

# Quasi-local mass and isometric embedding

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Recent Advances in Mathematical General Relativity

- ▶ Joint work with Po-Ning Chen and Shing-Tung Yau.
- ▶ The connection of “Rigidity of mass (when is mass equal to zero)” and “Uniqueness of isometric embedding”.
- ▶ A variational approach to the problem of mass in GR.
- ▶ An invariant mass theorem and a new mass loss formula.

- ▶ Consider  $(M, g, k)$  a spacelike hypersurface in a vacuum spacetime ( $Ric = 0$ ) with induced metric  $g$  and second fundamental form  $k$ .
- ▶  $(M, g, k)$  is asymptotically flat if outside a compact subset, there exists an asymptotically flat coordinate system  $(x^1, x^2, x^3)$  on each end, such that

$$g = \delta + O_2(r^{-q}) \text{ and } k = O_1(r^{-q-1}), \text{ with } q > \frac{1}{2}.$$

- ▶ ADM mass (energy) of an end of  $(M, g, k)$  is defined to be:

$$E = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,j} - g_{jj,i}) \nu^i,$$

where  $S_r$  is a coordinate sphere of coordinate radius  $r$  on the end and  $\nu^i$  is the unit outward normal of  $S_r$ .

- ▶ Schoen-Yau's PMT (Witten, Parker-Taubes, Bartnik, Chrusciel, Eichmair-Lee-Huang-Schoen):  $E \geq 0$  and  $E = 0$  if and only if  $(M, g, k)$  is a spacelike hypersurface in  $\mathbb{R}^{3,1}$ .

- ▶ The model space is a totally geodesic hypersurface in the Schwarzschild spacetime. The Lorentz metric on the spacetime is

$$-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), m > 0.$$

- ▶ On the  $t = 0$  slice,  $g = \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$ ,  $k = 0$  (totally geodesic), and  $E = m$ .
- ▶ The expression of the ADM mass depends on the choice of the coordinate system, but Bartnik proved that it is a geometric invariant.

- ▶ However, the asymptotic condition  $q > 1/2$  is important as there exist
  - (1) a spacelike hypersurface in  $\mathbb{R}^{3,1}$  with  $g = \delta + O(r^{-1/2})$  and **non-zero** ADM mass.
  - (2) a spacelike hypersurface in  $Sch^{3,1}$  with flat induced metric and thus **zero** ADM mass, the second fundamental form  $k = O(r^{-3/2})$ . This is the graph of  $t = f(r)$ , with  $f(r) = O(r^{1/2})$ .
- ▶ PMT is sharp! But does the notion of mass make sense on a spacelike hypersurface that does not satisfy the asymptotic conditions?

- ▶ Fundamental difficulties of mass in GR:
- ▶ No “mass density” in GR.

In Newtonian gravity, the equation is  $\Delta\phi = 4\pi\rho$  where  $\rho$  is the mass density, and the total mass enclosed in a region  $\Omega \subset \mathbb{R}^3$  is simply  $\int_{\Omega} \rho$ . Such a formula does not hold in GR. We can turn the Newtonian mass into a boundary integral  $\frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}$  though.

- ▶ No reference system in GR.

Only on the asymptotically flat region where gravity is weak can we find an asymptotically flat coordinate system. This also makes the definition of other physical quantities such as angular momentum and center of mass very difficult.

- ▶ The expression of the ADM mass is a boundary integral at infinity.

- ▶ A description of mass or energy on a bounded region is extremely useful for many purposes.
- ▶ Once such a description is available, one can take the limit to infinity on spacelike hypersurface to allow more general asymptotics.
- ▶ In 1982, Penrose proposed a list of major unsolved problems in general relativity, and the first was “find a suitable definition of quasilocal mass (energy)”.
- ▶ Given a spacelike bounded region  $\Omega^3$  in spacetime, find the enclosed mass in terms of the boundary data on  $\partial\Omega = \Sigma^2$ .
- ▶ After reviewing several previous definitions of quasilocal mass, I shall discuss a new proposal which anchors the question of reference system at the same time.
- ▶ There are different approaches (Bartnik, Penrose, Dougan-Mason, Ludvigsen-Vickers, etc.) to the quasilocal mass problem. I shall restrict my review to two of the most well-known definitions of quasi-local mass.

