Lorentzian N-Bakry-Émery cosmological singularity and splitting theorems

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Collaborators

Talk based on

- EW and William Wylie, *Cosmological singularity theorems and splitting theorems for N-Bakry-Émery spacetimes*, arxiv:1509:05734.
- Gregory J Galloway and EW, *Cosmological singularities in Bakry-Émery spacetimes*, J Geom Phys 86 (2014) 359–369.

Related work

- Matthew Rupert and EW, *Bakry-Émery black holes*, Class Quantum Gravit 31 (2014) 025008.

Anniversaries

- Hundredth anniversary of Einstein's Berlin Academic lectures: final form of field equations.
- Hundredth anniversary of Hilbert's variational derivation.
- Fiftieth anniversary of Penrose's singularity theorem.
- Almost the fiftieth anniversary of Hawking's 3 papers on cosmological singularity theorems.
- Thirtieth anniversary of the publication of Bakry and Émery's seminal paper on Markovian diffusions.

Prototype Singularity/Splitting theorems

- Hawking (1967)
  If
  - $R_{ab}t^a t^b \geq 0$ for every timelike t^a,
  - S is a compact spacelike hypersurface without edge, and
  - the (future) mean curvature of S is $H < 0$ everywhere,

  then spacetime is not timelike geodesically complete.

- Indeed, if S is a Cauchy surface, then no timelike geodesic is future-complete.

- Geroch (1966)
  If we relax the mean curvature condition to $H \leq 0$, and if the spacetime is future-timelike geodesically complete, then it is flat.

Riemannian prototypes

- Myers (1941)
  - Let (M, g) be a Riemannian manifold with $\mathrm{Ric} \geq kg$ for some $k > 0$. Then (M, g) has finite diameter and finite fundamental group.
- Cheeger-Gromoll (1971)
  - If $\mathrm{Ric} \geq 0$ and (M, g) contains a complete, maximal geodesic, then it is isometric to $(\mathbb{R} \times S, dt^2 + g_S)$.

Note: These theorems are not true prototypes because they make no hypersurface assumption, giving the proofs a different character.

Extensions

Can one relax the assumptions of these theorems?

- "Averaged" conditions $\int_0^\infty \mathrm{Ric}(\gamma', \gamma') ds \geq 0$.
- Bakry-Émery: Replace pointwise sign condition on Ric, e.g.,

  $$\mathrm{Ric}(X, X) \geq 0$$

  for all X (or for a class of X), with a similar pointwise sign condition on

  $$\mathrm{Ric}_f^N := \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{(N-n)} df \otimes df \ ,$$

  f is the *weight function*, N is the *synthetic dimension*; conditions on f, N?
- Both? e.g.:
  - $\int_0^\infty \mathrm{Ric}_f^N(\gamma', \gamma') ds \geq 0$, $N > n$
  - $\int_0^\infty e^{2f(s)/(n-1)} \mathrm{Ric}_f^N(\gamma', \gamma') ds \geq 0$, $f \leq k$, $N < 1$ or $N = \infty$.

Ubiquity of Bakry-Émery

- Harmonic Einstein equation: $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = 0$: special case $X = df$.
- Gradient Ricci soliton equation: $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$.
- Lichnérowicz (1970): Cheeger-Gromoll-type splitting, assuming
  - $N = n + 1$
  - $\mathrm{Ric}_f^{n+1} \geq 0$
  - f bounded (can relax to f bounded above).
- Homage à Monge:
  - Dimension-curvature condition: $\mathrm{Ric}_f^N \geq \lambda g$.
  - Use optimal transportation to prove analytical results: e.g., displacement convexity of entropy.
    - Bakry and Émery (1985).
    - Otto and Villani (2000).
    - Lott and Villani (2009): Synthetic Ricci curvature.
    - Cordero-Erausquin, McCann, and Schmuckenschläger (2006).

