On the stability of the wave-map equation in Kerr spaces

Alexandru D. Ionescu

We are interested in the question of the global stability of a stationary axially-symmetric solution of the wave map equation in Kerr spaces of small angular momentum.

We consider the domain of outer communications of the Kerr spacetime $\mathcal{K}(M, a)$, $0 \leq a < M$, in standard Boyer–Lindquist coordinates,

$$\mathbf{g} = \mathbf{g}_{M,a} = -\frac{q^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{q^2}\Big(d\phi - \frac{2aMr}{\Sigma^2}dt\Big)^2 + \frac{q^2}{\Delta}(dr)^2 + q^2(d\theta)^2,$$

where

$$\begin{cases} \Delta = r^2 - 2Mr + a^2; \\ q^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)q^2 + 2Mra^2(\sin\theta)^2. \end{cases}$$

Let $\mathbf{T} := d/dt$ denote the stationary Killing vector-field of the spacetime, $\mathbf{Z} := d/d\phi$ the axial Killing vector-field.

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Let $\mathbf{T} := d/dt$ denote the stationary Killing vector-field of the spacetime, $\mathbf{Z} := d/d\phi$ the axial Killing vector-field.

Let

$$A + iB := \frac{\Sigma^2(\sin\theta)^2}{q^2}$$
$$- i\Big[2aM(3\cos\theta - (\cos\theta)^3) + \frac{2a^3M(\sin\theta)^4\cos\theta}{q^2}\Big],$$

denote the Ernst potential associated to the Killing vector-field $\mathbf{Z}$, where $A = \mathbf{g}(\mathbf{Z}, \mathbf{Z})$,

$$\mathbf{D}_\mu(A + iB) = \mathbf{Z}^\beta(\mathbf{D}_\mu\mathbf{Z}_\beta + i \in_{\mu\beta\gamma\delta} \mathbf{D}^\gamma\mathbf{Z}^\delta).$$

It is known that (A, B) verify the $\mathbb{H}^2$-valued Wave Map Equation

$$A\Box A = \mathbf{D}^\mu A\mathbf{D}_\mu A - \mathbf{D}^\mu B\mathbf{D}_\mu B,$$
$$A\Box B = 2\mathbf{D}^\mu A\mathbf{D}_\mu B,$$
$$\text{or } A\Box(A + iB) = \mathbf{D}_\mu(A + iB)\mathbf{D}^\mu(A + iB),$$

where $\Box = \Box_{\mathbf{g}}$ denotes the wave operator with respect to the metric.

Question (Global stability of the WM Equation) : The stationary solution $\Phi = (A, B) : \mathcal{K}(M, a) \to \mathbb{R}^2$ of the WM Equation is future asymptotically stable in the domain of outer communication of $\mathcal{K}(M, a)$, for all smooth, axially symmetric perturbations.

Global regularity in Euclidean spaces : Klainerman–Machedon, Tataru, Tao, Shatah–Struwe, Klainerman–Rodnianski, Sterbenz–Tataru, Krieger–Schlag, Tao.

Motivation : In the case of axially symmetric solutions of the Einstein vacuum equations, there is a link between the WM Equation and the Einstein vacuum equations. More precisely, assume $\mathbf{g}$ is a Lorentzian metric satisfying the Einstein vacuum equations

$$\mathbf{Ric}_{\mathbf{g}} = 0$$

in an open domain O, and V is a Killing vector-field for $\mathbf{g}$ in O.

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in an open domain O, and V is a Killing vector-field for $\mathbf{g}$ in O.

