Solutions of the Constraint Equations with Conformal Killing Fields

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October 27, 2015

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II Describe two seemingly unrelated difficulties with the conformal method:

1. Nonexistence/nonuniqueness phenomena for (certain) non-CMC conformal data.
2. Obstructions caused by conformal Killing fields for near-CMC solutions.

III Description of a variation of the conformal method [M '14c] that addresses at least one facet of 1 and nearly completely, if not fully, addresses 2. [HMM '15]

Part I

The Conformal Method

Einstein Constraint Equations

Initial data for the Cauchy problem:

- Riemannian manifold (M^3, g_{ab})
- Second fundamental form K_{ab} (i.e. a symmetric $(0,2)$-tensor)

$$
\begin{aligned}
R - |K|^2 + \operatorname{tr} K^2 &= 2\rho + 2\Lambda && \text{[Hamiltonian constraint]} \\
-\operatorname{div}(K - \operatorname{tr} K\, g) &= j && \text{[momentum constraint]}
\end{aligned}
$$

The energy density ρ and momentum density j_a are functions of g_{ab} and the matter fields. The cosmological constant Λ is a constant.

Structure of set of solutions of the constraints

Regarding the set of pairs (g_{ab}, K_{ab}) solving the constraints:

Fischer, Marsen, Moncrief 1980

The vacuum solutions of the constraints on a compact manifold are, when given a suitable topology, a Banach manifold away from the solutions of the constraints that generate spacetimes with Killing fields.

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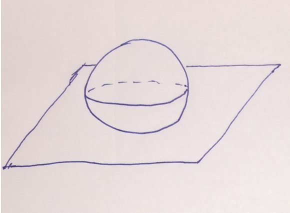
Helpful Parable



$S^2 \subseteq \mathbb{R}^3$

Functions x and y on $\mathbb{R}^3$ yield (tentative) coordinates on S^2

Coordinate Functions of the Conformal Method

$$\begin{tikzcd} & \mathbf{g} = [g] \\ (g, K) \arrow[ur] & \end{tikzcd}$$

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Conformal Momentum

A symmetric tensor σ_{ab} is **transverse traceless** with respect to g_{ab} if

1. $g^{ab}\sigma_{ab} = 0 \qquad$ (traceless)
2. $\nabla^a \sigma_{ab} = 0 \qquad$ (transverse)

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If $\phi > 0$ is a conformal factor:

$$\sigma_{ab} \quad \text{is TT w.r.t} \quad g_{ab} \quad \Longleftrightarrow \quad \phi^{-2}\sigma_{ab} \quad \text{is TT w.r.t} \quad \phi^4 g_{ab}$$

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Definition

Let $\mathbf{g}$ be a conformal class. A **conformal momentum** at $\mathbf{g}$ is an equivalence class of pairs (g_{ab}, σ_{ab}) where $g_{ab} \in \mathbf{g}$, σ_{ab} is transverse traceless with respect to g_{ab}, and where

$$(\phi^4 g_{ab}, \phi^{-2}\sigma_{ab}) \sim (g_{ab}, \sigma_{ab})$$

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A conformal momentum $\boldsymbol{\sigma}$ is a cotangent vector to conformal classes (modulo diffeomorphisms).

Measurement of Conformal Momentum

Although (g, K) determines unambiguously a conformal class and a mean curvature, it does not determine unambiguously a conformal momentum. We must make a gauge choice, which we can represent by the choice of a volume form α.

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For each gauge choice α we obtain a candidate chart Φ_α

$$(g, K) \xrightarrow{\Phi_\alpha} (\mathbf{g}, \boldsymbol{\sigma}, \tau).$$

The Gauge Choice

The volume form α is related to the lapse N used when evolving the data. In effect, we will not prescribe the lapse in advance, but will determine it from α and the current (spatial) metric according to

$$N = \frac{\omega_g}{\alpha}$$

where ω_g is the volume form of g.

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Modifying the ADM Lagrangian to use α instead of N, one can compute that $\boldsymbol{\sigma}$ is the momentum of (g, K) conjugate to the conformal class.

The Conformal Method

Fix a volume gauge α. Let $(\mathbf{g}, \boldsymbol{\sigma}, \tau)$ be desired conformal coordinates. Find all solutions (g, K) of the vacuum constraint equations with

$$\Phi_\alpha(g, K) = (\mathbf{g}, \boldsymbol{\sigma}, \tau).$$

Ideally there will be one and only one.