Physics: Kaluza-Klein theorems

- Warped product $\mathcal{N}^N = \mathcal{M}^n \times_{\varepsilon e^{-f/(N-n)}} \mathcal{F}$

$$g_{\mathcal{N}} = g_{\mathcal{M}} \oplus \varepsilon^2 e^{-2f/(N-n)} g_{\mathcal{F}}$$

- Then

$$\begin{aligned} \mathrm{Ric}(g) = & \left[\mathrm{Ric}(g_{\mathcal{M}}) + \mathrm{Hess}_{g_{\mathcal{M}}} f - \frac{1}{(N-n)} df \otimes df \right] \\ & \oplus \left[\mathrm{Ric}(g_{\mathcal{F}}) + \frac{1}{(N-n)} e^{-2f/(N-n)} g_{\mathcal{F}} L(f) \right] \\ L(f) = & \Delta_{g_{\mathcal{M}}} f - \frac{(N-n+1)}{(N-n)} df \oplus df \ , \end{aligned}$$

- Justifies the term *synthetic dimension*.

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Physics: Scalar-tensor gravity

- Prototype: the Brans-Dicke family in $n = 4$ spacetime dimensions

$$\begin{aligned}
\mathrm{Ric} - \frac{1}{\varphi}\,\mathrm{Hess}\,\varphi - \frac{\omega}{\varphi^2} d\varphi \otimes d\varphi &= \frac{8\pi}{\varphi}\tau \ , \\
\tau := T - \frac{(1+\omega)}{(3+2\omega)}(\mathrm{tr}\ T)g \ , \\
\Delta\varphi = \frac{8\pi}{(3+2\omega)}\,\mathrm{tr}\ T \ .
\end{aligned}$$

- $\omega \in \left(-\frac{3}{2}, \infty\right)$ is family parameter, $\varphi \sim 1/G_{\mathrm{Newton}}$.
- For $\varphi := e^{-f}$, get

$$\begin{aligned}
\mathrm{Ric}_f^N \equiv \mathrm{Ric} + \mathrm{Hess}\,f - \frac{1}{(N-4)} df \otimes df &= 8\pi e^f \tau \ , \\
N = \frac{5+4\omega}{1+\omega} \ .
\end{aligned}$$

Negative synthetic dimension arises

We have

- $n = 4$.
- $N = \frac{5+4\omega}{1+\omega}$.
- $\omega \in \left(-\frac{3}{2}, \infty\right)$.

Then ω and N are related via:

- $N \in (4, \infty) \Leftrightarrow \omega \in (-1, \infty)$.
- $N = \infty \Leftrightarrow \omega = -1$ (dilaton gravity).
- $N \in (-\infty, 2) \Leftrightarrow \omega \in \left(-\frac{3}{2}, -1\right)$.

Timelike curvature-dimension condition

- Fix some $N \in \mathbb{R} \cup \{\infty\}$, $\lambda \in \mathbb{R}$.
- The *timelike curvature-dimension condition* $\mathrm{TCD}(\lambda, N)$ is

$$\mathrm{Ric}_f^N(X, X) \geq \lambda \in \mathbb{R}$$

  for every unit timelike vector X.
- The $\mathrm{TCD}(0, N)$ condition reduces to $\mathrm{Ric}(X, X) \geq 0$ if f is constant.
- In general relativity:
  - $\mathrm{Ric}(X, X) \geq 0$ follows from the *strong energy condition*.
  - $\lambda = -\Lambda/(n-1)$, $\Lambda =$ cosmological constant.

Typical conditions on f when $N = \infty$ or $N \leq 1$

These conditions are only needed when $N = \infty$ or $N \leq 1$ (or $N \leq 2$ for certain Lorentzian theorems)

(a) The "classic" condition: $f \leq k$.

(b) Wylie's weaker condition: $\int_0^\infty e^{-2f(t)/(n-1)} dt = \infty$ along (certain) complete geodesics.[1]

(c) Sometimes need a stronger condition: ∇f future-timelike to the future of a Cauchy surface S.

Note that

(1) If (a) holds, then (b) holds for every complete geodesic.

(2) If (c) holds and S is compact, then (a) holds to the future of S.

(3) (b) says that complete geodesics are assumed also complete in the parameter $s = s(t) = \int_0^t e^{-2f(u)/(n-1)} du$.

[1] $f(t)$ is short-hand for $f \circ \gamma(t)$ where γ is a geodesic.