Then we consider the induced metric

$$h_{\alpha\beta} = X\mathbf{g}_{\alpha\beta} - V_\alpha V_\beta, \qquad \text{where } X = \mathbf{g}(V, V),$$

on a hypersurface Π passing through the point p and transversal to V. The metric h is nondegenerate (Lorentzian) as long as $X > 0$ in Π. The Einstein vacuum equations together with stationarity $\mathcal{L}_Y \mathbf{g} = 0$ are equivalent to the system of equations

$$^h\mathbf{Ric}_{ab} = \frac{1}{2X^2}(\nabla_a X \nabla_b X + \nabla_a Y \nabla_b Y),$$
$$^h\Box(X + iY) = \frac{1}{X} h^{ab}\partial_a(X + iY)\partial_b(X + iY),$$

in Π, where Y is the Ernst potential associated to Y,

$$\mathbf{D}_\mu Y = \in_{\mu\beta\gamma\delta} V^\beta \mathbf{D}^\gamma V^\delta.$$

This procedure is reversible : the metric $\mathbf{g}$ can be reconstructed if one is given h and $X + iY$ (up to gauge invariance). Therefore, the dynamical variable in the full Einstein vacuum equations in the axially symmetric case is the complex-valued solution $(X + iY)$ of the WM Equation.

Stability of the solution $A + iB$ associated to the axially symmetric vector-field $\mathbf{Z}$ is "consistent" with the full nonlinear stability of the Kerr family of solutions, in the case of axially-symmetric perturbations.

Main nonlinear stability question : global stability of the Kerr family with small angular momentum, in the case of small axially symmetric perturbations (one can further simplify to the polarized case, no angular momentum, $a = 0$, $B = 0$).

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The $\mathbb{H}^2$-valued WM Equation

$$A\Box A = \mathbf{D}^\mu A \mathbf{D}_\mu A - \mathbf{D}^\mu B \mathbf{D}_\mu B,$$
$$A\Box B = 2\mathbf{D}^\mu A \mathbf{D}_\mu B,$$

where $\Box = \Box_{\mathbf{g}_{M,a}}$ denotes the wave operator with respect to the fixed Kerr metric $\mathbf{g}_{M,a}$, is a partial linearization of the axially symmetric Einstein vacuum equations. Other linearizations have been studied : the wave equation, Maxwell equations, linearization of the null structure equations, in Schwazschild spaces and in Kerr spaces.

Kay–Wald, Blue–Soffer, Blue–Sterbenz, Finster–Kamran–Smoller–Yau, Dafermos–Rodnianski, Marzuola–Metcalfe–Tataru–Tohaneanu, Tataru–Tohaneanu, Andersson–Blue, Luk, Sterbenz–Tataru, Dafermos–Rodnianski–Shlapentokh-Rothman, Dafermos–Holzegel–Rodnianski.

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We are looking for solutions (A', B') of the WM equation, of the form $(A', B') = (A, B) + (\varepsilon A\phi, \varepsilon A\psi)$, where ϕ and ψ are real-valued **Z**-invariant functions. Simple calculations show that the functions (ϕ, ψ) have to satisfy a system of the form

$$\Box\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu A}{A^2}\psi = \varepsilon\mathcal{N}_\phi^\varepsilon,$$
$$\Box\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\psi = \varepsilon\mathcal{N}_\psi^\varepsilon,$$

where $\mathcal{N}_\phi^\varepsilon$ and $\mathcal{N}_\psi^\varepsilon$ are nonlinearities that can be calculated explicitly.

The nonlinearities are

$$\mathcal{N}_\phi^\varepsilon = \frac{A^2\mathbf{D}^\mu\phi\mathbf{D}_\mu\phi - A^2\mathbf{D}^\mu\psi\mathbf{D}_\mu\psi - 2A\psi\mathbf{D}^\mu A\mathbf{D}_\mu\psi}{A^2(1+\varepsilon\phi)}$$
$$+ \frac{\mathbf{D}^\mu B\mathbf{D}_\mu B\phi^2 - \mathbf{D}^\mu A\mathbf{D}_\mu A\psi^2}{A^2(1+\varepsilon\phi)}$$
$$+ \frac{\phi}{A^2(1+\varepsilon\phi)}[2A\mathbf{D}^\mu B\mathbf{D}_\mu\psi - 2\mathbf{D}^\mu B\mathbf{D}_\mu B\phi + 2\mathbf{D}^\mu B\mathbf{D}_\mu A\psi],$$