- ▶ Given a closed embedded 2-surface  $\Sigma$  in a spacetime. Both definitions depend on  $\sigma$ , the induced metric on  $\Sigma$ , and  $H$  the mean curvature vector field on  $\Sigma$ .
- ▶ The Hawking-Geroch mass is defined by

$$m = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} |H|^2\right)$$

- ▶ There is a time symmetric version in which  $|H|$  is replaced by the mean curvature of  $\Sigma$  in a time-symmetric slice.
- ▶ Time symmetric case: The Hawking mass has the amazing monotone property (Eardley, Geroch, Jang-Wald) along the inverse mean curvature flow, which was instrumental in Huisken-Ilmanen's proof of Riemannian Penrose conjecture for a single black hole. The positivity of Hawking mass for stable CMC surfaces was proved by Christodoulou-Yau.

- ▶ Another important quasi-local mass construction was initiated by Brown-York, and the isometric embedding problem enters the consideration through the Hamilton-Jacobi theory.
- ▶ The Brown-York-Liu-Yau mass is defined to be

$$M = \frac{1}{8\pi} \left( \int_{\Sigma} H_0 - \int_{\Sigma} |H| \right)$$

where  $H_0$  is the mean curvature of the isometric embedding of  $\sigma$  into  $\mathbb{R}^3$ .

- ▶ (Nirenberg-Pogorelov) Any Riemannian metric of positive Gauss curvature on  $S^2$  can be uniquely isometrically embedded into  $\mathbb{R}^3$  up to Euclidean motions.  $H_0$  is uniquely determined by  $\sigma$  in this case.
- ▶ The isometric embedding into  $\mathbb{R}^3$  plays the role of a reference system and  $\frac{1}{8\pi} \int_{\Sigma} H_0$  is the reference Hamiltonian.
- ▶ The BYLY mass has the desirable positivity property for a mass which is proved by Shi-Tam (time-symmetric) and Liu-Yau.

- ▶ However, a very important criterion (rigidity property) for a valid definition of quasilocal mass is that **for a surface in  $\mathbb{R}^{3,1}$ , the quasilocal mass should be zero**, as there is no gravitation and no matter field.
- ▶ We can compute the HG ( $m$ ) and BYLY ( $M$ ) mass for a surface  $\Sigma$  in the Minkowski spacetime  $\mathbb{R}^{3,1}$ .
- ▶ If  $\Sigma \subset \mathbb{R}^3 \subset \mathbb{R}^{3,1}$ , then  $M = 0$  (by the uniqueness of isometric embedding) but  $m < 0$  except for a standard round sphere.
- ▶ If  $\Sigma$  is in a light cone in  $\mathbb{R}^{3,1}$ , then  $m = 0$  but  $M > 0$  except for a standard round sphere.

- ▶ This suggests that in the Hamilton-Jacobi approach one needs to consider isometric embeddings of  $\sigma$  into the Minkowski spacetime, i.e.  $X : S^2 \rightarrow \mathbb{R}^{3,1}$  such that  $\langle dX, dX \rangle = \sigma$ .
- ▶ Such an embedding can be obtained in the following way. Take any smooth function  $\tau$  on  $S^2$  and isometrically embed  $\sigma + d\tau \otimes d\tau$  as a surface  $\hat{\Sigma}$  in  $\mathbb{R}^3$  (Nirenberg-Pogorelov). Take the graph of  $\tau$  over  $\hat{\Sigma}$  and denote the embedding of the graph by  $X$ . The induced metric on  $X(\Sigma)$  is exactly  $\sigma$ .
- ▶  $\tau = -\langle X, \partial_t \rangle$ . We can replace  $\partial_t$  by any future timelike unit Killing field  $T_0$  in  $\mathbb{R}^{3,1}$ .
- ▶ Among all such isometric embeddings, is there a “best match” of the physical surface  $\Sigma$  with physical data  $(\sigma, H)$ ?

- ▶ Nash's theorem guarantees the existence of isometric embedding but the issue of uniqueness is not well-understood except for convex surfaces in  $\mathbb{R}^3$ .
- ▶ We assume  $H$  is spacelike and extract from it a function  $|H|$  and a connection one-form  $\alpha_H$ .
- ▶ The quasilocal mass we proposed is defined in terms of the data  $(\sigma, |H|, \alpha_H)$  and is obtained through a variational approach to single out an optimal isometric embedding among the space of pairs  $(X, T_0)$ .
- ▶ Definition: the quasilocal energy  $E(\Sigma, X, T_0)$  with respect to a pair  $(X, T_0)$  is defined to be:

$$(*) \frac{1}{8\pi} \int_{\hat{\Sigma}} \hat{H} - \frac{1}{8\pi} \int_{\Sigma} [\sqrt{|H|^2(1 + |\nabla \tau|^2) + (\Delta \tau)^2} + \theta \Delta \tau - \alpha_H(\nabla \tau)]$$

where  $\sinh \theta = \frac{-\Delta \tau}{|H| \sqrt{1 + |\nabla \tau|^2}}$  and  $\tau = -\langle X, T_0 \rangle$ .

