On the cosmic no-hair conjecture in the Einstein-Vlasov setting

Hans Ringström
KTH, Royal Institute of Technology, Stockholm

Recent Advances in Mathematical General Relativity
Institut Henri Poincaré, Paris September 23-25, 2015

Introduction

Observations of supernovae of type Ia, 1998: the universe is expanding at an accelerated rate.

One possible mechanism: positive cosmological constant Λ.

de Sitter space

Model solution: **de Sitter space**

$$g_{\mathrm{dS}} = -dt^2 + e^{2Ht}\bar{g}$$

on $\mathbb{R} \times \mathbb{T}^3$, where $H = \sqrt{\Lambda/3}$.

Solves

$$G + \Lambda g = 0,$$

where G is the **Einstein tensor**

$$G = \mathrm{Ric} - \frac{1}{2}Sg.$$

Cosmic no-hair, rough formulation

Cosmic no-hair conjecture: In a spacetime solving Einstein's equations with a positive cosmological constant, the geometry appears de Sitter like to late time observers.

In particular: solutions are expected to homogenise and isotropise.

Minkowski space

Let $\gamma(t) = (t, 0, 0, 0)$. Then γ is an observer in Minkowski space. How much of the $t = 0$ hypersurface does γ see?

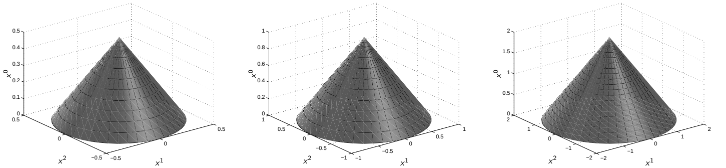

Figure : The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$, $t = 1$ and $t = 2$.

de Sitter space

Consider the metric

$$g_{\mathrm{dS}} = -dt^2 + e^{2t}\bar{g}.$$

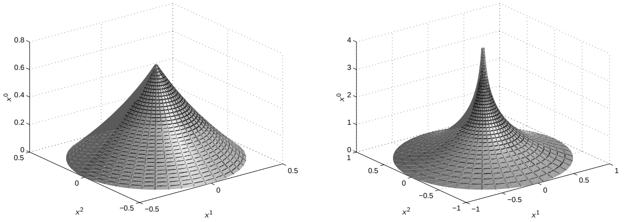

Figure : The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$ and for all t.

Regions of interest

Relevant spacetime regions:

$$C_{\Lambda,K,T} = \{(t,\bar{x}) : t > T,\ |\bar{x}| < KH^{-1}e^{-Ht}\},$$

where $H = \sqrt{\Lambda/3}$.

Role of T: to specify what is meant by 'late times'.
Role of $K \geq 1$: to provide a margin in the general case.

Cosmic no-hair, formal definition

Let (M, g) be a time oriented, globally hyperbolic Lorentz mfd which is future causally geodesically complete. Assume, moreover, that (M, g) is a solution to Einstein's equations with a positive cosmological constant Λ. Then (M, g) is said to be *future asymptotically de Sitter like* if there is a Cauchy hypersurface Σ in (M, g) such that for every future oriented and inextendible causal curve γ in (M, g), the following holds:

- there is an open set D in (M, g), such that $J^-(\gamma) \cap J^+(\Sigma) \subset D$, and D is diffeomorphic to $C_{\Lambda,K,T}$ for a suitable choice of $K \geq 1$ and $T > 0$,

...continued

- using $\psi : C_{\Lambda,K,T} \to D$ to denote the diffeomorphism; letting $R(t) = KH^{-1}e^{-Ht}$; using $\bar{g}_{\mathrm{dS}}(t,\cdot)$ and $\bar{k}_{\mathrm{dS}}(t,\cdot)$ to denote the metric and second fundamental form induced on $S_t = \{t\} \times B_{R(t)}(0)$ by g_{dS}; using $\bar{g}(t,\cdot)$ and $\bar{k}(t,\cdot)$ to denote the metric and second fundamental form induced on S_t by $\psi^* g$; and letting $N \in \mathbb{N}$, the following holds:

$$\lim_{t\to\infty} \Big(\|\bar{g}_{\mathrm{dS}}(t,\cdot) - \bar{g}(t,\cdot)\|_{C^N_{\mathrm{dS}}(S_t)} + \|\bar{k}_{\mathrm{dS}}(t,\cdot) - \bar{k}(t,\cdot)\|_{C^N_{\mathrm{dS}}(S_t)} \Big) = 0.$$