and

$$\mathcal{N}_\psi^\varepsilon = \frac{2A^2\mathbf{D}^\mu\phi\mathbf{D}_\mu\psi + (\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B)\phi\psi + 2A\psi\mathbf{D}^\mu A\mathbf{D}_\mu\phi}{A^2(1+\varepsilon\phi)}$$
$$- \frac{\phi}{A^2(1+\varepsilon\phi)}[2A\mathbf{D}^\mu B\mathbf{D}_\mu\phi + (\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B)\psi].$$

These nonlinearities are well-defined only if ψ vanishes on the axis. They are semilinear and have standard null structure.

In the Schwarzschild case $a = 0$, $B = 0$, the linearized system is

$$\Box\phi = 0,$$
$$\Box\psi - \left(\frac{4}{r^2(\sin\theta)^2} - \frac{8M}{r^3}\right)\psi = 0.$$

To study the system in the general case we need, at least :

- a "good" notion of energy ;
- good function spaces.

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Theorem (I.–Klainerman) : Assume (ϕ, ψ) is a $\mathbf{Z}$-invariant C^1_{loc} solution of the system of linear equations

$$\Box\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B \mathbf{D}_\mu A}{A^2}\psi = 0,$$

$$\Box\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A \mathbf{D}_\mu A + \mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2}\psi = 0,$$

in an open set $\mathcal{O} \subseteq \mathcal{K}(M, a)$. Assume, in addition that ψ vanishes on the axis. Then the solution $\Psi = (\phi, \psi)$ admits a quadratic energy-momentum tensor $Q_{\mu\nu}$ such that

(a) $Q(X, Y) > 0$ for any future-oriented timelike vector-fields X, Y;

(b) $\mathbf{D}^\mu Q_{\mu\nu} = J_\nu$;

(c)$\mathbf{T}^\nu J_\nu = 0$;

(d)$Q(\mathbf{Z}, X) = 0$ for any vector-field X that satisfies $\mathbf{g}(\mathbf{Z}, X) = 0$.

Let

$$
\begin{aligned}
E_\mu &:= \mathbf{D}_\mu \phi + \psi A^{-1} \mathbf{D}_\mu B, \\
F_\mu &:= \mathbf{D}_\mu \psi - \phi A^{-1} \mathbf{D}_\mu B, \\
M_\mu &:= \frac{\phi \mathbf{D}_\mu B - \psi \mathbf{D}_\mu A}{A}, \\
Q_{\mu\nu} &:= \left(E_\mu E_\nu - \frac{1}{2} \mathbf{g}_{\mu\nu} E_\alpha E^\alpha \right) + \left(F_\mu F_\nu - \frac{1}{2} \mathbf{g}_{\mu\nu} F_\alpha F^\alpha \right) \\
&\quad + \left(M_\mu M_\nu - \frac{1}{2} \mathbf{g}_{\mu\nu} M_\alpha M^\alpha \right).
\end{aligned}
$$

Then

$$
\mathbf{D}^\mu Q_{\mu\nu} =: J_\nu = \frac{2\mathbf{D}_\nu B M^\mu E_\mu - 2\mathbf{D}_\nu A M^\mu F_\mu}{A}.
$$

New system of coordinates : we fix first a smooth function $\chi : \mathbb{R} \to [0,1]$ supported in the interval $(-\infty, 5M/2]$ and equal to 1 in the interval $(-\infty, 9M/4]$, and define $g_1, g_2 : (r_h, \infty) \to \mathbb{R}$ such that

$$g_1'(r) = \chi(r)\frac{2Mr}{\Delta}, \qquad g_2'(r) = \chi(r)\frac{a}{\Delta}.$$

We define the functions

$$u_+ := t + g_1(r), \qquad \phi_+ := \phi + g_2(r).$$

The metric is smooth in this system of coordinates beyond the horizon

Function space : $(\phi, \psi) \in \mathbf{H}^1(\Sigma_t^c)$ is $\phi, \psi \in H^1$ and $\psi/(\sin\theta) \in L^2$.