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- Same results if τ is near constant, has no zeros, **and** if $\mathbf{g}$ has no conformal Killing fields. [Isenberg, Moncrief, Clausen, Allen, O'Murchadha, M]
- If $Y_{\mathbf{g}} > 0$, τ is arbitrary, and if $\boldsymbol{\sigma} \neq \mathbf{0}$ is sufficiently small there is at least one solution. [Holst-Nagy-Tsogtergel, M]

The Conformal Method as a PDE

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Find ϕ and W^a solving

$$-\Delta\phi + R\phi - \left|\sigma + \frac{1}{N}\mathcal{D}W\right|^2 \phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 2\Lambda\phi^5 + 2\rho(\phi)$$

$$\mathcal{D}^*\left[\frac{1}{N}\mathcal{D}W\right] + \frac{2}{3}\phi^6\, d\tau = j(\phi).$$

where

$$(\mathcal{D}W)_{ab} = \frac{1}{2}\left[\nabla_a W_b + \nabla_b W_a\right] - \frac{1}{3}\nabla^c W_c\, h_{ab}$$

and

$$\mathcal{D}^* = -\operatorname{div}.$$

The Conformal Method as a PDE (II)

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If a solution is found,

$$g = \phi^4 h$$

$$K_{ab} = \phi^{-2}\left[\sigma_{ab} + \frac{1}{N}(\mathcal{D}W)_{ab}\right] + \frac{\tau}{3}g_{ab}.$$

satisfies the constraints and

$$\Phi_\alpha(g, K) = (\mathbf{g}, \boldsymbol{\sigma}, \tau).$$

All such solutions can be found this way.

Part II

Difficulties With The Conformal Method

Issue # 1: Conformal Killing Fields

Suppose (g_{ab}, K_{ab}) solves the vacuum momentum constraint:

$$\nabla^a (K_{ab} - \tau g_{ab}) = 0.$$

Let Q^a be a conformal Killing field.

$$\begin{aligned} 0 &= \int \nabla^a (K_{ab} - \tau g_{ab}) Q^b \ \omega_g \\ &= -\int (K_{ab} - \tau g_{ab}) \nabla^a Q^b \ \omega_g \\ &= -\int (K_{ab} - \tau g_{ab}) \frac{1}{3} \nabla_c Q^c g^{ab} \ \omega_g \\ &= \int \frac{2}{3} \tau \nabla_c Q^c \ \omega_g. \end{aligned}$$

Hence

$$\int Q(\tau) \ \omega_g = 0.$$

Conformal Killing Field Compatability

The compatability condition

$$\int Q(\tau)\ \omega_g = 0$$

reads

$$\int \phi^6 Q(\tau)\ \omega_h = 0$$

with respect to h_{ab}. Is the conformal method always clever enough to pick a ϕ satisfying this condition?

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Potential obstruction appears during construction:

$$\mathcal{D}^* \left[\frac{1}{N} \mathcal{D} W_{k+1} \right] + \phi_k^6\ d\tau = 0.$$

Conformal Killing Field Compatability (II)

The condition

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Theorem (Host, M, Mazzeo '15)

There exist Yamabe positive CMC conformal data $(\mathbf{g}, \boldsymbol{\sigma}, \tau_0)$ with $\tau_0 \neq 0$ and a perturbation $\dot{\tau}$ such that there exists a solution of the vacuum constraint equations with conformal coordinates

$$(\mathbf{g}, \boldsymbol{\sigma}, \tau_0 + \epsilon\dot{\tau})$$

if and only if $\epsilon = 0$.

Issue #2: Nonexistence/Nonuniqueness

$$
\begin{aligned}
-\Delta\phi + R\phi - \left|\sigma + \frac{1}{N}\mathcal{D}W\right|^2 \phi^{-7} + \frac{2}{3}\tau^2\phi^5 &= 0 \\
\mathcal{D}^*\left[\frac{1}{N}\mathcal{D}W\right] + \frac{2}{3}\phi^6\, d\tau &= 0.
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Problematic term in the system is the ϕ^5 term; this is the critical exponent coming from the continuous but non-compact embedding of H^1 into L^6.

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The term $\tau^2\phi^5$ has a good sign. But $\mathcal{D}W \sim \phi^6$, so

$$
\left|\sigma + \frac{1}{N}\mathcal{D}W\right|^2 \phi^{-7} \sim \phi^{12}\phi^{-7} = \phi^5.
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This yields a term that is effectively ϕ^5, but with the unfavorable sign.

