$

Proposition

On a concentration point $(x_\alpha)_\alpha$ there holds: $\rho_\alpha \to 0$ and the following estimates hold in $B_{x_\alpha}(\rho_\alpha)$:

$$|\mathcal{L}_g W_\alpha|_g \sim \mu_\alpha^{n-1} \rho_\alpha^{-n} \left(\mu_\alpha^2 + d_g(x_\alpha, x)^2\right)^{-\frac{n-1}{2}}$$

$$u_\alpha \sim \mu_\alpha^{\frac{n-2}{2}} \left(\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} d_g(x_\alpha, x)^2\right)^{-\frac{n-2}{2}}.$$

Here, $\mu_\alpha = \left(\max_{B_{x_\alpha}(\rho_\alpha)} u_\alpha\right)^{-\frac{2}{n-2}}$ controls the size of the explosion at x_α. It is a **characteristic radius** of the problem ($\mu_\alpha << \rho_\alpha$).

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Improving the estimates:

The sharp pointwise estimates are obtained by iteratively and simultaneously improving the available estimates on u_α and on W_α *via* a **ping-pong game**.

Problems arising in the analysis:

- One needs precise enough initial estimates to start the analysis. **And the weak estimates of Step** 1 **are not enough!**
- The two equations blow-up at different rates. The scalings to bring explosion profiles to a finite size are *non compatible*: the unknowns u_α and W_α only interact in the *intermediate region* $\mu_\alpha << r \le \rho_\alpha \Rightarrow$ No way to recover estimates on this region by scaling.
- Loss of regularizing effect of the equations: no *a priori* Harnack inequality, neither for u_α nor for W_α. In particular: estimates on u_α *do not imply anymore* estimates on ∇u_α!
- Because of the Kernel invariance, we use representation formulas for the Lamé operator $\overrightarrow{\triangle}_g$ in $B_{x_\alpha}(8\rho_\alpha)$ with Neumann boundary-type conditions.

  But with the coupling and the structure of the system: this adds a new **local loss of compactness**: since $\rho_\alpha \to 0$, the Green vector fields of $\overrightarrow{\triangle}_g$ in $B_{x_\alpha}(8\rho_\alpha)$ blow-up as $\alpha \to +\infty$ (except in the locally conformally flat case!)

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Step 3: Contradiction and conclusion

Step 2 gives: there are at least 2 concentration points. We define:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

and order the $x_{i,\alpha}$ so as to have $d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha})$. We then show that there simultaneously holds:

$$d_\alpha \to 0 \text{ and } d_\alpha \not\to 0.$$

Proof:

- $d_\alpha \to 0$: it is given by the sharp asymptotics around $x_{1,\alpha}$ obtained in Step 2.
- $d_\alpha \not\to 0$: we assume that the contrary holds and we look at u_α around $x_{1,\alpha}$ at a d_α scale:

$$\hat{u}_\alpha = d_\alpha^{\frac{n-2}{2}} u_\alpha(d_\alpha x).$$

Using Step 2 we show that $\hat{u}_\alpha$ is locally bounded. In particular: $\hat{u}_\alpha$ converges in $C^0_{loc}(\mathbb{R}^n)$ to some $\hat{u}_\infty$ satisfying:

$$\triangle_\xi \hat{u}_\infty = f_0(x_1)\hat{u}_\infty^{2^*-1} \quad \text{in } \mathbb{R}^3.$$

Thus $\hat{u}_\infty = C \cdot \left(1 + \frac{f_0(x_1)}{n(n-2)}|x|^2\right)^{-(n-2)/2}$ (Caffarelli-Gidas-Spruck classification result). This is impossible since $\hat{u}_\infty$ has by definition at least two distinct critical points. □

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- $d_\alpha \not\to 0$: we assume that the contrary holds and we look at u_α around $x_{1,\alpha}$ at a d_α scale:

$$\hat{u}_\alpha = d_\alpha^{\frac{n-2}{2}} u_\alpha(d_\alpha x).$$

Using Step 2 we show that $\hat{u}_\alpha$ is locally bounded. In particular: $\hat{u}_\alpha$ converges in $C^0_{loc}(\mathbb{R}^n)$ to some $\hat{u}_\infty$ satisfying:

$$\triangle_\xi \hat{u}_\infty = f_0(x_1) \hat{u}_\infty^{2^*-1} \quad \text{in } \mathbb{R}^3.$$

Thus $\hat{u}_\infty = C \cdot \left(1 + \frac{f_0(x_1)}{n(n-2)} |x|^2 \right)^{-(n-2)/2}$ (Caffarelli-Gidas-Spruck classification result). This is impossible since $\hat{u}_\infty$ has by definition at least two distinct critical points. □

Thank you for your attention.