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Let

$$L := \chi_{\geq 4M}(r)\Big(\partial_r + \frac{r}{r-2M}\partial_t\Big),$$

For any $t \in \mathbb{R}$ and $(\phi, \psi) \in \mathbf{H}^1(\Sigma_t^c)$ we define the *outgoing energy density* $(e(\phi), e(\psi))$,

$$e(\phi)^2 := \frac{(\partial_\theta \phi)^2}{r^2} + (L\phi)^2 + \frac{M^2\left[(\partial_r\phi)^2 + (\partial_t\phi)^2\right]}{r^2} + \frac{\phi^2}{r^2},$$

$$e(\psi)^2 := \frac{(\partial_\theta \psi)^2 + \psi^2(\sin\theta)^{-2}}{r^2} + (L\psi)^2 + \frac{M^2\left[(\partial_r\psi)^2 + (\partial_t\psi)^2\right]}{r^2} + \frac{\psi^2}{r^2}.$$

We work in the axially symmetric case, therefore the relevant trapped null geodesics are still confined to a codimension 1 set. Assuming that $a \ll M$, it is easy to see that the equation

$$r^3 - 3Mr^2 + a^2 r + Ma^2 = 0$$

has a unique solution $r^* \in (c_0, \infty)$. Moreover, $r^* \in [3M - a^2/M, 3M]$.

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Main Theorem (I.–Klainerman) : Assume that $M \in (0, \infty)$, $a \in [0, \overline{\varepsilon} M]$ and $c_0 \in [r_h - \overline{\varepsilon} M, r_h]$, where $\overline{\varepsilon} \in (0, 1]$ is a sufficiently small constant. Assume that $T \geq 0$, and $(\phi, \psi) \in C^k([0, T] : \mathbf{H}^{1-k}(\Sigma_t^{c_0}))$, $k \in [0, 1]$, is a solution of the system

$$\Box\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B \mathbf{D}_\mu A}{A^2}\psi = \mathcal{N}_\phi,$$
$$\Box\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A \mathbf{D}_\mu A + \mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2}\psi = \mathcal{N}_\psi,$$

satisfying $\mathbf{Z}(\phi, \psi) = 0$. Then, for any $\alpha \in (0, 2)$ and any $t_1 \leq t_2 \in [0, T]$,

$$\begin{aligned}
&\mathcal{B}_\alpha^{c_0}(t_1, t_2) + \int_{\Sigma_{t_2}^{c_0}} \frac{r^\alpha}{M^\alpha}\left[e(\phi)^2 + e(\psi)^2\right] d\mu_t \\
&\leq \overline{C}_\alpha \int_{\Sigma_{t_1}^{c_0}} \frac{r^\alpha}{M^\alpha}\left[e(\phi)^2 + e(\psi)^2\right] d\mu_t \\
&+ \overline{C}_\alpha \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \left[|\mathcal{N}_\phi| + |\mathcal{N}_\psi|\right] \cdot \frac{r^\alpha}{M^\alpha}\left[e(\phi)^2 + e(\psi)^2\right]^{1/2} d\mu,
\end{aligned}$$

where $\overline{C}_\alpha$ is a large constant that may depend on α, and

$$\mathcal{B}_\alpha^{c_0}(t_1,t_2) := \int_{\mathcal{D}^{c_0}_{[t_1,t_2]}} \frac{r^\alpha}{M^\alpha}\Big\{ \frac{(r-r^*)^2}{r^3}\frac{|\partial_\theta\phi|^2+|\partial_\theta\psi|^2+\psi^2(\sin\theta)^{-2}}{r^2} + \frac{1}{r}\big[(L\phi)^2+(L\psi)^2\big] + \frac{1}{r^3}\big(\phi^2+\psi^2\big) + \frac{M^2}{r^3}\big[(\partial_r\phi)^2+(\partial_r\psi)^2\big] + \frac{M^2(r-r^*)^2}{r^5}\big[(\partial_t\phi)^2+(\partial_t\psi)^2\big]\Big\}\,d\mu.$$