Lorentzian results

- JS Case (2010)
  - $N \in (n, \infty]$
  - Hawking-Penrose theorem
  - Timelike splitting theorem
- GJ Galloway and EW
  - $N = \infty$, $f \leq k$.
  - Hawking's cosmological singularity theorem for nonnegative cosmological constant.
  - Splitting theorem for non-positive CMC Cauchy surface.
- EW and W Wylie
  - Generalize GJG and EW to $N \in (n, \infty] \cup (-\infty, 1]$.
  - For $N = 1$, splitting theorem yields a warped product.
- EW and WW in progress:
  - Generalize Case's Hawking-Penrose theorem to $N \in (-\infty, 2]$.
  - Generalize Case's timelike splitting theorem to $N \in (-\infty, 1]$.
  - Galloway's null splitting theorem.

Case's splitting conjecture

Take $N \in (n, \infty]$.

- *Case's hard question*:
  - (M, g) is globally hyperbolic with compact Cauchy surface S.
  - (M, g) is timelike geodesically complete.
  - $\mathrm{TCD}(0, N)$ holds.
  - If $N = \infty$ then $f \leq k$.

  Then does (M, g) split isometrically as $(\mathbb{R} \times S, -dt^2 \oplus h)$, with f constant in time?

- *Why it's hard*: If *a priori* we set $f = const$, this is *Bartnik's splitting conjecture*.

- *Case's tractable question*:
  - With the above assumptions and
  - if S has f-mean curvature $H_f = 0$,

  then does (M, g) split isometrically as $(\mathbb{R} \times S, -dt^2 \oplus h)$, with f constant in time?

Hawking-type cosmological singularity theorem

Assume that

- $\mathrm{TCD}(0, N)$ holds for some fixed $N \in (-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f < 0$ everywhere, and
- if $N \in [-\infty, 1]$ then $\int_0^\infty e^{-2f(s)/(n-1)} ds$ diverges along every complete timelike geodesic orthogonal to S.

Then no timelike geodesic is future-complete.

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Then no timelike geodesic is future-complete.

- Recall TCD$(0, N) \Rightarrow \text{Ric}(X, X) + \text{Hess}(X, X) f - \frac{1}{(N-n)} \langle df, X \rangle^2 \geq 0$.
- When $N > n$, the $\langle df, X \rangle^2$ term *helps*: no control of f required.
- When $N \leq 1$, the $\langle df, X \rangle^2$ term *hinders*: but can still obtain a theorem if we have mild control of f.

Splitting theorem

Assume that

- TCD$(0, N)$ holds for some fixed $N \in (-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f \leq 0$ everywhere,
- if $N \in [-\infty, 1]$ then $\int_0^\infty e^{-2f(s)/(n-1)} ds$ diverges along every complete timelike geodesic orthogonal to S, and
- the geodesics orthogonal to S are future-complete.

Then,

- if $N \in (-\infty, 1) \cup (n, \infty]$, the future of S is isometric to $-dt^2 \oplus h$ and f is independent of t (answers Case's tractable question).
- if $N = 1$, the future of S is isometric to $-dt^2 \oplus e^{2\psi(t)/(n-1)} h$ and $f = \psi(t) + \phi(y)$, $y \in S$.

Positive cosmological constant singularity theorem

Assume that

- $\text{TCD}(-(n-1), N)$ holds for some fixed $N \in (-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f < -(n-1)$ everywhere, and
- if $N \in [-\infty, 1]$ then ∇f is future-causal to the future of S.

Then no timelike geodesic is future-complete.

Alternative version for $N \in [-\infty, 1]$

There are a number of versions that do not require that ∇f be future-causal, but require other assumptions to be strengthened; e.g., If

- $\mathrm{TCD}(-(n-1)e^{-4f/(n-1)}, N)$ holds for some $N \in [-\infty, 1]$,
- $H_f < -(n-1)e^{-2\inf_S f/(n-1)}$ on compact Cauchy surface S, and
- $\int_0^\infty e^{-2f(s)/(n-1)}ds$ diverges along every complete timelike geodesic orthogonal to S,

then no timelike geodesic is future-complete.