Norms

Here:

$$\|h\|_{C^N_{\mathrm{dS}}(S_t)} = \left(\sup_{S_t} \sum_{l=0}^{N} \bar{g}_{\mathrm{dS},i_1 j_1} \cdots \bar{g}_{\mathrm{dS},i_l j_l} \bar{g}_{\mathrm{dS}}^{im} \bar{g}_{\mathrm{dS}}^{jn} \overline{\nabla}_{\mathrm{dS}}^{i_1} \cdots \overline{\nabla}_{\mathrm{dS}}^{i_l} h_{ij} \overline{\nabla}_{\mathrm{dS}}^{j_1} \cdots \overline{\nabla}_{\mathrm{dS}}^{j_l} h_{mn}\right)^{1/2}$$

for a covariant 2-tensor field h on S_t, where $\overline{\nabla}_{\mathrm{dS}}$ denotes the Levi-Civita connection associated with $\bar{g}_{\mathrm{dS}}(t, \cdot)$.

Cosmic no-hair conjecture

Cosmic no-hair conjecture: Let A denote the class of initial data such that the corresponding maximal Cauchy developments are future causally geodesically complete solutions to Einstein's equations with a positive cosmological constant Λ (for some fixed matter model). Then generic elements of A yield maximal Cauchy developments that are *future asymptotically de Sitter like*.

Some results in the Einstein-Vlasov setting

- Spatially homogeneous setting; H. Lee '04.
- General perturbations thereof; H. R. '13.
- The surface symmetric setting; S. B. Tchapnda, A. D. Rendall '03.
- General perturbations thereof; E. Nungesser '14.
- The $\mathbb{T}^3$-Gowdy symmetric setting and general perturbations thereof; H. Andréasson and H. R. '13.

Vlasov matter

Vlasov matter: collection of particles, where

- the particles all have unit mass,
- collisions are neglected,
- the particles follow geodesics,
- collection described statistically by a distribution function.

Stress energy tensor, Vlasov matter

In the Vlasov setting, the relevant mathematical structures are

- the **mass shell** P; the future directed unit timelike vectors in (M, g),
- the **distribution function** $f : P \to [0, \infty)$,
- the **stress energy tensor**

$$T_{\alpha\beta}|_{\xi} = \int_{P_{\xi}} f p_{\alpha} p_{\beta} \mu_{P_{\xi}},$$

- the **Vlasov equation**

$$\mathcal{L}f = 0.$$

The Einstein-Vlasov system

The Einstein-Vlasov system consists of the equations

$$\begin{aligned} G + \Lambda g &= T, \\ \mathcal{L} f &= 0 \end{aligned}$$

for g and f. Note that the second equation corresponds to the requirement that f be constant along timelike geodesics.

$\mathbb{T}^2$-symmetry

Metric:

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + \alpha^{-1}d\theta^2) + te^P[dx + Qdy + (G + QH)d\theta]^2 + te^{-P}(dy + Hd\theta)^2,$$

Here $\alpha > 0$, λ, P, Q, G and H only depend on t and θ.

$\mathbb{T}^3$-Gowdy: H and G time-independent. Frame:

$$e_0 = t^{1/4}e^{-\lambda/4}\partial_t, \quad e_1 = t^{1/4}e^{-\lambda/4}\alpha^{1/2}(\partial_\theta - G\partial_x - H\partial_y),$$
$$e_2 = t^{-1/2}e^{-P/2}\partial_x, \quad e_3 = t^{-1/2}e^{P/2}(\partial_y - Q\partial_x).$$

Matter components and equations

Matter components

$$\rho = T(e_0, e_0), \quad J_i = -T(e_0, e_i), \quad P_i = T(e_i, e_i), \quad S_{ij} = T(e_i, e_j),$$

Main equations

$$\begin{aligned}
\partial_t(t\alpha^{-1/2}P_t) &= \partial_\theta(t\alpha^{1/2}P_\theta) + t\alpha^{-1/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2) + t^{1/2}e^{\lambda/2}\alpha^{-1/2}(P_2 - P_3), \\
\partial_t(t\alpha^{-1/2}e^{2P}Q_t) &= \partial_\theta(t\alpha^{1/2}e^{2P}Q_\theta) + 2t^{1/2}\alpha^{-1/2}e^{\lambda/2+P}S_{23}, \\
\lambda_t - 2\frac{\alpha_t}{\alpha} &= t\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\right] + 4t^{1/2}e^{\lambda/2}(\rho + \Lambda), \\
\lambda_t &= t\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\right] + 4t^{1/2}e^{\lambda/2}(P_1 - \Lambda), \\
\lambda_\theta &= 2t(P_t P_\theta + e^{2P}Q_t Q_\theta) - 4t^{1/2}e^{\lambda/2}\alpha^{-1/2}J_1.
\end{aligned}$$

