Conformal scattering

Recent advances in general relativity
IHP, Paris, trimester on mathematical relativity, 100 years of GR
Jean-Philippe Nicolas, Brest, France.

1 Introduction

1. The purpose of conformal scattering is to use conformal methods to construct scattering theories, not to re-interpret existing scattering theories in conformal terms.

2. This talk focuses on asymptotically flat situations with null conformal infinity ; the techniques involved in this situation are quite different from the asymptotically De Sitter or anti De Sitter cases

3. General principle. View the evolution of the field on the spacetime as an old-fashioned scattering experiment : incoming plane wave, hits an obstacle, propagates out as a superposition of plane waves. The scattering data (analogue to the incoming and outgoing plane waves) are the radiation fields (restrictions of the conformally rescaled solution on past and future null infinities). Complete scattering : past (resp. future) scattering data characterize the field uniquely. Like a Cauchy problem but set on a null hypersurface at infinity : a Goursat problem at infinity. Scattering operator goes from past scattering data to future scattering data.

4. History of conformal scattering.

   - 1965 Penrose Zero rest-mass fields including gravitation : asymptotic behaviour. Description of his conformal approach to infinity and its application to the study of the asymptotic behaviour of conformally invariant equations. The notion of conformal scattering is explicitely introduced and cited as one major motivation for the conformal approach.
   - 1967 Lax-Phillips Scattering theory. This is a spectral analytic approach to scattering theory by an obstacle. A plane wave decomposition is used in order to describe the evolution away from the obstacle as a simple translation representer of the solution. This translation representer is then reinterpreted as a radiation field, i.e. a trace of the conformally rescaled field at null infinity. Lax and Phillips also obtain an integral representation of the solution of the free wave equation in terms of its translation representer. This is in fact Whittaker's formula from 1907, put in its rightful context as an integral formula for the resolution of the initial value problem for the wave equation from null infinity.

- 1970's Friedlander's study of radiation fields. 1980 link with the Lax-Phillips theory, first conformal scattering construction, conformal wave equation on a static spacetime. In a way it is surprising that Friedlander chose to work on a static spacetime. The structure of conformal scattering only relies on having a smooth conformal boundary and is therefore insensitive to time dependence. I think there's two reasons for his choice : 1. he wanted to be able to construct null infinity rigorously and for a static spacetime with sufficiently fast fall-off of the metric, this is easy ; he wanted to extend the Lax-Phillips theory in its analytic richness, particularly to recover the translation representer, and this is tied in with at least stationarity of the underlying spacetime.
- Late 1980's-early 1990's : series of papers by Baez and collaborators, static frameworks, non linear equations.
- 2002, 2010 : Mason-Nicolas, Joudioux. These are constructions on generically non stationnary spacetimes, allowing for the presence of energy.
- 2013 : Nicolas. Wave equation on the Schwarzschild metric.

2 Spectral approach to scattering theory

- Spectral analytic scattering theory in general relativity started in the early 1980's with Dimock, Kay, then Bachelot and his students. The first studies were in spherically symmetric situations (Schwarzschild metric) and the spherical symmetry was crucially used in order to simplify the problem to a $1+1$ dimensional one and apply trace class perturbation methods. Then, under the impulse of Christian Gérard, people started using commutator methods and Mourre theory. This led to scattering theories on the Kerr metric (Häfner, Häfner-Nicolas).

  These theories were crucial in the rigorous analytic study of the Hawking effect by Bachelot, then Melnyk and Häfner.

  All these scattering theories had a common structure, a common formulation (whether in the construction or as a final re-formulation) in terms of wave operators.

- Scattering theory in terms of wave operators (Reed-Simon volume 3 for instance). H and H_0 two self-adjoint operators on two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$.

$$H \text{ full dynamics, associated equation : } \partial_t \Psi = iH\Psi \,,$$
$$H_0 \text{ simplified dynamics, associated equation : } \partial_t \Phi = iH_0\Phi \,.$$

  Complete scattering between the two :

$$\forall \Psi_0 \in \mathcal{H} \,,\ \exists ! \Phi_0^\pm \in \mathcal{H}_0 \,;\ \lim_{t\to\pm\infty} \| e^{itH}\Psi_0 - \mathcal{J} e^{itH_0}\Phi_0^\pm \|_{\mathcal{H}} = 0 \,,$$

  $\mathcal{J} \,:\, \mathcal{H}_0 \to \mathcal{H}$ identifying operator (could be a natural identification, or also involve some cut-off).

