

Properties of axially symmetric stationary solutions of the Einstein-Vlasov system and the Vlasov-Poisson system

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Joint work with Ellery Ames, Anders Logg and Gerhard Rein

Two projects

I will present results from two different projects.

In the first part I will discuss properties of axially symmetric stationary solutions of the Einstein-Vlasov system. This is a joint work with Ellery Ames and Anders Logg.

In the second part I will discuss properties of the rotation curves for disk solutions of the axially symmetric Vlasov-Poisson system. This is a joint work with Gerhard Rein.

Shapiro and Teukolsky (Abrahams and Cook) studied numerically the axially symmetric Vlasov-Poisson system and Einstein-Vlasov system intensively during the time period 1985-1995.

They found many interesting and also remarkable results.

- Static and stationary solutions were obtained.
- The stability of these solutions were studied.
- Gravitational collapse was investigated and they claimed that cosmic censorship is violated.

Since then no other study exists in the literature. Our goal is to make an independent investigation of these issues.

Up to now we have studied static and stationary solutions and these results will be presented in the first part of this talk.

- Can we find solutions which are far from spherically symmetric?
- Can we obtain highly relativistic solutions?
- What is the effect of angular momentum?
- Can we model the large variety of galaxies and star clusters observed?

The Einstein-Vlasov system

Vlasov distribution function

The mass-shell condition

$$g_{\mu\nu} p^\mu p^\nu = -m^2$$

Einstein's Equations

$$Ric(g)_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Vlasov equation:

$$p^\alpha \partial_{x^\alpha} f - \Gamma_{\mu\nu}^a p^\mu p^\nu \partial_{p^a} f = 0$$

Energy Momentum

$$T_{\mu\nu} = \int p_\mu p_\nu f(x, p) \frac{\sqrt{|g|}}{-p_0} dp^1 dp^2 dp^3$$

\mathbb{R}^+

(M, g)

f

The metric and the boundary conditions

We use axial coordinates $t \in \mathbb{R}$, $\rho \in [0, \infty[$, $z \in \mathbb{R}$, $\varphi \in [0, 2\pi]$ and write the metric in the form

$$ds^2 = -e^{2\nu} dt^2 + e^{2\mu} d\rho^2 + e^{2\mu} dz^2 + \rho^2 B^2 e^{-2\nu} (d\varphi - \omega dt)^2,$$

for functions ν, B, μ, ω depending on ρ and z .

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$$\lim_{|(\rho,z)| \rightarrow \infty} (\nu(\rho, z), \mu(\rho, z), B(\rho, z), \omega(\rho, z)) = (0, 0, 1, 0).$$

In addition we need to require the condition that the metric is locally flat at the axis of symmetry, i.e.,

$$\nu(0, z) + \mu(0, z) = \ln B(0, z), \quad z \in \mathbb{R}.$$

Conserved quantities

The symmetries of the metric imply that the quantities

$$E := -g(\partial/\partial t, p^\alpha) = c^2 e^{2\nu/c^2} p^0 + \rho^2 B^2 \omega e^{-2\nu/c^2} (p^3 - \omega p^0),$$

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are constant along geodesics.

Here E can be thought of as a local or particle energy and L_z is the angular momentum of a particle with respect to the axis of symmetry.

The ansatz

Hence we make the ansatz

$$f = \phi(E, L_z)$$

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Inserting this ansatz into the definition of the energy momentum tensor we get that $T_{\alpha\beta}$ becomes a functional of the metric functions ν, B, μ and ω .

The Einstein equations

$$\Delta\nu = 4\pi (\Phi_{00} + \Phi_{11} + (1 + (\rho B)^2 e^{-4\nu} \omega^2) \Phi_{33} + 2e^{-4\nu} \omega \Phi_{03}) \\ - \frac{1}{B} \nabla B \cdot \nabla \nu + \frac{1}{2} e^{-4\nu} (\rho B)^2 \nabla \omega \cdot \nabla \omega$$

$$\Delta B = 8\pi B \Phi_{11} - \frac{1}{\rho} \nabla \rho \cdot \nabla B$$

$$\Delta\mu = -4\pi (\Phi_{00} + \Phi_{11} + ((\rho B)^2 e^{-4\nu} \omega^2 - 1) \Phi_{33} + 2e^{-4\nu} \omega \Phi_{03}) \\ + \frac{1}{B} \nabla B \cdot \nabla \nu - \nabla \nu \cdot \nabla \nu + \frac{1}{\rho} \nabla \rho \cdot \nabla \mu + \frac{1}{\rho} \nabla \rho \cdot \nabla \nu \\ + \frac{1}{4} e^{-4\nu} (\rho B)^2 \nabla \omega \cdot \nabla \omega$$

$$\Delta\omega = \frac{16\pi}{(\rho B)^2} (\Phi_{03} + (\rho B)^2 \omega \Phi_{33}) - \frac{3}{B} \nabla B \cdot \nabla \omega + 4 \nabla \nu \cdot \nabla \omega \\ - \frac{2}{\rho} \nabla \rho \cdot \nabla \omega$$

Here Δ, ∇ are the Laplacian and gradient in cartesian coordinates.