- ▶ Such an expression comes from the Hamilton-Jacobi analysis of gravitational action (Brown-York, Hawking-Horowitz) and vanishes for a surface in the Minkowski spacetime (projections through any  $T_0$ ).

- ▶ The general expression depends on a gauge choice. Our expression  $(*)$  is defined with respect to a canonical gauge that matches the area element variations along the surfaces  $\Sigma$  and  $X(\Sigma)$ .
- ▶ The choice of this canonical gauge was at first justified by the positivity of the quasi-local energy (W.-Yau). It was in fact closely related to the following gravitational conservation law which later played a critical role in several other applications.

$$\int_{\partial B} \pi(T_0, u) = \int_B Ric(T_0, v),$$

where  $\pi$  is the conjugate momentum of the timelike hypersurface  $B$  and  $v$  is the outward unit normal of  $B$

- ▶ To find a “best match” of the physical data  $(\sigma, |H|, \alpha_H)$ , we minimize  $E(\Sigma, X, T_0)$  over the pairs  $(X, T_0)$ .
- ▶ By Lorentz invariance, it suffices to consider variations with respect to  $\tau = -\langle X, T_0 \rangle$ .

- The Euler-Lagrange equation is:

$$\begin{aligned}
 & -(\hat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} \\
 & + \operatorname{div}_\sigma\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\right) \cosh\theta|H| - \nabla\theta - \alpha_H = 0,
 \end{aligned} \tag{0.1}$$

$$\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}.$$

- This is again an identity that is satisfied by any surface in  $\mathbb{R}^{3,1}$  with respect to the projection along any  $T_0$ .
- In order to better understand the energy and the equation, we rewrite them in terms of data on  $X(\Sigma)$ .
- Let  $H_0$  be the mean curvature vector of  $X(\Sigma)$  and again we extract  $|H_0|$  and  $\alpha_{H_0}$ .

- ▶ Consider a function  $\rho$  and a 1-form  $j_a$  on  $\Sigma$ :

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}.$$

$$j_a = \rho \nabla_a \tau - \nabla_a [\sinh^{-1}(\frac{\rho \Delta\tau}{|H_0||H|})] - (\alpha_{H_0})_a + (\alpha_H)_a.$$

- ▶ It can be checked that the quasi-local energy becomes

$$E(\Sigma, \tau) = \frac{1}{8\pi} \int_{\Sigma} (\rho + j_a \nabla^a \tau).$$

- ▶ The E-L equation becomes  $\nabla^a j_a = 0$ .
- ▶ The quasi-local mass is defined to be

$$\frac{1}{8\pi} \int_{\Sigma} \rho,$$

and  $\rho$  is called the quasi-local mass density.

- ▶ The optimal isometric embedding system of  $(\sigma, |H|, \alpha_H)$  seeks for  $(X, T_0)$  with  $\tau = -\langle X, T_0 \rangle$  such that the following system is satisfied:

$$(**) \quad \begin{cases} \langle dX, dX \rangle = \sigma \\ \nabla^a j_a = 0 \end{cases}$$

- ▶ This is a system of four unknowns and four equations. We do not know how to solve this system in general, but we do have the following theorems.

- ▶ **Theorem (Chen-W.-Yau)** Let  $(\sigma, |H|, \alpha_H)$  be the data of a spacelike surface  $\Sigma$  in the Minkowski spacetime. Suppose the projection of  $\Sigma$  onto the orthogonal complement of  $T_0$  is convex. Then the kernel of the linearized optimal isometric embedding system  $(**)$  consists precisely of Lorentz motions. Moreover, the second variation of the quasilocal energy  $E(\Sigma, \tau)$  is non-negative definite.
- ▶ Comparing with Cohn-Vossen's local rigidity theorem of isometric embedding of convex surfaces in  $\mathbb{R}^3$ , the theorem implies that surfaces with prescribed  $(\sigma, |H|, \alpha_H)$  are locally rigid in  $\mathbb{R}^{3,1}$ .

