Causality, hyperbolicity, and shock formation in Lovelock theories

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HSR, N. Tanahashi and B. Way, arXiv:1406.3379, 1409.3874
G. Papallo, HSR arXiv:1508.05303

Lovelock's theorem (1971)

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

LHS is most general symmetric tensor that is

- a function of g, ∂g, $\partial^2 g$
- divergence-free

This assumes $d = 4$ dimensions. For $d > 4$, extra terms can appear on LHS. These were determined by Lovelock.

Lovelock theories

$$E_{ab} = 8\pi T_{ab}$$

$$E^a{}_b \equiv \sum_{p\geq 0} k_p \delta^{ac_1\ldots c_{2p}}_{bd_1\ldots d_{2p}} R_{c_1c_2}{}^{d_1d_2} \ldots R_{c_{2p-1}c_{2p}}{}^{d_{2p-1}d_{2p}}$$

Antisymmetry: $p \leq [(d-1)/2]$

$$k_0 = \Lambda \qquad k_1 = -\frac{1}{4}$$

Einstein equation is obtained if we demand linearity in $\partial^2 g$, i.e., *quasilinearity*. General Lovelock theories are not quasilinear.

Why should I care about Lovelock theories?

- There has been interest in classical GR in $d > 4$ dimensions. Classically, Lovelock theories are as well-motivated as GR.
- Wave equation $\Box\phi = 0$: can be (i) generalised to d dimensions; (ii) made nonlinear $\Box\phi = \mathcal{F}(\phi, \partial\phi, \partial^2\phi)$. There has been considerable interest in understanding such equations. Doing the same for Einstein equation gives Lovelock theories uniquely.
- Interesting mathematical question: how do properties of such theories differ from GR? Is GR special? Are Lovelock theories pathological in some way?

Causality in Lovelock theories

Causality of a hyperbolic PDE is determined by its *characteristic surfaces*.

In GR, a hypersurface is characteristic if, and only if, it is null so causality is determined by the lightcone.

Characteristic hypersurfaces of Lovelock theories are generically non-null (Aragone 1987, Choquet-Bruhat 1988) so gravity can propagate faster or slower than light

Motivation

- How do characteristic hypersurfaces behave in Lovelock theories?
- Are Lovelock theories hyperbolic? (Necessary for well-posed initial value problem.)
- What happens when we evolve generic initial data in Lovelock theories?
- Focus on vacuum solutions: $T_{ab} = 0$

Characteristic surfaces

Consider hypersurface Σ and coordinates (x^0, x^i) so that Σ is $x^0 = 0$

Metric components $g_{0\mu}$ non-dynamical (e.g. lapse, shift)

Eq of motion gives constraints and evolution equation:

$$A^{ijkl}\partial_0^2 g_{kl} + \ldots = 0$$

Linear in $\partial_0^2 g$ (but A^{ijkl} depends on $\partial_i \partial_j g$) (Aragone 1987)

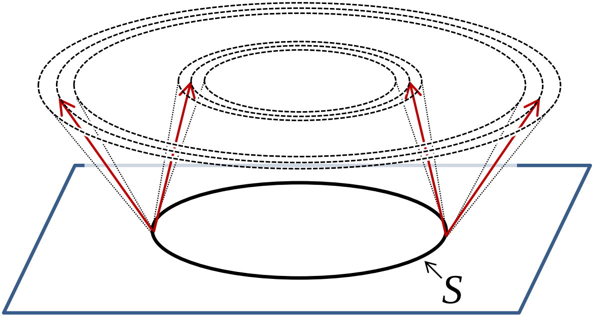
Characteristic surfaces

$$A^{ijkl}\partial_0^2 g_{kl} + \ldots = 0$$

- Σ is *non*characteristic iff knowing the fields and their first derivatives on Σ determines uniquely the second derivatives on Σ
- Σ characteristic iff $\exists r_{ij} \neq 0$ such that $A^{ijkl} r_{kl} = 0$: "graviton polarization r_{ij} propagates along Σ"
- GR: $A^{ijkl} \sim g^{00}\delta^{k(i}\delta^{j)l}$ so Σ characteristic iff null ($g^{00} = 0$), null hypersurfaces characteristic for *all* graviton polarizations
- Lovelock: $A^{ijkl} \sim g^{00}\delta^{k(i}\delta^{j)l} + \mathcal{R}^{ijkl}$. Expect different graviton polarizations to propagate along different characteristic hypersurfaces: "multirefringence"

Hyperbolicity

Pick some "initial" hypersurface Σ (non-characteristic) and a compact $(d-2)$-dimensional surface $S \subset \Sigma$.



$N = d(d-3)/2$ independent graviton polarizations. Theory is *hyperbolic* if there are N "ingoing" and N "outgoing" characteristic hypersurfaces through S (allow for degeneracy).