  Wave operators. $W^\pm \,:\, \Phi_0^\pm \mapsto \Psi_0$, $\tilde{W}^\pm \,:\, \Psi_0 \mapsto \Phi_0^\pm$. They are isometries, inverse of one another.

Meaning. Each solution of the full equation asymptotically approaches a solution to the simplified one and the data for the simplified evolution characterize the full solution completely.

Scattering data. The scattering data are the data for the simplified equation in the future and the past, i.e. $\Phi_0^\pm$. They are the remote past of future data for the field.

- Example : wave equation on an asymptotically flat static spacetime ($\mathcal{M} = \mathbb{R} \times \Sigma\,,\ g = \mathrm{d}t^2 - h$).

$$\partial_t^2 u - \Delta_h u = 0$$

  At least two natural choices for the comparison dynamics.

  1. Wave equation on flat spacetime : $\mathcal{H}_0 = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$;
  2. Damped transport along radial null geodesics in Minkowski spacetime : data for H_0 live on $\mathbb{R} \times S^2$ which is the space of outgoing (incoming) radial null geodesics, $\mathcal{H}_0 = \dot{H}^1(\mathbb{R}\,;\ L^2(S^2))$ and $\mathcal{J}$ will be a cut-off on the positive values of r.

The two choices are not equivalent, yet, provided the metric decays fast enough at infinity, we have a complete scattering for both choices.

3 The principles of conformal scattering

Explained on the conformal wave equation on 3 types of globally hyperbolic asymptotically flat spacetimes.

3.1 Explained on a conformally compact spacetime.

Spacetime $(\mathbb{R}^4, g)$ admitting a smooth conformal compactification (i^0 and $i^\pm$ smooth). The compactified spacetime is

$$\overline{\mathcal{M}} = \mathcal{M} \cup \mathscr{I}^\pm \cup \{i^\pm\} \cup \{i^0\}$$

$\mathscr{I}^\pm \simeq \mathbb{R} \times S^2$ are smooth null hypersurfaces, past and future lightcones of $i^\pm$. Rescaled metric $\hat{g} = \Omega^2 g$ smooth and non degenerate on $\overline{\mathcal{M}}$.
$u \in \mathcal{D}'(\mathbb{R}^4)$ solution to $\Box_g u + \frac{1}{6}\mathrm{Scal}_g u = 0 \Leftrightarrow \hat{u} := \Omega^{-1} u$ solution of $\Box_{\hat{g}} \hat{u} + \frac{1}{6}\mathrm{Scal}_{\hat{g}} \hat{u} = 0$.

Work entirely for the rescaled equation, on a picture. $\Sigma \simeq \mathbb{R}^3$ Cauchy hypersurface, $\partial\Sigma = i^0$. Smooth data on Σ, the solution $\hat{u}$ extends as a smooth function up to the boundary. The scattering data will be the restrictions of $\hat{u}$ to $\mathscr{I}^+$ and $\mathscr{I}^-$. They are the data for a Goursat (characteristic Cauchy) problem on future and past null infinities.
Method of construction of the scattering theory.

Step 1. Define trace operator $T^\pm$ that to smooth data on Σ associate $\Phi_0^\pm = \hat{u}|_{\mathscr{I}^\pm} \in \mathcal{C}^\infty(\mathscr{I}^\pm)$.

Step 2. Prove uniform energy equivalence between data on Σ and scattering data. This is done by chosing a timelike (causal) vector field τ on $\bar{\mathcal{M}}$ and using it to define an energy current $\tau^b {T_b}^a$ via the standard stress-energy tensor for the wave equation. Only approximate conservation but compact domain, energy equivalence is easy, simply the divergence theorem (Stokes theorem) on the closed hypersurface $\Sigma \cup \mathscr{I}^+$ (resp. $\Sigma \cup \mathscr{I}^-$) for J^a plus some a priori estimates.