Vlasov equation

Vlasov equation:

$$\frac{\partial f}{\partial t} + \frac{\alpha^{1/2} v^1}{v^0}\frac{\partial f}{\partial \theta} - \left[\frac{1}{4}\alpha^{1/2}\lambda_\theta\, v^0 + \frac{1}{4}\left(\lambda_t - \frac{2\alpha_t}{\alpha} - \frac{1}{t}\right) v^1 - \alpha^{1/2} e^P Q_\theta \frac{v^2 v^3}{v^0} + \frac{1}{2}\alpha^{1/2} P_\theta \frac{(v^3)^2 - (v^2)^2}{v^0}\right]\frac{\partial f}{\partial v^1} - \left[\frac{1}{2}\left(P_t + \frac{1}{t}\right) v^2 + \frac{1}{2}\alpha^{1/2} P_\theta \frac{v^1 v^2}{v^0}\right]\frac{\partial f}{\partial v^2} - \left[\frac{1}{2}\left(\frac{1}{t} - P_t\right) v^3 - \frac{1}{2}\alpha^{1/2} P_\theta \frac{v^1 v^3}{v^0} + e^P v^2\left(Q_t + \alpha^{1/2} Q_\theta \frac{v^1}{v^0}\right)\right]\frac{\partial f}{\partial v^3} = 0,$$

where $f : I \times S^1 \times \mathbb{R}^3 \to [0, \infty)$ is the distribution function, $v^0 = [1 + (v^1)^2 + (v^2)^2 + (v^3)^2]^{1/2}$. Matter quantities:

$$\rho = \int_{\mathbb{R}^3} v^0 f\, dv, \quad P_k = \int_{\mathbb{R}^3} \frac{(v^k)^2}{v^0} f\, dv, \quad J_k = \int_{\mathbb{R}^3} v^k f\, dv, \quad S_{jk} = \int_{\mathbb{R}^3} \frac{v^j v^k}{v^0} f\, dv,$$

An estimate for α

Note that

$$\lambda_t - \frac{\alpha_t}{\alpha} = t\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\right] + 2t^{1/2}e^{\lambda/2}(\rho + P_1) \geq 0.$$

In particular, there is a $0 < c_0 \in \mathbb{R}$ such that $\alpha^{-1/2}e^{\lambda/2} \geq c_0$. Since

$$-\frac{\alpha_t}{\alpha} \geq 4t^{1/2}e^{\lambda/2}\Lambda,$$

we obtain

$$\partial_t \alpha^{-1/2} \geq 2t^{1/2}\alpha^{-1/2}e^{\lambda/2}\Lambda \geq c_1 t^{1/2},$$

whence $\alpha(t,\theta) \leq Ct^{-3}$ for $t \geq t_1$.

Asymptotics for λ

Letting

$$\hat{\lambda} = \lambda + 3 \ln t - 2 \ln \left(\frac{3}{4\Lambda} \right),$$

it can be estimated that

$$\partial_t \hat{\lambda} \geq \frac{3}{t}(1 - e^{\hat{\lambda}/2}).$$

For every $\epsilon > 0$, there is thus a T such that

$$\lambda(t, \theta) \geq -3 \ln t + 2 \ln \frac{3}{4\Lambda} - \epsilon$$

for all $t \geq T$.

Energy

$$E = \int_{S^1} t^2 \alpha^{-1/2} \left(P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) + 4t^{-1/2} e^{\lambda/2} \rho \right) d\theta.$$

Lower bound on $\lambda \Rightarrow$ upper bound on E of the form $E(t) \leq C_a t^a$, $a > 1/2$. This bound on $E \Rightarrow \hat{\lambda} \to 0$. In the end:

$$\begin{aligned} \left\| \lambda(t, \cdot) + 3 \ln t - 2 \ln \frac{3}{4\Lambda} \right\|_{C^0} &\leq Ct^{-1/2}, \\ E(t) &\leq Ct^{1/2}. \end{aligned}$$

Moreover P and Q are bounded in C^0.