- The method of simultaneous estimates of Marzuola–Metcalfe–Tataru–Tohaneanu ;

- The r-weighted estimates along null hypersurfaces of Dafermos–Rodnianski ;

- The main point is to get simultaneous pointwise decay ; the outgoing energies decay at rate almost $t^{-(2-\alpha)}$;

- We use energy estimates. The main new issue is the presence of the J-term in the identity $\mathbf{D}^\mu Q_{\mu\nu} = J_\nu$.

Corollary. Assume that $N_1 = 8$ and $(\phi, \psi) \in C^k([0, T] : \mathbf{H}^{N_1-k}(\Sigma_t^{c_0}))$, $k \in [0, N_1]$, is a solution of the wave-map system with $\mathcal{N}_\phi = \mathcal{N}_\psi = 0$. Then, for any $t \in [0, T]$ and $\beta < 2$,

$$\int_{\Sigma_t^{c_0}} \left[e(\phi)^2 + e(\psi)^2\right] d\mu_t \lesssim_\beta (1 + t/M)^{-\beta}$$
$$\sum_{k=0}^{4} M^{2k} \int_{\Sigma_0^{c_0}} \frac{r^2}{M^2} \left[e(\mathbf{T}^k\phi)^2 + e(\mathbf{T}^k\psi)^2\right] d\mu_t.$$

For simplicity, we consider only the equation for ψ in the Schwarzschild case $a = 0$, $B = 0$, $A = r^2(\sin\theta)^2$. The equation is

$$\Box\psi - \Big(\frac{4}{r^2(\sin\theta)^2} - \frac{8M}{r^3}\Big)\psi = 0.$$

Let

$$F_\mu := \mathbf{D}_\mu\psi, \quad M_\mu := \frac{-\psi\mathbf{D}_\mu A}{A},$$
$$Q_{\mu\nu} := F_\mu F_\nu + M_\mu M_\nu - \frac{1}{2}\mathbf{g}_{\mu\nu}(F_\alpha F^\alpha + M_\alpha M^\alpha).$$

For suitable triplets (X, w, m') we define

$$\widetilde{P}_\mu = \widetilde{P}_\mu[X, w, m'] := Q_{\mu\nu}X^\nu + \frac{w}{2}\psi F_\mu$$
$$- \frac{\psi^2}{4}\mathbf{D}_\mu w + \frac{\psi^2}{4}m'_\mu - \frac{X^\nu\mathbf{D}_\nu A}{A}\frac{\mathbf{D}_\mu A}{A}\psi^2.$$

Notice the correction $-\frac{X^\nu\mathbf{D}_\nu A}{A}\frac{\mathbf{D}_\mu A}{A}\psi^2$, which is needed to partially compensate for the source term J.

Then we have the divergence identity

$$2\mathbf{D}^{\mu}\widetilde{P}_{\mu} = \sum_{j=1}^{4} L^{j},$$

where

$$\begin{aligned}
L^1 = L^1[X, w, m'] &:= Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + w(F_\alpha F^\alpha + M_\alpha M^\alpha),\\
L^2 = L^2[X, w, m'] &:= \psi m'^{\mu}\mathbf{D}_\mu\psi,\\
L^3 = L^3[X, w, m'] &:= \frac{1}{2}\psi^2(\mathbf{D}^\mu m'_\mu - \Box w),\\
L^4 = L^4[X, w, m'] &:= -2\mathbf{D}^\mu\Big[\frac{X^\nu\mathbf{D}_\nu A}{A}\frac{\mathbf{D}_\mu A}{A}\Big]\psi^2.
\end{aligned}$$