- ▶ Proof: In the special case when the reference isometric embedding lies in a totally geodesic  $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ , the theorem was proved by Miao-Tam-Xie by a different method. Suppose  $\Omega$  is a compact domain in  $\mathbb{R}^3$  with mean convex boundary  $\partial\Omega$ , then for any smooth function  $\eta$  on  $\partial\Omega$

$$\int_{\partial\Omega} \left[ \frac{(\Delta\eta)^2}{H} - h(\nabla\eta, \nabla\eta) \right] \geq 0$$

and equality holds if and only if  $\eta$  is the restriction of a Cartesian coordinate function on  $\mathbb{R}^3$ .

- ▶ To prove the general case, we need to go back to the positivity proof of the quasilocal energy and trace when equality holds. The ingredients of the proof consist of: (1) Schoen-Yau-Jang equation, (2) Bartnik's quasi-spherical construction, (3) Shi-Tam's monotonicity formula, and (4) Witten's spinor proof of PMT.

- ▶ **Theorem (Chen-W.-Yau)** Let  $(\sigma, |H|, \alpha_H)$  be the data of a spacelike surface  $\Sigma$  in a general spacetime. Suppose that  $\tau_0$  is a critical point of the quasi-local energy  $E(\Sigma, \tau)$  and that the corresponding quasilocal mass density  $\rho$  (with respect to  $\tau_0$ ) is positive, then  $\tau_0$  is a local minimum for  $E(\Sigma, \tau)$ .
- ▶ The special case when  $\tau_0 = 0$  follows from Miao-Tam-Xie's work and the general case relies on the following comparison theorem.
- ▶ **Theorem (Chen-W.-Yau)** Under the assumption of the above theorem, for any time function  $\tau$  such that  $\sigma + d\tau \otimes d\tau$  has positive Gaussian curvature, we have

$$E(\Sigma, \tau) \geq E(\Sigma, \tau_0) + E(\Sigma_{\tau_0}, \tau). \quad (0.2)$$

Moreover, equality holds if and only if  $\tau - \tau_0$  is a constant.

- ▶ The above theorems allow us to solve the optimal isometric embedding system for configurations that limit to surfaces in the Minkowski spacetime. This is in particular sufficient for calculations at infinity of an asymptotically flat initial data set.
- ▶ Suppose the ADM mass of  $(M, g, k)$  is positive, then there is a unique, locally energy-minimizing, optimal isometric embedding of  $S_r$ , whose image approaches a large round sphere in  $\mathbb{R}^3$ .
- ▶ In each step, it suffices to solve linear elliptic equations on standard  $S^2$  of the following form:

$$\Delta(\Delta + 2)f = g.$$

$\Delta(\Delta + 2)$  is the positive definite second variation operator in this case.

- ▶ The limit of the quasilocal mass on  $S_r$  is defined to be the total mass of the asymptotically flat hypersurface.

- ▶ **Theorem (Chen-W.-Yau, in progress):** Every strictly spacelike hypersurface in the Schwarzschild spacetime has the same new total mass  $m$ .
- ▶ This invariant mass theorem holds in more general spacetime (cf. Chrusciel).
- ▶ “Strictly spacelike” means outside a compact subset the hypersurface is defined by  $t = f(x^1, x^2, x^3)$  such that  $\limsup_{r \rightarrow \infty} \frac{f}{r} < 1$ .
- ▶ The calculation is robust in the sense that we can use an arbitrary family of convex surface foliation.
- ▶ A key ingredient in the proof is a local conservation law of the momentum tensor of a timelike hypersurface.
- ▶ We apply this to both the physical spacetime and the reference spacetime  $(\mathbb{R}^{3,1})$ , and evaluate the difference which determines the total mass.
- ▶ The optimal embedding helps cancel any possible wild asymptotic of the physical hypersurface.

- ▶ In special relativity, the conserved quantities come from symmetry, or Killing fields in  $\mathbb{R}^{3,1}$ . The reference system from optimal isometric embedding provides the symmetry for conserved quantities.
- ▶ For an optimal isometric embedding  $(X, T_0)$  with  $X : \Sigma \rightarrow \mathbb{R}^{3,1}$ , the quasi-local conserved quantity corresponding to a Killing field in  $\mathbb{R}^{3,1}$  is defined to be:

$$\frac{(-1)}{8\pi} \int_{\Sigma} (\langle K, T_0 \rangle \rho + (K^{\top})^a j_a),$$

where  $K^{\top}$  is the component of  $K$  that is tangential to  $X(\Sigma)$ .