The matter terms

$$\begin{aligned}\Phi_{00} &= e^{2\mu-2\nu} T_{00} \\ &= \frac{2\pi}{B} e^{2\mu-2\nu} \int_{e^\nu}^{\infty} \int_{-s_I}^{s_I} E^2 \Phi(E, \rho s) ds dE, \\ \Phi_{11} &= T_{\rho\rho} + T_{zz} \\ &= \frac{2\pi}{B^3} e^{2\mu+2\nu} \int_{e^\nu}^{\infty} \int_{-s_I}^{s_I} (s_I^2 - s^2) \Phi(E, \rho s) ds dE.\end{aligned}$$

... the matter terms

$$\begin{aligned}\Phi_{33} &= (\rho B)^{-2} e^{2\mu+2\nu} T_{\varphi\varphi} \\ &= \frac{2\pi}{B^3} e^{2\mu+2\nu} \int_{e^\nu}^{\infty} \int_{-s_l}^{s_l} s^2 \Phi(E, \rho s) ds dE, \\ \Phi_{03} &= e^{2\mu+2\nu} T_{0\varphi} \\ &= -2\pi \rho B^{-1} e^{2\mu+2\nu} \int_{e^\nu}^{\infty} \int_{-s_l}^{s_l} s E \Phi(E, \rho s) ds dE,\end{aligned}$$

where

$$s_l := B e^{-\nu} \sqrt{e^{-2\nu} (E - \omega \rho s)^2 - 1}.$$

Existence for small deviation from spherical symmetry

Existence of axially symmetric stationary solutions of the Einstein-Vlasov system has been shown:

- H.A., M. KUNZE AND G. REIN Rotating, stationary, axially symmetric spacetimes with collisionless matter. *Commun. Math. Phys.* **329**, 787–808 (2014).

However, the method of proof only guarantees existence of solutions which are close to being spherically symmetric and which have small angular momentum. The main purpose of the present, numerical study, is to go beyond this restriction.

Numerical method

- Choose ansatz $f = K\Phi(E, L_z)$.
- Give initial guess of metric functions $\{\nu^0, \mu^0, B^0, \omega^0\}$.
- Compute T_{ab}
- Rescale K so that total ADM mass is M .
- Use Finite Element Method (FEniCS) to solve linear system of PDEs to obtain $\{\nu^1, \mu^1, B^1, \omega^1\}$ applying "numerical" boundary conditions.
- Iterate until residual $<$ tolerance.

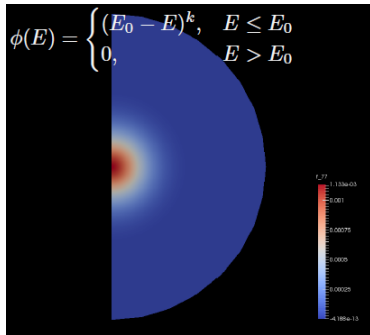
It is important to choose the initial guess carefully.

Convergence and stability

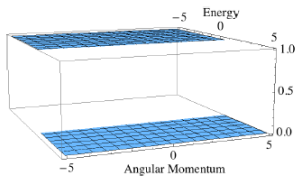
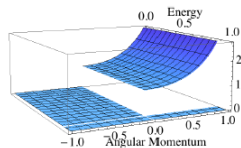
There are indications that the solutions we obtain are in fact dynamically stable, i.e., the algorithm seems only to converge to a solution if it is dynamically stable.

This claim is supported by applying the algorithm to the spherically symmetric case.

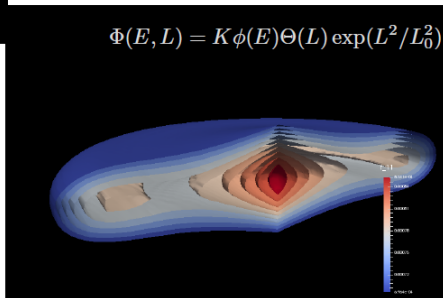
However, Abrahams, Cook, Shapiro and Teukolsky have studied the stability of the static and stationary solutions they obtained and they do find a couple of solutions which are unstable.



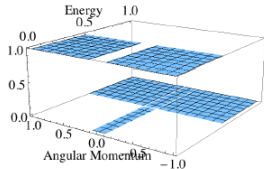
Numerical Menu of Solutions



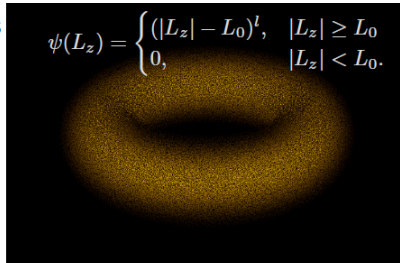
$$\Phi(E, L) = K \phi(E) \Theta(L) \exp(L^2/L_0^2)$$



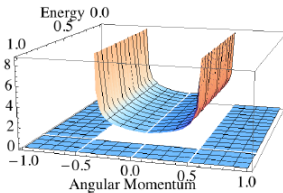
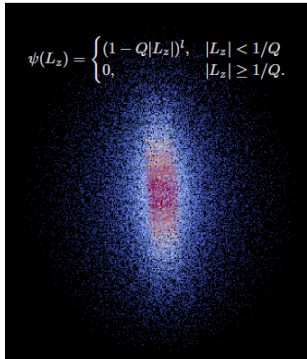
Numerical Menu of Solutions



$$\psi(L_z) = \begin{cases} (|L_z| - L_0)^l, & |L_z| \geq L_0 \\ 0, & |L_z| < L_0. \end{cases}$$