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But τ is involved everywhere in this battle. Maybe we get lucky?

Enforcing a bad sign...

What if we introduce a bad sign via $\Lambda > 0$? Key term is

$$\left(\frac{2}{3}\tau^2 - 2\Lambda\right)\phi^5.$$

Theorem (Premoselli '14)

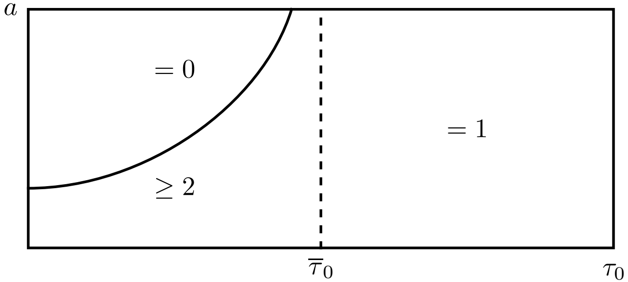
Consider CMC conformal coordinates $(\mathbf{g}, a\boldsymbol{\sigma}_0, \tau_0)$ *where* $\mathbf{g}$ *is Yamabe positive,* $\boldsymbol{\sigma} \neq \mathbf{0}$, $a > 0$, *and where*

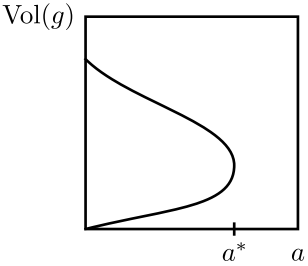
$$\frac{2}{3}\tau_0^2 - 2\Lambda < 0$$

There is an $a_* > 0$ *such that*

- *If* $a > a_*$ *there are no solutions.*
- *If* $a = a_*$ *there is exactly one solution.*
- *If* $0 < a < a_*$ *there are at least two solutions.*

Premoselli, schematically





Nonexistence/Nonuniqueness from Coupling

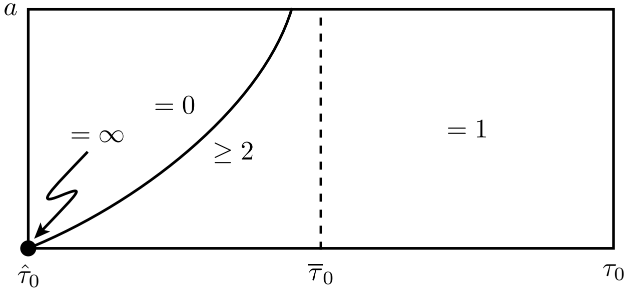
In fact, folds were observed in [M '11] in the **vacuum** setting with $\Lambda = 0$. The culprit is the effective leakage of a term with a bad sign on ϕ^5 from the coupled system.

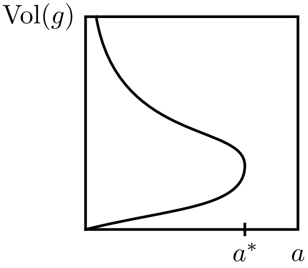
The conformal data involved a certain collection $(\mathbf{g}, a\boldsymbol{\sigma}_0, \tau)$ where $Y_{\mathbf{g}} = 0$, and

$$\tau = \tau_0 + \xi$$

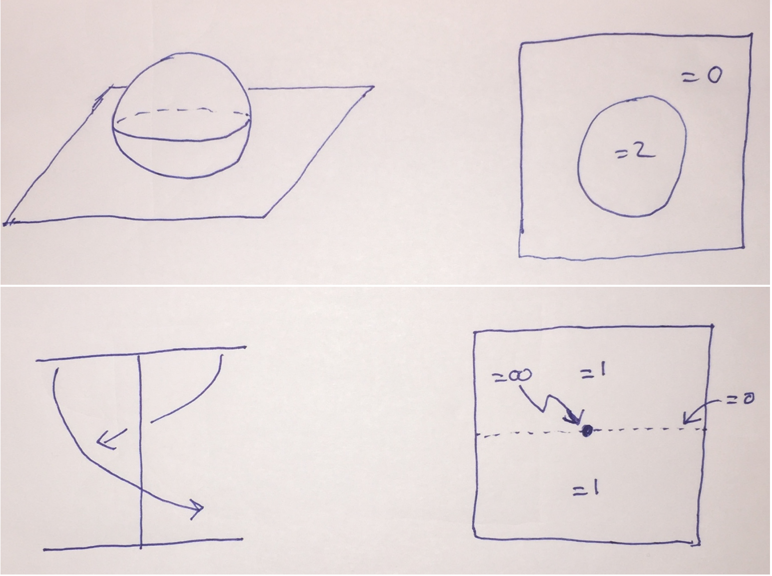
ξ is a particular fixed function and τ_0 is an adjustable constant.