Hyperbolicity

- GR is hyperbolic in any background.
- Lovelock: $A^{ijkl} \sim g^{00}\delta^{k(i}\delta^{j)l} + \mathcal{R}^{ijkl}$. Hyperbolicity not obvious when curvature comparable to scale set by coupling constants. Hyperbolicity depends on background.
- A^{ijkl} can be read off from equation of motion for linearized perturbations. Linearized eqs hyperbolic iff nonlinear eqs hyperbolic.

Example 1: Ricci flat type N spacetime

Type N: $\exists$ null ℓ^a such that $\ell^a C_{abcd} = 0$ (e.g. pp-wave).
Solves Lovelock eq. of motion with $\Lambda = 0$.
A hypersurface is characteristic iff it is null w.r.t. one of $N = d(d-3)/2$ "effective metrics" of form

$$G_{(I)ab} = g_{ab} - k_2 \omega_{(I)} \ell_a \ell_b \qquad I = 1, \ldots, N$$

where ω_I is homogeneous (degree 1) function of curvature.

- "Total multirefringence"
- Lovelock theories are hyperbolic in such backgrounds *for arbitrarily large curvature*
- Null cones of $G_{(I)ab}$ form a nested set, tangent along ℓ^a, causality determined by outermost cone

Example 2: Killing horizon

Gravitational signals can travel faster than light. Can they escape from inside a black hole?

Izumi (2014): a *Killing horizon* is characteristic for all graviton polarizations in Einstein-Gauss-Bonnet theory. We generalized this to any Lovelock theory.

If we deform the metric inside a Killing horizon, the deformation cannot escape the horizon.

Event horizon of a static BH must be a Killing horizon. True also for stationary BHs in GR - what about Lovelock?

Non-stationary BHs?

Example 3: static black hole spacetimes

Lovelock theories admit static, spherically symmetric, solutions (Boulware & Deser 1985, Wheeler 1986)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$$

Characteristic hypersurfaces determined by two-derivative terms in eqs for linear perturbations.

Linear perturbations can be classified into scalar, vector or tensor types, each satisfying a "master equation" (Dotti & Gleiser 2004-5, Takahashi & Soda 2009)

Example 3: static black hole spacetimes

Master equation for each type: $G^{\mu\nu}_{(I)}\partial_\mu\partial_\nu h + \ldots$ ($I =$ scalar, vector or tensor) where $G^{\mu\nu}_{(I)}$ is smooth.

Invert $G^{\mu\nu}_{(I)}$ to define "effective metrics", which have form

$$G_{(I)\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{c_I(r)}d\Omega^2$$

for certain functions $c_I(r)$

A surface is characteristic iff it is null w.r.t. one of these effective metrics.

NB: effective metrics non-generic

Example 3: static black hole spacetimes

$$G_{(I)\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{c_I(r)} d\Omega^2$$

Null cone of $G_{(I)}$ lies outside that of g iff $c_I > 1$ (gravity travels faster than light)

More serious issue: $c_I(r)$ can *vanish* at $r = r_*$ outside event horizon for small black holes $\Rightarrow$ violation of hyperbolicity

When this happens, $\exists$ exponentially growing linear perturbations. Previously this was interpreted as an *instability* of the black hole.

It is much worse. The initial value problem is ill-posed. For a generic initial perturbation there is no solution of the linearized equations.

Discussion

Lovelock theories are not always hyperbolic: depends on background.

Can one set up initial data so that theory is initially hyperbolic but becomes non-hyperbolic after some time?

Yes: consider *large* black hole: hyperbolicity violated to future of surface Σ of constant r *inside* black hole.

Is this generic? Generic linear perturbations cannot be evolved to future of Σ. Suggests nonlinear instability may ensure preservation of hyperbolicity (cf strong cosmic censorship). (Work in progress.)

Shock formation in Lovelock theories

"Speed of gravity" can vary in spacetime

Can we make a wavepacket so that back of wavepacket travels faster than front?

cf compressible perfect fluid: speed of sound depends on pressure $\Rightarrow$ wave steepening $\Rightarrow$ shock!

Shock formation in 1d

Canonical example: Burgers' equation

$$u_t + uu_x = 0$$

Solve by method of characteristics: u constant along characteristics

If $u_x(x_0) < 0$ then characteristic through $x = x_0$ will intersect its neighbours $\Rightarrow u_x$ diverges: shock

Shock formation in 3d fluids

Compressible perfect fluid

Initial data: fluid at rest for $r > R$, non-trivial in $r < R$

Sideris (1985, non-relativistic): blow-up of solutions for small initial data

Christodoulou (2007, relativistic): blow-up for small data occurs because "density" of outgoing characteristic hypersurfaces diverges

Propagation of curvature discontinuity

Consider a spacetime with a curvature discontinuity across a hypersurface Σ, metric C^1 across Σ.

$\partial^2 g$ not uniquely determined on $\Sigma \Rightarrow \Sigma$ is characteristic.