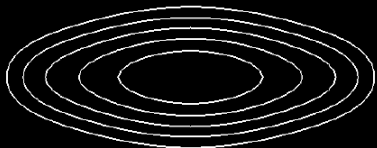
$$\psi(L_z) = \begin{cases} (1 - Q|L_z|)^l, & |L_z| < 1/Q \\ 0, & |L_z| \geq 1/Q. \end{cases}$$



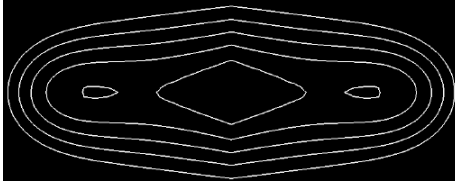
Disk-like galaxies. A comparison.

Does
Relativity
Matter?

Newtonian
gravity

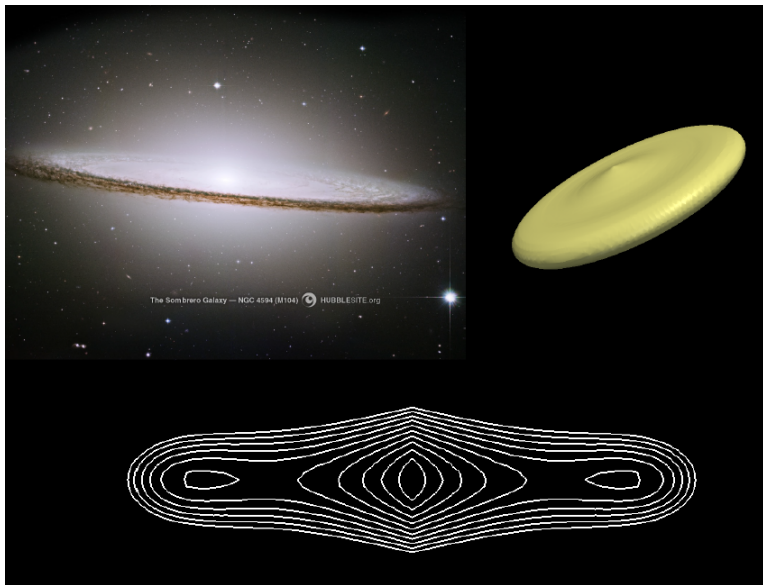


Non-rotating
relativistic disk
galaxy



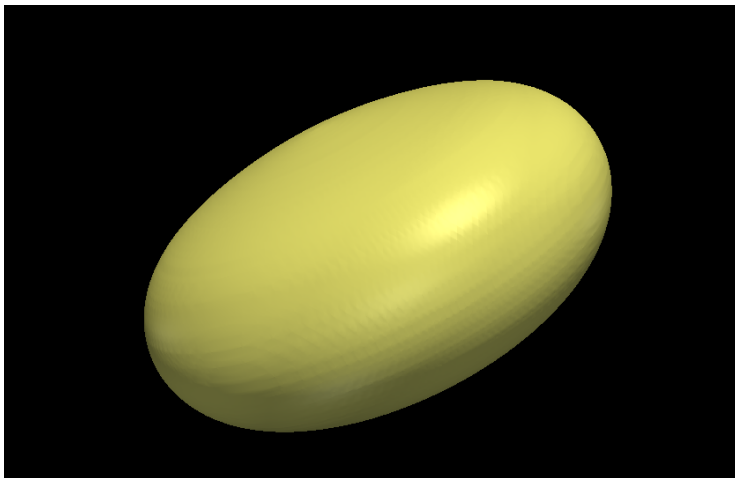
Rotating
relativistic disk
galaxy

The rotating relativistic disk



A less flat and less flattering contour...

Contour is here at 20% of peak value.



Highly relativistic, extremely thin, tori solutions

A general challenge is to obtain highly relativistic solutions. Shapiro and Teukolsky measure how relativistic a solution is by its cut-off energy E_0 . In spherical symmetry E_0 is related to the fraction $2M/R$, where R is the areal radius of the support of the solution by the formula

$$E_0 = \sqrt{1 - \frac{2M}{R}}.$$

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By increasing the resolution we believe that in the limit an infinitely thin torus is obtained. These highly relativistic tori might even be stable!

Numerical method to improve convergence

Assume we want to solve the equation $x = g(x)$, $x \in \mathbb{R}$, with fixed point iteration. If the Lipschitz constant $L_g < 1$ then it converges. If it does not converge we can in some cases make it converge by damped fixed point iteration. We modify g by

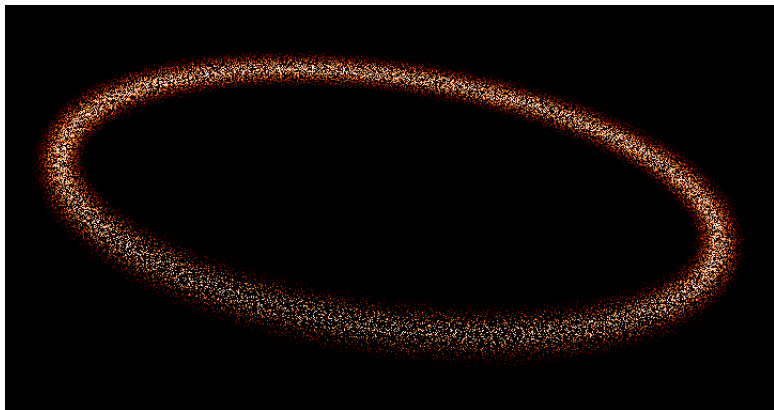
$$\tilde{g}(x) = (1 - \theta)x + \theta g(x).$$