The divergence identity gives

$$\int_{\Sigma^c_{t_1}} \widetilde{P}_\mu n_0^\mu \, d\mu_{t_1} = \int_{\Sigma^c_{t_2}} \widetilde{P}_\mu n_0^\mu \, d\mu_{t_2} + \int_{\mathcal{N}^c_{[t_1,t_2]}} \widetilde{P}_\mu k_0^\mu \, d\mu_c + \int_{\mathcal{D}^c_{[t_1,t_2]}} \mathbf{D}^\mu \widetilde{P}_\mu \, d\mu,$$

where $t_1, t_2 \in [0, T]$, $c \in (c_0, 2M]$, and the integration is with respect to the natural measures induced by the metric $\mathbf{g}$. To prove the main theorem we need to choose a suitable multiplier triplet (X, w, m') in a such a way that all the terms in the identity above are nonnegative.

We use four multipliers $(X_{(j)}, w_{(j)}, m'_{(j)})$:

• a multiplier in the trapped region around $r = r^*$;

• a multiplier in the region near the horizon $r = r_h$, using also the redshift vector-field of Dafermos–Rodnianski ;

• a new multiplier close to the trapped region $r \in [r^*, 4M]$, to deal with the extra term in the divergence identity ;

• a new multiplier at ∞, with a vector-field of the form

$$f \partial_r + (f + g)\partial_t,$$

where g is very large for small values of r (to make the surface integrals positive) but small at ∞ to preserve the character of outgoing energies.

This gives a good Morawetz estimate. In principle, one could use this to study the global regularity for the full semilinear problem

$$\begin{aligned}\Box\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu A}{A^2}\psi &= \varepsilon\mathcal{N}^\varepsilon_\phi,\\ \Box\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\psi &= \varepsilon\mathcal{N}^\varepsilon_\psi.\end{aligned}$$

Work in progress of John Stogin.

Main nonlinear stability question : global stability of the Kerr family with small angular momentum, in the case of small axially symmetric perturbations.

- Several simplifications are possible :

(1) consider only the polarized case, (no angular momentum, $a = 0$, $B = 0$) ;
(2) construct the solution on quadratic time $c\varepsilon^{-2}$.

- Several linearizations have been studied and are well understood.

This gives a good Morawetz estimate. In principle, one could use this to study the global regularity for the full semilinear problem

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$$\Box\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A \mathbf{D}_\mu A + \mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2}\psi = \varepsilon\mathcal{N}^\varepsilon_\psi.$$

Work in progress of John Stogin.

Main nonlinear stability question : global stability of the Kerr family with small angular momentum, in the case of small axially symmetric perturbations.

- Several simplifications are possible :

(1) consider only the polarized case, (no angular momentum, $a = 0,\ B = 0$) ;
(2) construct the solution on quadratic time $c\varepsilon^{-2}$.

- Several linearizations have been studied and are well understood.

- The rigidity (uniqueness) problem for stationary solutions is well understood (Carter–Robinson, Hawking, Chrusciel–Costa, Alexakis-I.-Klainerman).

- **Theorem** (Alexakis-I.-Klainerman) : Assume $(\mathbf{g}, \mathbf{T})$ is a regular stationary solution of the Einstein-vacuum equations, which is "close" (smallness of the Mars–Simon tensor $\mathcal{S} = \mathcal{S}((\mathbf{g}, \mathbf{T}))$ to a Kerr solution. Then $(\mathbf{g}, \mathbf{T})$ coincides with that Kerr solution.

- **Asymptotic uniqueness question :** Assume $(\mathbf{g}, \mathbf{T})$ is a regular asymptotically-stationary solution of the Einstein-vaccum equations, in the sense that $\mathcal{L}_{\mathbf{T}}\mathbf{g} \to 0$ as $t \to \infty$ at a suitable rate. If $(\mathbf{g}, \mathbf{T})$ is "close" to a Kerr solution, then it converges to a Kerr solution.