Nonexistence/Nonuniqueness from Coupling





Folds and Spikes



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Very recently, fold-like behaviour for the **vacuum** constraints with $\Lambda = 0$:

Theorem (Nguyen '15)

There exist conformal parameters $(\mathbf{g}, a\boldsymbol{\sigma}, \tau)$ with $Y_{\mathbf{g}} > 0$ such that

- *There are no solutions of the vacuum constraints if a is large enough.*
- *There is a sequence $a_k \to 0$ such that for each $(\mathbf{g}, a_k\boldsymbol{\sigma}, \tau)$ there are at least two solutions of the constraints.*

Spikes

In [M '11] and [M '14b] the spikes appear for a conformal class with $Y_{\mathbf{g}} = 0$ and when $\boldsymbol{\sigma} = \mathbf{0}$ and $\tau_* = 0$ where

$$\tau_* = \frac{\int N_g \tau \, \omega_g}{\int N_g \omega_g}$$

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The CMC spike for $Y_{\mathbf{g}} = 0$, $\boldsymbol{\sigma} = \mathbf{0}$, $\tau_0 = 0$ is well known. The slices have $K_{ab} \equiv 0$ and the resulting family of spacetimes are all homothetically related.

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The non-CMC versions of the spikes are more complicated. It is no longer true that $K_{ab} \equiv 0$, and the generated spacetimes are not homothetically related.

The Trouble with τ_*

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Given a gauge choice α and coords $(\mathbf{g}, \boldsymbol{\sigma}, \tau)$, can you tell if a spike will be generated?

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With respect to $h \in \mathbf{g}$,

$$\tau_* = \frac{\int \phi^{12} N_h \tau \, \omega_h}{\int \phi^{12} N_h \omega_h}$$

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If τ doesn't change sign, you can be sure that $\tau^* \neq 0$. But otherwise?

Part III

Alternative to the Conformal Method

Volumetric Momentum

The quantity τ_* has an interpretation very similar to that of $\boldsymbol{\sigma}$:

$$-\frac{4}{3}\tau_*$$

is the momentum conjugate to the volume form ω_g, modulo diffeomorphisms, as measured by α [M '14c].

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$$\begin{aligned} \boldsymbol{\sigma} &\longleftrightarrow \mathbf{g} \\ -\frac{4}{3}\tau_* &\longleftrightarrow \omega_g \end{aligned}$$

This suggests finding variations of the conformal method where $\boldsymbol{\sigma}$ and τ_* are both explicit parameters.

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Volume is fixed.

But volume can be reallocated relative to the conformal class.

Represent the infinitesimal reallocation of volume using a vector field X^a.

If $\operatorname{div} X = 0$, there is no reallocation.

If $\mathcal{D}X = 0$, there is no reallocation.

Drifts (II)

Definition

A **drift** at a metric g_{ab} is a vector field, modulo $\operatorname{Ker}\operatorname{div}_g$ and modulo $\operatorname{Ker}\mathcal{D}_g$.

Central Anzatz

Given (g_{ab}, K_{ab}) decompose the second fundamental form as follows.

$$K_{ab} = A_{ab} + \frac{\tau}{n} g_{ab}$$

with $g^{ab} A_{ab} = 0$

$$A_{ab} = \sigma_{ab} + \frac{1}{N} (\mathcal{D} W)_{ab}$$

$$\tau = \tau_* + \frac{1}{N} \operatorname{div} V$$

Each of W^a and V^a represent a drift.

Momentum constraint for drift variables

Momentum constraint:

$$-\mathcal{D}^* \left(\frac{1}{N} \mathcal{D} W \right) = d \left(\frac{1}{N} \operatorname{div} V \right)$$

Idea: specify V^a, and solve for W^a. But conformal Killing fields are an obstacle:

$$\int \frac{1}{N} \operatorname{div} V \operatorname{div} R \; \omega_g = 0$$

for all conformal Killing fields R^a.