Note that \tilde{g} has the same fixed point(s) as g . If we are lucky and $g' < -1$, then there is a $0 < \theta < 1$ such that

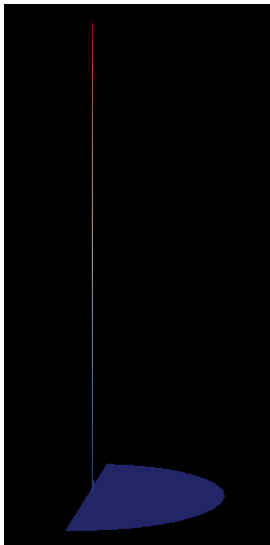
$$|\tilde{g}'(x)| = |1 - \theta + \theta g'| < 1.$$

In the case of the very thin torus, this modification is necessary.

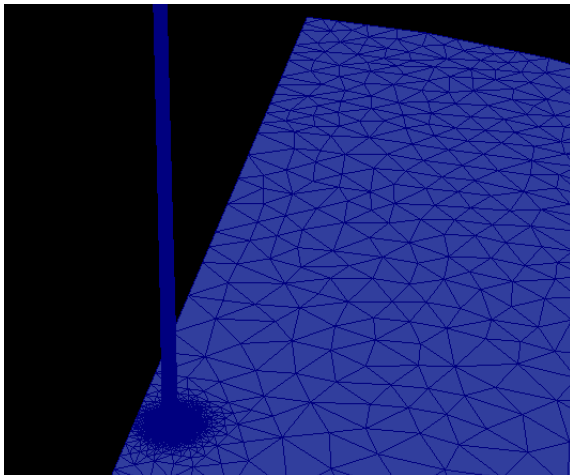
Thin torus



Thin torus

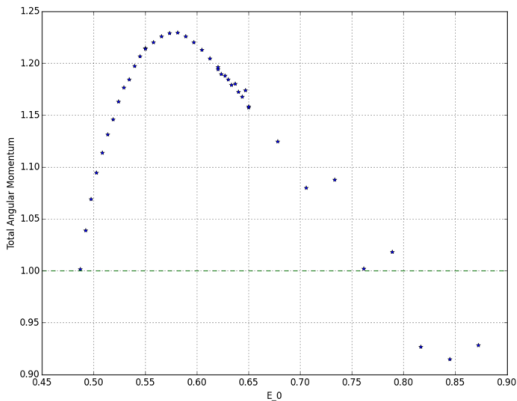


Thin torus



J versus E_0 . A very thin torus requires $J \neq 0$!

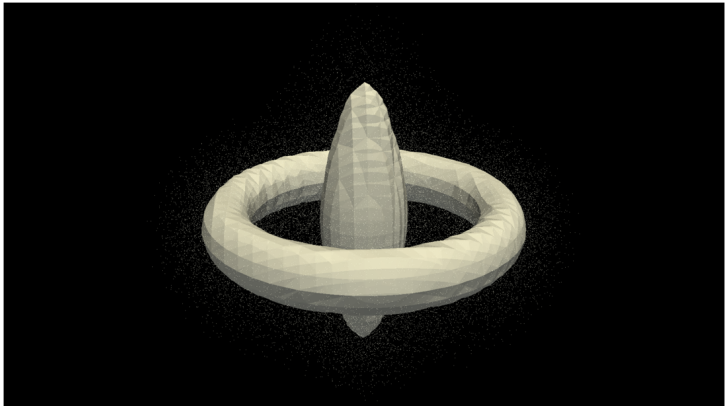
Recall that for a Kerr black hole the inequality $J^2 < M$ holds. Below $M = 1$. In the limit it holds that $M/R = 1/3$.



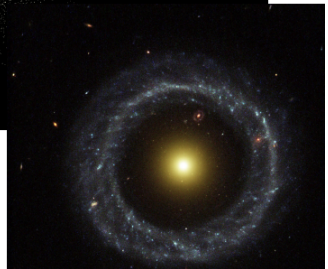
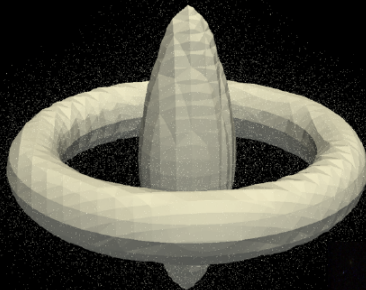
The thin tori solutions may thus be stable.

Numerical Solutions: Ring galaxy

$$\Phi(E, L) \propto C_{ring} \Phi_{ring}(E, L) + C_{center} \Phi_{center}(E, L)$$



Numerical Solutions: Ring galaxy



- Investigate the properties of the infinitely thin torus
- It would be desirable to obtain more flattened disks
- Investigate the stability of the stationary solutions

The second part

In the second part I will present results on the rotation curves for axially symmetric disk-like solutions of the Vlasov-Poisson system.

This is a joint work with Gerhard Rein.