Asymptotics

There are smooth functions $\alpha_\infty > 0$, P_∞, Q_∞, G_∞ and H_∞ on $\mathbb{S}$, and, for every $0 \leq N \in \mathbb{Z}$, a constant $C_N > 0$ such that

$$\begin{aligned} t\|P_t(t,\cdot)\|_{C^N} + t\|Q_t(t,\cdot)\|_{C^N} + \|P(t,\cdot) - P_\infty\|_{C^N} + \|Q(t,\cdot) - Q_\infty\|_{C^N} &\leq C_N t^{-1}, && (1) \\ \left\|\frac{\alpha_t}{\alpha} + \frac{3}{t}\right\|_{C^N} + \left\|\lambda_t + \frac{3}{t}\right\|_{C^N} &\leq C_N t^{-2}, && (2) \\ \left\|t^3\alpha(t,\cdot) - \alpha_\infty\right\|_{C^N} + \left\|\lambda(t,\cdot) + 3\ln t - 2\ln\frac{3}{4\Lambda}\right\|_{C^N} &\leq C_N t^{-1}, && (3) \end{aligned}$$

for all $t \geq t_1$.

Asymptotics, continued

Define $f_{\rm sc}$ via $f_{\rm sc}(t,\theta,v) = f(t,\theta,t^{-1/2}v)$. Then there is a smooth, non-negative function with compact support, say $f_{\rm sc,\infty}$, on $\mathbb{S}\times\mathbb{R}^3$, such that

$$t\|\partial_t f_{\rm sc}(t,\cdot)\|_{C^N(\mathbb{S}\times\mathbb{R}^3)} + \|f_{\rm sc}(t,\cdot) - f_{\rm sc,\infty}\|_{C^N(\mathbb{S}\times\mathbb{R}^3)} \le C_N t^{-1}$$

for all $t \ge t_1$.

Cosmic no-hair conjecture, $\mathbb{T}^3$-Gowdy symmetric setting

Theorem

$\mathbb{T}^3$-Gowdy symmetric solutions to the Einstein-Vlasov system with a positive cosmological constant are future asymptotically de Sitter like.

Stability

Let $\bar{g}_{\mathrm{bg}}$ and $\bar{k}_{\mathrm{bg}}$ denote the metric and second fundamental form induced by a $\mathbb{T}^3$-Gowdy symmetric solution (to the Einstein-Vlasov system with a positive cosmological constant Λ) on a constant t hypersurface. Let, moreover, $\bar{f}_{\mathrm{bg}}$ denote the induced initial datum for the distribution function. Then there is an $\epsilon > 0$ such that if $(\mathbb{T}^3, \bar{g}, \bar{k}, \bar{f})$ are initial data satisfying

$$\|\bar{g} - \bar{g}_{\mathrm{bg}}\|_{H^5} + \|\bar{k} - \bar{k}_{\mathrm{bg}}\|_{H^4} + \|\bar{f} - \bar{f}_{\mathrm{bg}}\|_{H^4_{\mathrm{Vl},\mu}} \leq \epsilon,$$

then the associated maximal Cauchy development is

- future causally geodesically complete,
- future asymptotically de Sitter like.

Thank you!

Induced initial data

Let (M, g, f) be a solution and Σ be a spacelike hypersurface in (M, g). Then the **initial data induced on** Σ consist of

- the Riemannian metric induced on Σ by g, say $\bar{g}$,
- the second fundamental form induced on Σ by g, say $\bar{k}$,
- the induced distribution function $\bar{f} : T\Sigma \to [0, \infty)$.

Here

$$\bar{f} = f \circ \mathrm{proj}_{\Sigma}^{-1},$$

where $\mathrm{proj}_{\Sigma} : P_{\Sigma} \to T\Sigma$ represents projection orthogonal to the normal.

Function spaces

If Σ is a compact manifold, $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ denotes the space of smooth functions $\bar{f} : T\Sigma \to \mathbb{R}$ such that

$$\|\bar{f}\|_{H^l_{\mathrm{Vl},\mu}} = \left(\sum_{i=1}^{j} \sum_{|\alpha|+|\beta|\leq l} \int_{\bar{\mathsf{x}}_i(U_i)\times\mathbb{R}^n} \langle\bar{\varrho}\rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi}) (\partial^{\alpha}_{\bar{\xi}} \partial^{\beta}_{\bar{\varrho}} \bar{\mathsf{f}}_{\bar{\mathsf{x}}_i})^2 (\bar{\xi}, \bar{\varrho}) d\bar{\xi} d\bar{\varrho} \right)^{1/2}$$

is finite for every $l \geq 0$, where

$$\langle\bar{\varrho}\rangle = (1 + |\bar{\varrho}|^2)^{1/2}.$$