Characteristic surfaces are ruled by *bicharacteristic curves* (null geodesics in GR)

Can show that amplitude of discontinuity is governed by a transport equation: an ODE along these curves

e.g. in GR (Lichnerowicz 1960)

$$V^a \nabla_a [R_{bcde}] + \frac{1}{2} \nabla_a V^a [R_{bcde}] = 0$$

Propagation of curvature discontinuity: Lovelock

A curvature discontinuity across surface S in the initial data will resolve into curvature discontinuities propagating along the characteristic surfaces through S

Assume initial data outside S is that of known "background" spacetime (e.g. asymptotically flat)

Solution outside "outermost" characteristic surface will be the background spacetime - the discontinuity "invades" this spacetime

Background spacetime is known: we can determine the characteristic surface

Transport equation

In coordinates adapted to characteristic surface: $[\partial_0^2 g_{ij}] = \Pi r_{ij}$ $(A^{ijkl} r_{kl} = 0)$

Transport equation *nonlinear* (Lovelock theories are "genuinely nonlinear")

$$\frac{d\Pi}{ds} + N\Pi^2 + M\Pi = 0$$

s is parameter along bicharacteristic curve. N, M depend on extrinsic and intrinsic curvature of Σ (can be determined from background spacetime)

Main result:

$$N \sim k_2 K r^3 + \dots$$

Shock formation

GR is "exceptional": linear transport equation. Solution blows up only if hypersurface not smooth (caustic). Asymptotically flat: $\Pi(s)$ decays.

Lovelock: general solution (s is parameter along curve)

$$\Pi(s) = \frac{\Pi(0)e^{-\Phi(s)}}{1 + \Pi(0)\int_0^s N(s')e^{-\Phi(s')}ds'} \qquad \Phi(s) = \int_0^s M(s')ds'$$

Blows up when denominator vanishes: shock

If $N \neq 0$ then can arrange this by taking $\Pi(0)$ large enough: shock formation from "large" initial data (with curvature comparable to scale set by Lovelock couplings)

High frequency gravitational waves Choquet-Bruhat 1969

Two-time Ansatz:

$$g_{\mu\nu}(x,\eta) = \bar{g}_{\mu\nu}(x) + \omega^{-2} h_{\mu\nu}(x,\eta) + \mathcal{O}(\omega^{-3})$$

where $\eta = \omega\phi(x)$. Expand eqs of motion in ω^{-1}.

Surfaces of constant ϕ are characteristic surfaces of "background" spacetime $\bar{g}$.

Adapted coordinates: $\phi = x^0 \Rightarrow h_{ij} = \Omega(x,\eta) r_{ij}$

Ω obeys transport equation: nonlinear ODE with *same* nonlinear term as for curvature discontinuity

High frequency waves form shocks just as for curvature discontinuity

Examples

Nonlinear term *vanishes* for simplest examples:

- Any characteristic surface in flat spacetime
- Spherically symmetric characteristic surface in static, spherically symmetric, spacetime
- Killing horizon

Plane wave spacetime:

- shock formation for *arbitrarily small* initial amplitude
- focusing caused by caustic causes amplitude to grow, shock forms before caustic

More interesting example: axisymmetric characteristic surface in static, spherically symmetric, spacetime

Discussion

Smooth initial data

We've focused on curvature discontinuities and high frequency waves since these are easy to analyse

Do shocks form from *smooth* initial data?

Analogy with other "genuinely nonlinear" theories strongly suggests that they will, for a generic class of initial data with region where curvature is comparable to scale set by Lovelock coupling constants (numerics?)

Mechanism of shock formation (divergence in density of characteristic surfaces) seems to be the same as for perfect fluid

Weak cosmic censorship

Shocks are curvature singularities. Are these naked or hidden inside "black" holes?

"Naked": connected to $\mathcal{I}^+$ by bicharacteristic curve

Reduce amplitude of initial outgoing disturbance: takes longer for shock to form $\Rightarrow$ requires bigger black hole, but initial energy smaller...

Curvature discontinuity: choose asymptotically flat background spacetime and outgoing characteristic hypersurface Σ that extends to $\mathcal{I}^+$. Arrange for shock to form along Σ. Then this shock is "visible" to $\mathcal{I}^+$.

Small initial data

We've argued that shocks form for certain "large" initial data. What about "small" initial data, i.e., almost flat?

This is the question of nonlinear stability of Minkowski spacetime

Highly non-trivial in 4d GR (Christodoulou & Klainerman)

Expected to be much easier in $d > 4$ dimensions because linear field decays faster

Analogy with nonlinear wave equation $\Box\phi = f(\phi, \partial\phi, \partial^2\phi)$ suggests that Minkowski spacetime should be stable in Lovelock theories

Currently under investigation (J. Keir). Establishing local well-posedness is causing problems (cf Willison 2014).

Evolution of shocks

In fluid dynamics, shock formation is not the end of time evolution: can extend as a *weak* solution by allowing fields to be discontinuous. Rankine-Hugoniot junction conditions from conservation of energy-momentum and particle number. Shocks propagate along *non*characteristic hypersurfaces.

Analogous situation in Lovelock theories: once shock forms, allow ∂g to be discontinuous across hypersurface Σ. Weak solution: extremize action $\Rightarrow$ canonical momentum π^{ij} should be continuous across Σ. Does such a surface describes a dynamical shock?