Rotation curve of a galaxy

The rotation curve of a galaxy depicts the magnitude of the orbital velocities of visible stars or gas particles in the galaxy versus their radial distance from the center.

In the pioneering observations by Bosma (1981) and Rubin et al (1982) it was found that the rotation curves of spiral galaxies are approximately flat except in the inner region where the rotation curves rise steeply. Independent observations in more recent years agree with these conclusions.

Dark matter

The flat shape of the rotation curves is an essential reason for introducing the concept of dark matter. Let us cite from (Famaey & McGaugh 2012):

"Perhaps the most persuasive piece of evidence [for the need of dark matter] was then provided, notably through the seminal works of Bosma and Rubin, by establishing that the rotation curves of spiral galaxies are approximately flat (Bosma 1981, Rubin et al 1982). A system obeying Newton's law of gravity should have a rotation curve that, like the Solar system, declines in a Keplerian manner once the bulk of the mass is enclosed: $V_c \propto r^{-1/2}$."

The last statement is heuristic and it is therefore essential to construct self-consistent mathematical models which describe disk galaxies and study the corresponding rotation curves.

For this purpose it is natural to consider the Vlasov-Poisson system which is often used to model galaxies and globular clusters.

In fact, there exist well-known explicit solutions to the Vlasov-Poisson system describing axially symmetric disk galaxies which give rise to flat or even increasing rotation curves. The Mestel disks and the Kalnajs disks are examples of such solutions. However, these solutions are not considered physically realistic.

Mestel disks, which give rise to flat rotation curves, are infinite in extent and their density is singular at the center.

Kalnajs disks, which give rise to linearly increasing rotation curves, are (probably) dynamically unstable.

Physically realistic models

The approach in the present investigation is different. The aim is first to construct solutions which are realistic in the sense that they are dynamically stable, have finite extent and finite mass, and then to study the corresponding rotation curves.

However, the existence and stability theory for flat axially symmetric steady states of the Vlasov-Poisson system is much less developed than in the spherically symmetric case.

Nevertheless, motivated by a few analytic results on flat steady states we search for solutions which we expect to be physically realistic, but we emphasize that the solutions we construct are not covered by the present theory.

The Vlasov-Poisson system

The Vlasov-Poisson system is given by

$$\begin{aligned}\partial_t F + v \cdot \nabla_x F - \nabla_x U \cdot \nabla_v F &= 0, \\ \Delta U &= 4\pi\rho, \quad \lim_{|x| \rightarrow \infty} U(t, x) = 0, \\ \rho(t, x) &= \int_{\mathbb{R}^3} F(t, x, v) dv.\end{aligned}$$

Here $F = F(t, x, v)$ is the density function on phase space of the particle ensemble, and $t \in \mathbb{R}$ and $x, v \in \mathbb{R}^3$. ρ is the mass density and U the gravitational potential.

The mass of each particle in the ensemble is assumed to be equal and is normalized to one.

The potential U is given by

$$U(t, x) = - \int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x - y|} dy.$$

We are interested in extremely flattened axially symmetric galaxies where all the stars are concentrated in the (x_1, x_2) -plane.

We therefore assume that

$$F(t, x, x_3, v, v_3) = f(t, x, v) \delta(x_3) \delta(v_3),$$

where δ is the Dirac distribution, and where from now on $x, v \in \mathbb{R}^2$.

The flat Vlasov-Poisson system

The stars in the plane will only experience a force field parallel to the plane and the Vlasov-Poisson system for the density function $f = f(t, x, v)$, $x, v \in \mathbb{R}^2$, takes the form

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,$$

$$U(t, x) = - \int_{\mathbb{R}^2} \frac{\Sigma(t, y)}{|x - y|} dy,$$

$$\Sigma(t, x) = \int_{\mathbb{R}^2} f(t, x, v) dv;$$

where Σ is the surface density.

We emphasize that the system above is not a two dimensional version of the Vlasov-Poisson system but a special case of the three dimensional version where the density function is partially singular.

A reduced system of equations

We now derive a simplified form of the flat, static, axially symmetric Vlasov-Poisson system by using the ansatz

$$f = \Phi(E, L_z)$$

and the symmetry assumption. *Remark:* No Jeans' theorem exists.

For axially symmetric steady states the mass density Σ and the potential U are functions of $r := \sqrt{x_1^2 + x_2^2}$.

Let

$$K(\xi) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \xi^2 \sin^2 \theta}} = \int_0^1 \frac{dt}{\sqrt{1 - \xi^2 t^2} \sqrt{1 - t^2}}, \quad 0 \leq \xi < 1,$$

be the complete elliptic integral of the first kind.

The reduced system of equations

The change of variables $(v_1, v_2) \mapsto (E, L_z)$ implies that

$$\Sigma(r) = 2 \int_{U(r)}^{\infty} \int_{-\sqrt{2r^2(E-U(r))}}^{\sqrt{2r^2(E-U(r))}} \frac{\Phi(E, L_z) dL_z dE}{\sqrt{2r^2(E-U(r)) - L_z^2}}. \quad (1)$$

The potential U can be written in terms of K as

$$U(r) = -4 \int_0^{\infty} \frac{s \Sigma(s)}{r+s} K\left(\frac{2\sqrt{rs}}{r+s}\right) ds. \quad (2)$$

These equations constitute the system we use to numerically construct axially symmetric flat solutions of the Vlasov-Poisson system.