- If τ is only causal (timelike in $\mathcal{M}$ though) and tangent to $\mathscr{I}$, then norm on $\mathscr{I}^+$ is a weighted $H^1(\mathbb{R}; L^2(S^2))$.
- If τ timelike on $\bar{\mathcal{M}}$, then the norm on $\mathscr{I}$ is a full homogeneous H^1 norm.

The norms on $\mathscr{I}$ in the two cases are not equivalent, but also on Σ (τ tangent to $\mathscr{I}$).

This implies. $T^\pm$ extend as bounded linear operators, one-to-one and closed range.

Step 3. $T^\pm$ are surjective. Need to solve the Goursat problem for a dense subspace of the finite energy space on $\mathscr{I}\pm$. Results on spatially compact spacetimes can be used here, modulo an extension on $\bar{\mathcal{M}}$ and $\hat{g}$ to a cylinder for example. See Hörmander 1990, JFA for instance.

Conclusion. $T^\pm$ are isomorphisms. Analogue to $\tilde{W}^\pm$ (inverse wave operators).

3.2 On an asymptotically simple spacetime containing energy

Explain the geometrical framework. Corvino-Schoen / Chrusciel-Delay spacetimes. We have a singular i^0 to deal with. This is done using essentially the finite propagation speed. In the versions developed sofar, an exact symmetry is assumed near i^0 in order to obtain easily energy estimates that are uniform in the size of the support. In the resolution of the Goursat problem, again the propagation speed will play an important role for obtaining that the solution has finite energy. **Explain the trick for i^0 on a picture.**

3.3 On a black hole spacetime

Now i^+ is singular and finite propagation speed cannot help us.

Here are the required ingredients. Explain on a picture of the compactified spacetime with spacelike foliation, and asymptotically hyperbolic foliation. Closed hypersurface using initial data surface, hyperboloïd in the distant future (time $T >> 1$), bits of the horizon and null infinity. Corresponding spacelike slice at time T.

- Uniform energy estimates both ways between initial slice and surfaces above. Can be obtained as follows : uniform energy equivalence between any spacelike slice ; energy equivalence between Σ_T and S_T which can be easy to obtain if we have timelike Killing vectors outside a compact part of the exterior (e.g. very slow Kerr). **In this part, no loss of derivative is allowed!**

- Some way of proving that the energy flux across the cap tends to zero and $T \to +\infty$. Typically decay properties such as Price's law give the result, but they are too strong, in particular, they require a precise knowledge of the trapping, i.e. the local geometry, whereas scattering should not be sensitive to this. **Here we can allow any loss of derivatives.**

- That's it! The rest goes as in the asymptotically simple case.

4 Where is the comparison dynamics?

- **No comparison dynamics in conformal scattering.**

- **However.** Scattering data are functions on $\mathscr{I}$. Chosing a congruence of outgoing null geodesics in a neighbourhood of $\mathscr{I}$ allows to define a comparison dynamics :

  - the flow of the congruence with parameter defined by a spacelike foliation of the physical spacetime.
  - the free wave equation on Minkowski spacetime, using the congruence of null geodesics to glue the radial null geodesics of Minkowski spacetime.

- Choice of comparison dynamics allows to re-interpret a conformal scattering construction as a scattering theory defined in terms of wave operators **without need for a stationary background**. **Draw a picture** with geodesics, cut-off, foliation, etc...

- The question then naturally arises of the equivalence between a conformal scattering theory and a scattering theory defined in terms of wave operators. Not true in general of course : some spectral scattering theories cannot be re-interpreted as conformal scattering (massive cases for instance). But when $\mathscr{I}$ is available and the equation considered is conformally invariant, the question is valid. Partially addressed in Mason-Nicolas 2002 and Nicolas 2013.

- The energy norm of the data has some freedom (it is really the choice of energy current vector, i.e. the choice of observer measuring the energy). It has some importance when trying to extract wave operators from a conformal scattering theory. The norm on $\mathscr{I}$ should be compatible with the comparison dynamics chosen. Precisely : flow of null geodesics is natural for an observer tangent to $\mathscr{I}$ (weak norm), free wave equation natural for an observer transverse to $\mathscr{I}$.

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