Splitting theorem

For $N \in [-\infty, 1] \cup (n, \infty]$, assume that

- $\mathrm{TCD}(-(n-1), N)$ holds for some fixed $N \in (-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f \leq -(n-1)$ everywhere,
- ∇f is future-causal, and
- the geodesics orthogonal to S are future-complete.

Then the future of S splits as a warped product $-dt^2 \oplus e^{-2t}h$ and f is constant.

Alternative splitting theorem

For $N > n$, assume that

- $\mathrm{TCD}(-(N-1), N)$ holds for some fixed $N \in (-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f \leq -(N-1)$ everywhere, and
- the geodesics orthogonal to S are future-complete.

Then the future of S splits as a warped product $-dt^2 \oplus e^{-2t} h$ and $f = (N-n)t + f_S(y)$, $y \in S$.

The (timelike) f-Raychaudhuri equation

$$\frac{\partial H}{\partial t} = -\operatorname{Ric}(\gamma', \gamma') - |K|^2 = -\operatorname{Ric}(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)}$$

Use $H_f := H - f'$ and use definition of Ric_f^N. Get

$$\begin{aligned}\frac{\partial H_f}{\partial t} &= -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)} - \frac{f'^2}{(N-n)} \\ &= -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{1}{(n-1)}\left[H_f^2 + 2H_f f' + \frac{(1-N)}{(n-N)} f'^2\right]\end{aligned}$$

Analyse this. Use that H_f diverges along γ at finite t iff H diverges.

- First line: If $N > n$ each term on right is ≤ 0 (assuming $\operatorname{TCD}(0, N)$).
- Second line: Coefficient of f'^2 has same sign for $N < 1$ as for $N > n$, but must deal with $H_f f'$ term.

Example of a focusing argument: TCD$(0, N)$ case

- For $N > n$, an easy identity yields

$$\frac{\partial H_f}{\partial t} \le -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{H_f^2}{(N-1)}$$

$$\Rightarrow \frac{\partial x}{\partial t} \le -x^2 \ , \ x := H_f/(N-1) \ , \ \text{using } \ \mathrm{TCD}(0, N) \ .$$

- Otherwise, use an integrating factor to eliminate $H_f f'$ term:

$$\frac{\partial}{\partial t}\left(e^{\frac{2f}{(n-1)}} H_f\right) = -e^{\frac{2f}{(n-1)}}\left[\operatorname{Ric}_f^N(\gamma', \gamma') + |\sigma|^2 + H_f^2 + \frac{(1-N)f'^2}{(n-N)(n-1)}\right]$$

$$\Rightarrow \frac{\partial x}{\partial t} \le -e^{-\frac{2f}{(n-1)}} x^2 \ , \ x := e^{\frac{2f}{(n-1)}} H_f \ , \ \text{using } \ \mathrm{TCD}(0, N) \ .$$

- Now $x(0) \le x_0 < 0$. $\begin{cases} x(t) \le \frac{1}{t + 1/x_0}, & N > n \\ x(t) \le \frac{1}{\int_0^t e^{-2f(s)/(n-1)} ds + 1/x_0}, & N \in [-\infty, 1] \end{cases}$
- Thus $x(t) \to -\infty$ as $t \nearrow t_0$.

Completion of the argument.

- $x \to -\infty$ as $t \to t_0$ for some $t_0 \leq T(x_0) \leq T$.
- Thus $H \to -\infty$ as $t \to t_0$ for some $t_0 \leq T(x_0) \leq T$.
- Thus no future-timelike geodesic orthogonal to S can maximize beyond $t = T$.
- If there were a future-complete timelike geodesic γ, there would be a sequence of maximizing geodesics from S to γ, meeting S orthogonally and of unbounded length.
- Thus there can be no future-complete timelike geodesic. QED.

Example of a splitting argument

- Now $H_f \leq 0$, and we assume future completeness.
- If $H_f < 0$ on S, cannot be future complete, so $H_f = 0$ at least somewhere on S.
- If H_f is not *identically* zero on S, do short f-mean curvature flow.

$$\frac{\partial X}{\partial s} = -H_f \nu \ .$$

- Strong maximum principle implies that $H_f(s) < 0$ for $s > 0$ (and still Cauchy).
- Therefore must have $H_f \equiv 0$ on S.
- And must have $H_f(t) \equiv 0$, so each term on right in Raychaudhuri equation must vanish.