Momentum constraint for drift variables (II)

Theorem (M '14c)

Let V^a be given. There is a conformal Killing field Q^a and a vector field W^a such that

$$-\mathcal{D}^* \left(\frac{1}{N} \mathcal{D} W \right) = d \left(\frac{1}{N} \operatorname{div}(V + Q) \right)$$

Moreover, Q^a is unique up to addition of a true Killing field, and W^a is unique up to addition of a conformal Killing field.

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Moreover, Q^a is unique up to addition of a true Killing field, and W^a is unique up to addition of a conformal Killing field.

The data V^a really represents a drift. Adding a divergence free vector field won't change W^a, nor will adding a conformal Killing field.

Momentum constraint for drift variables (III)

Theorem (M '14c)

Let W^a be given. There is a divergence free vector field E^a and a vector field V^a such that

$$-\mathcal{D}^*\left(\frac{1}{N}\mathcal{D}(W+E)\right) = d\left(\frac{1}{N}\operatorname{div} V\right)$$

Moreover, E^a is unique up to addition of a true Killing field, and V^a is unique up to addition of a divergence-free vector field.

The data W^a really represents a drift. Adding a conformal Killing field won't change V^a, nor will adding a divergence-free vector field.

One drift formulation of constraint equations

Gauge: N

Data: $(h_{ab}, \sigma_{ab}, \tau_*, V^a)$

Find ϕ, W^a and a conformal Killing field Q^a solving

$$-\Delta\phi + R\phi - \left|\sigma + \frac{1}{N}\mathcal{D}W\right|^2 \phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 2\Lambda\phi^5 + 2\rho(\phi)$$

$$\mathcal{D}^*\left[\frac{1}{N}\mathcal{D}W\right] + \frac{2}{3}\phi^6\, d\tau = j(\phi).$$

where

$$\tau = \tau_* + \phi^{-12}\frac{1}{N}\operatorname{div}(\phi^6(V + Q))$$

Near-CMC Existence

Equations to solve for (ϕ, W^a, Q^a):

$$-\Delta\phi + R\phi - \left|\sigma + \frac{1}{N}\mathcal{D}W\right|^2 \phi^{-7} + \frac{2}{3}\tau^2\phi^5 = 2\Lambda\phi^5$$

$$\mathcal{D}^*\left[\frac{1}{N}\mathcal{D}W\right] + \frac{2}{3}\phi^6\, d\tau = 0.$$

with

$$\tau = \tau_* + \phi^{-12}\frac{1}{N}\,\mathrm{div}(\phi^6(V + Q))$$

Near-CMC Existence

Theorem (HMM '15)

Fix N. Suppose $(h_{ab}, \hat{\sigma}_{ab}, \hat{\tau}_*, \hat{V}^a \equiv 0)$ lead to a CMC solution of the constraints. Suppose also that all of the following hold:

1. *The CMC solution metric does not admit any true Killing fields.*
2. $(2/3)\tau_*^2 \geq 2\Lambda$.
3. *Either $(2/3)\tau_*^2 > 2\Lambda$, or $\hat{\sigma}_{ab} \not\equiv 0$*

Then there exists $\epsilon > 0$ such that all conformal data $(h_{ab}, \sigma_{ab}, \tau_*, V^a)$ satisfying

$$\|\sigma - \hat{\sigma}\|_{W^{2,p}} + |\tau_* - \hat{\tau}_*| + \|V\|_{W^{2,p}} < \epsilon$$

generate a unique solution of system of the drift-parameterized constraint equation in a neighbourhood of the CMC solution.

Some difficulties and questions

- The PDEs are more technical than the standard conformal method. Lichnerowiz equation is no longer semilinear. Second derivatives of ϕ appear in both equations.
- If g_{ab} and g'_{ab} are representatives of $\mathbf{g}$, how are the drifts at g_{ab} and g'_{ab} related?
- We shouldn't be representing drifts with a vector field V^a. E.g., solutions are overcounted.
- Unclear if the broader difficulites with non-uniqueness/non-existence (i.e. folds) are addressed at all.

References

M '14a *The Conformal Method and the Conformal Thin Sandwich Method are the Same*

M '14b *Conformal Parameterizations of Flat Kasner Spacetimes*

M '14c *Initial Data in General Relativity Parameterized by Conformal Deformation, Expansion, and Drift*

HMM '15 *(with M. Holst and R. Mazzeo) Conformal Killing Fields and the Space of Solutions of the Einstein Constraint Equations*