No circular orbits

Let R_b denote the boundary of the steady state. We have the following result which we state and prove in the flat, axially symmetric case as well as in the spherically symmetric case.

Theorem

Consider a non-trivial compactly supported steady state of the flat, axially symmetric Vlasov-Poisson system or the spherically symmetric Vlasov-Poisson system for which the ansatz function satisfies the cut-off property. Then there is an $\epsilon > 0$ such that for $r \in [R_b - \epsilon, R_b]$ no circular orbits exist in the particle distribution given by f .

- (a) Numerically we find that in the flat case and for our ansatz functions the typical interval where there are no stars in circular orbits is approximately given by $[0.6R_b, R_b]$.
- (b) It should be stressed that if $U'(r) > 0$ for some radius $r > 0$ then **test particles** with the proper circular velocity do travel on the circle of radius r . The result above only says that *the particle distribution given by the steady state* does not contain such particles, i.e., stars. We nevertheless compute the rotation curves below by using the equation $v_c^2 = rU'(r)$ and compare these with observational data. Hence the interpretation of our rotation curves is that they correspond to the circular orbits of test particles in the gravitational field generated by the steady state of the Vlasov-Poisson system.

The ansatz and the numerical algorithm

We consider the following ansatz function:

$$\Phi(E, L_z) = A(E_0 - E)_+^k (1 - Q|L_z|)_+^l. \quad (3)$$

Here $A > 0$, $E_0 < 0$, $Q \geq 0$, $k \geq 0$, and l are constants.

The constant A is merely a normalization constant which controls the total mass of the solution.

Hence, when a solution is depicted its total mass M is given rather than the value of A .

The ansatz (3) has the property that Φ is decreasing as a function of E for fixed L_z . In the regular three dimensional case this property is well-known to be essential for stability. The ansatz for the [Kalnajs disk](#)

$$f = \begin{cases} A[(\Omega_0^2 - \Omega^2)a^2 - 2(E - \Omega L_z)]^{-1/2}, & [...] > 0 \\ 0, & [...] \leq 0. \end{cases}$$

is on the other hand increasing in E which indicates that these solutions are unstable.

The numerical algorithm

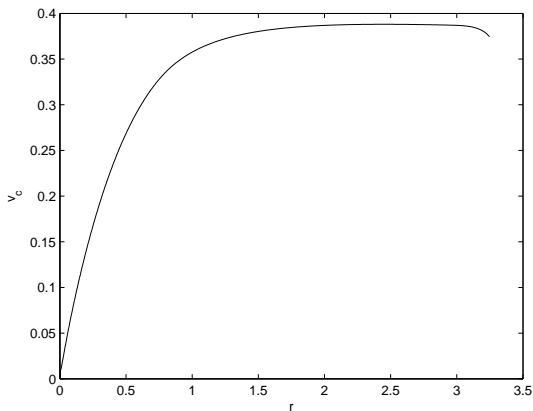
The system of equations is solved by an iteration scheme of the following type (clearly not FEM in this case):

- Choose some start-up density Σ_0 which is non-negative and has a prescribed mass M .
- Compute the potential U_0 induced by Σ_0 ; the complete elliptic integral appearing there is computed using the gsl package.
- Compute the spatial density $\tilde{\Sigma}$ from the density U_0 .
- Define $\Sigma_1 = c\tilde{\Sigma}$ where c is chosen such that Σ_1 again has mass M .
- Return to the first step with $\Sigma_0 := \Sigma_1$.

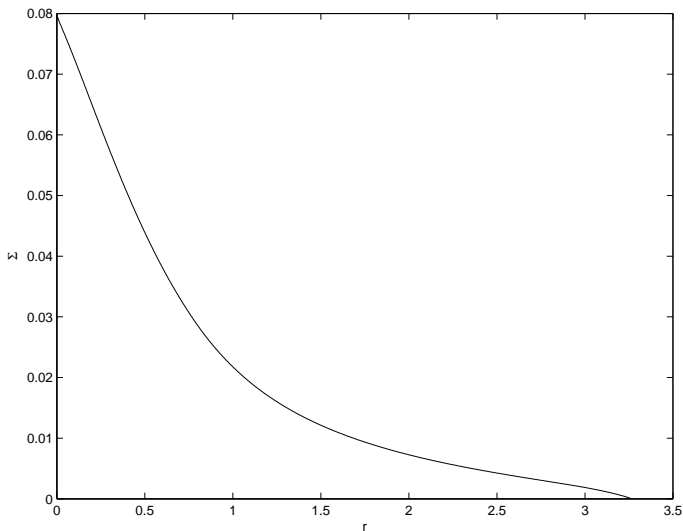
The parameters

- (a) To obtain flat rotation curves it is essential that the value of $k \geq 0$ is small and for simplicity it is set to zero but the results are not affected in an essential way as long as it is small.
- (b) The cut-off energy E_0 determines the extent of the support of the solution and the influence of this parameter will not affect the qualitative behavior of the solutions. Hence, the essential parameters in this study are I and Q .