- ▶  $K = x^i \partial_j - x^j \partial_i, i < j$  defines an angular momentum with respect to  $\partial_t$ .

- ▶ Given an asymptotically flat initial data set  $(M, g, k)$  (with positive ADM mass), consider the coordinate sphere  $S_r$ .
- ▶ Evaluate with respect to the unique solution of the optimal isometric embedding system and take the limit as  $r \rightarrow \infty$  of the quasi-local conserved quantities on  $S_r$ , we obtain  $(E, P_i, J_i, C_i)$  where  $(E, P_i)$  is the same as the ADM energy -momentum.

- ▶ **Asymptotically hyperbolic case**
- ▶ The limiting case of strictly spacelike correspond to an asymptotically hyperbolic or null hypersurface. In this case, the energy (mass) can radiate away along the null directions. Our analysis applies to the dynamics of a family of asymptotically hyperbolic hypersurfaces  $(M, g, k)$  and captures this radiation phenomenon along the Einstein equation.
- ▶ Definition of mass by X. Wang, Chrusciel-Herzlich, Nagy-Sakovich, X. Zhang, etc.
- ▶ The model case is an umbilical slice in  $Sch^{3,1}$  ( $k = g$ ).
- ▶ In previous definitions,  $k$  is sometimes assumed to be  $g$ .
- ▶ It is nevertheless important to take into account of  $k$  for dynamical considerations. In fact, there exists an asymptotically umbilical slice in  $Sch^{3,1}$  that is isometric to  $\mathbb{H}^3$ . There is no way to read off the mass from the induced metric.

- ▶ Suppose  $g_{\mathbb{H}^3} = \frac{1}{r^2+1} dr^2 + r^2 \tilde{\sigma}_{ab} du^a du^b$  is the hyperbolic 3-metric.
- ▶  $(M, g, k)$  is said to be an asymptotically hyperbolic initial data set if outside a compact subset  $K$ ,  $M$  is diffeomorphic to a finite union of ends  $\cup_i \mathbb{H}^3 \setminus B_i$ . Under the diffeomorphism, we have

$g = g_{rr} dr^2 + 2g_{ra} dr du^a + g_{ab} du^a du^b$  and  $k = g + p$ , where

$$g_{rr} = \frac{1}{r^2} - \frac{1}{r^4} + \frac{g_{rr}^{(-5)}}{r^5} + \text{l.o.t.} \quad g_{ra} = \frac{g_{ra}^{(-3)}}{r^3} + \text{l.o.t.}$$

$$g_{ab} = r^2 \tilde{\sigma}_{ab} + g_{ab}^{(0)} + \frac{g_{ab}^{(-1)}}{r} + \text{l.o.t.}$$

$$p_{rr} = \frac{p_{rr}^{(-4)}}{r^4} + \text{l.o.t.} \quad p_{ra} = \frac{p_{ra}^{(-3)}}{r^3} + \text{l.o.t.},$$

$$p_{ab} = p_{ab}^{(0)} + \frac{p_{ab}^{(-1)}}{r} + \text{l.o.t.}$$

- ▶ We solve the optimal isometric equation equation on each coordinate sphere and evaluate the limit of quasilocal conserved quantities.
- ▶ Define the mass aspect function  $m$  to be

$$m = \frac{3}{2} \operatorname{tr}_{S^2} g_{ab}^{(-1)} + \operatorname{tr}_{S^2} p_{ab}^{(-1)} + g_{rr}^{(-5)}. \quad (0.3)$$

The energy-momentum is

$$E = \frac{1}{8\pi} \int_{S^2} m dS^2$$

$$P^i = \frac{1}{8\pi} \int_{S^2} \tilde{X}^i m dS^2, \quad i = 1, 2, 3$$

where  $\tilde{X}^i, i = 1, 2, 3$  are the three standard coordinate functions on  $S^2$ .

- ▶  $(E, P^i)$  is non-spacelike by the positivity of quasi-local mass. Set  $\mathfrak{M} = \sqrt{E^2 - \sum_{i=1}^3 (P^i)^2}$ , then along the vacuum Einstein equation with  $(M, g(t), k(t))$ , we derive

$$\partial_t \mathfrak{M}(t) = -\frac{1}{8\pi} \int_{S^2} |p_{ab}^{(0)} + g_{ab}^{(0)}|^2 dS^2.$$

# Thank you!