Splitting argument: continued

- For $N > n$, recall

$$\frac{\partial H_f}{\partial t} = -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)} - \frac{f'^2}{(N-n)}\ .$$

- Must have $H_f \equiv 0$ on $(0, t)$.
- Thus $\sigma = 0$, $H = 0$, $f' = 0$ on $(0, t)$.
- $g = -dt^2 \oplus h$, $f' = 0$, and since the γ are future-complete, the splitting is global.

Splitting argument: continued

- For $N \in [-\infty, 1]$, had

$$\frac{\partial}{\partial t}\left(e^{\frac{2f}{(n-1)}} H_f\right) = -\, e^{\frac{2f}{(n-1)}} \Bigg[\mathrm{Ric}_f^N(\gamma', \gamma') + |\sigma|^2 + H_f^2 + \frac{(1-N)f'^2}{(n-N)(n-1)} \Bigg]$$

- Must have $H_f \equiv 0$ on $(0, t)$.
- Thus $\sigma = 0$, $H = f'$, and *either* $f' = 0$ or $N = 1$, on $(0, t)$.
- If $N \neq 1$, get $H = 0$ and get global product splitting as before.
- If $N = 1$, use also that $\mathrm{Ric}_f^1(\gamma', \gamma') = 0$ on $(0, t)$.
- A computation then yields the warped product of the theorem.

Further thoughts: Conjugate pairs

The timelike *f-generic condition*:

- Define $\mathcal{R}_{ijkl} := R_{ijkl} + \left[g \odot \left(\frac{1}{(n-1)} \operatorname{Hess} f + \frac{1}{(n-1)^2} df \otimes df \right) \right]_{ijkl}$.
- The timelike f-generic condition is said to hold if along each future-complete timelike geodesic $\gamma(t)$ there is a t_0 such that

$$w^q w^r w_{[i} \mathcal{R}_{j]qr[k} w_{l]} \neq 0 \ , \ \ w := \gamma'(0) \ .$$

- The shear σ and expansion θ_f (previously H_f) of a twist-free timelike congruence can be combined into $B_f := \sigma + \frac{1}{(n-1)} \theta_f \, \mathrm{id}$.
- It obeys a matrix Riccati equation

$$B_f' + B_f \cdot B_f = -\bar{R}_f \ \ ...(*)$$

- $\bar{R}_f$ is non-zero $\Leftrightarrow w^q w^r w_{[i} \mathcal{R}_{j]qr[k} w_{l]} \neq 0 \Leftrightarrow$ $(*)$ nonhomogeneous.

Conjugate pairs: continued

Assume that

- TCD$(0, N)$ holds for some $N \in (-\infty, 1] \cup (n, \infty]$,
- the timelike f-generic condition holds, and
- if $N \in [-\infty, 1]$ then along each complete timelike geodesic, $\int_0^\infty e^{-f(s)/(n-1)} ds = \infty$ and $\int_{-\infty}^0 e^{-f(s)/(n-1)} ds = \infty$.

Then

- each complete timelike geodesic has a pair of conjugate points, so
- an inextendible maximal timelike geodesic is necessarily incomplete.
- All this also holds for *null geodesics*, except that the domain $N \in [-\infty, 1]$ will now extend to $N \in [-\infty, 2]$.

Another question of Case

- Is $tcd(0, N)$ weaker than the strong energy condition $\mathrm{Ric}(X, X) \geq 0$?
  - That is, say
    - (M, g) is future timelike geodesically complete,
    - $\mathrm{Ric}_f^N(g)(X, X) \geq 0$ for all timelike X,
    - $f \leq k$ if $N \in [-\infty, 1]$.
  - Does M admit a future-timelike complete metric g_1 with $\mathrm{Ric}(g_1)(X, X) \geq 0$?
  - If it also admits an $H_f \leq 0$ compact Cauchy surface, we know the answer is yes.
  - Posed in Riemannian setting by Wei-Wylie arxiv:0706.1120.
  - Posed in Lorentzian phrasing by Case arxiv:0712:1321.