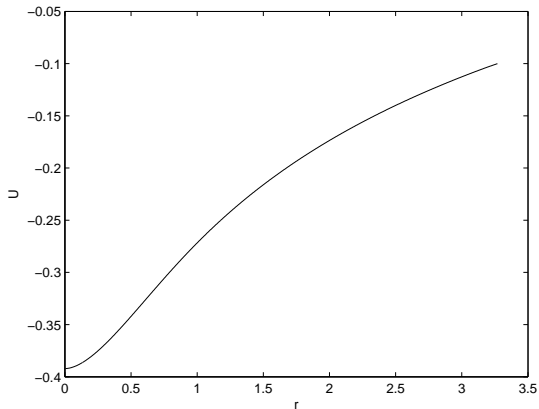
Rotation velocity v_c when $I = 1.0$ and $Q = 2.0$



Corresponding density Σ ($I = 1.0$ and $Q = 2.0$)



Corresponding potential U ($I = 1.0$ and $Q = 2.0$)



Rotation velocity v_c when $I = 1.0$ and $Q = 1.5$

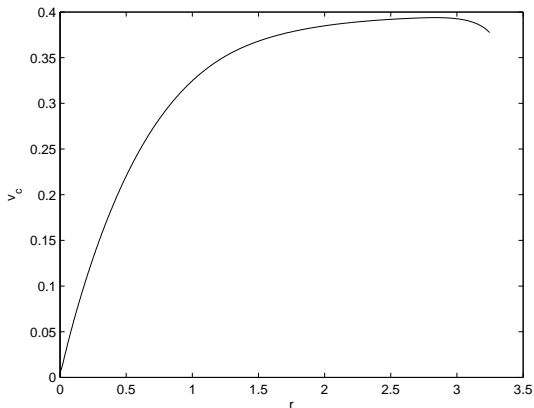
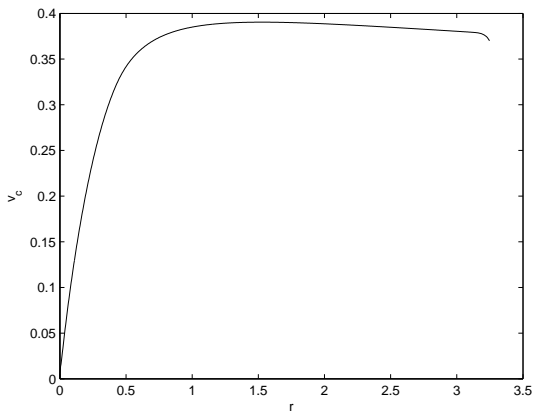
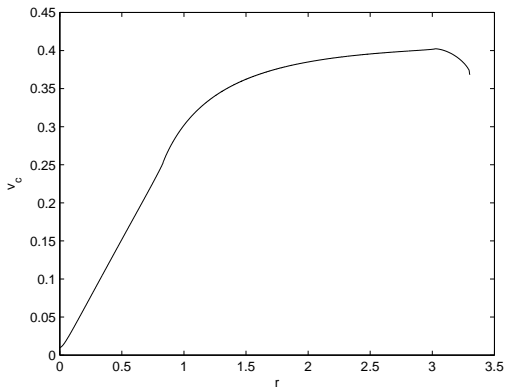


Figure : Rotation velocity v_c versus radius r

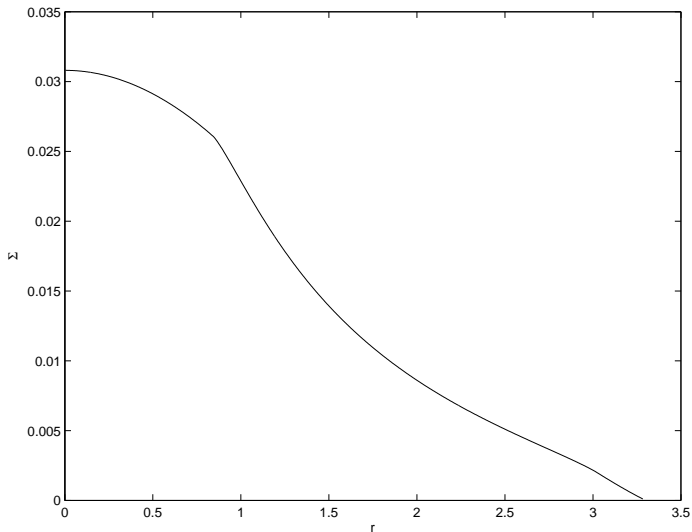
Rotation velocity v_c when $I = 1.0$ and $Q = 3.0$



Rotation velocity v_c when $I = 0.0$ and $Q = 2.0$



Corresponding density Σ ($I = 0.0$ and $Q = 2.0$)



Comparison to observations

Let us now consider data for some spiral galaxies which belong to the Ursa Major cluster.

The aim is to find solutions of the flat Vlasov-Poisson system which match these data.

We have stated the Vlasov-Poisson system in dimensionless form, but in order to compare our results with observations we need to attach proper units.

Since the gravitational constant G is the only physical constant which enters the system and since we have normalized this to unity we can choose any set of units for time, length, and mass in which the numerical value for G equals unity.

Mass predictions

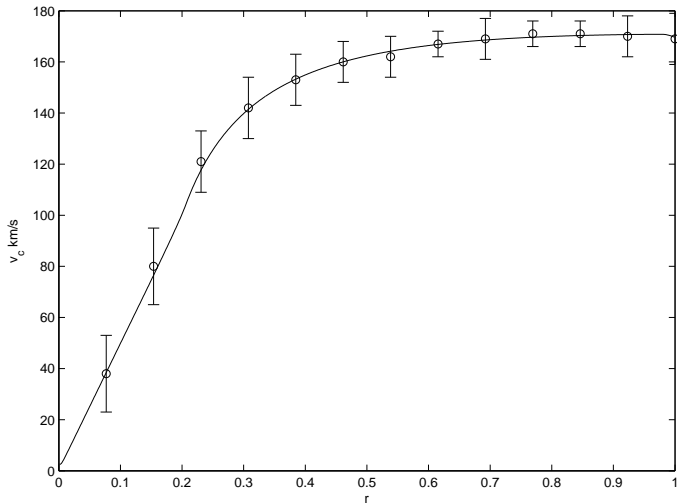
We first choose the unit for length such that the numerically obtained value for its radius corresponds to the observed radius.

Then we choose the unit of time (and hence velocity) in such a way that the numerically predicted rotation curve fits the observations—of course this only works if the mathematical model we consider produces a rotation curve with the proper shape, which we try to achieve by varying the parameters in the ansatz (3).

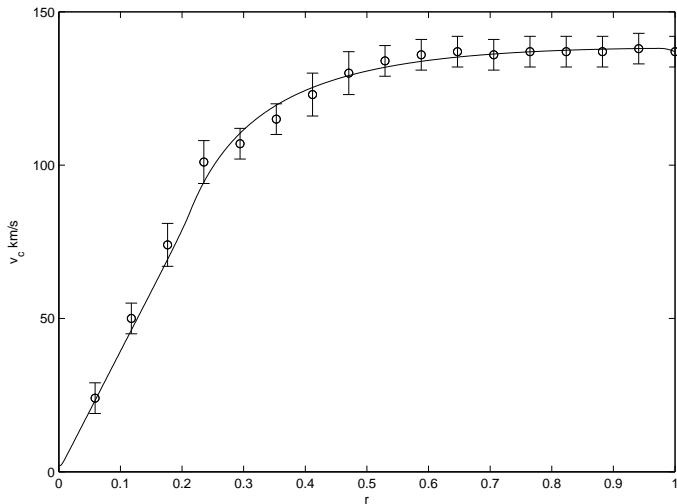
Once the units for length and time are chosen in this way, the condition $G = 1$ fixes the unit for mass.

We can then transform the numerically obtained value for the mass into a predicted mass of the galaxy under consideration in units of solar masses.

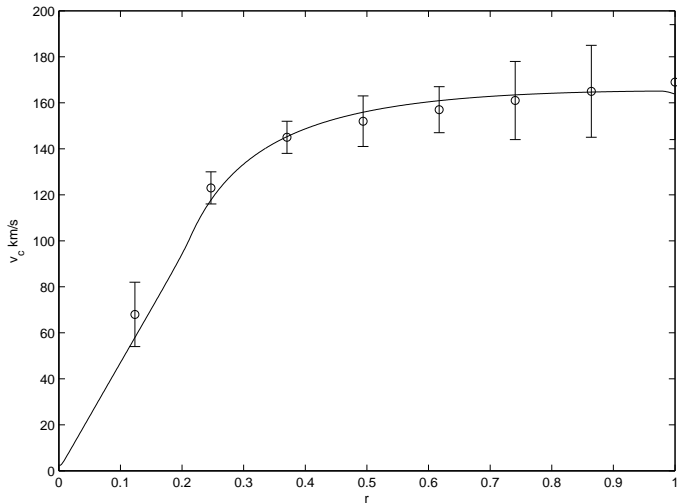
The galaxy NGC3877 ($I = 0.0$ and $Q = 2.40$)



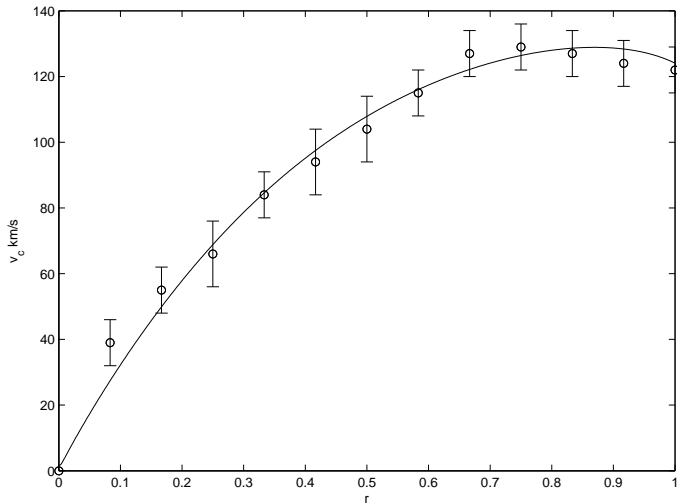
The galaxy NGC3917 ($I = 0.0$ and $Q = 2.30$)



The galaxy NGC3949 ($I = 0.0$ and $Q = 2.35$)



The galaxy NGC4010 ($I = 1.0$ and $Q = 0.65$)



Predicted masses

As explained above we can predict the total mass of these galaxies from the numerical results, and we find the following results.

galaxy	predicted mass in solar masses
NGC3877	$4.7 \cdot 10^{10} M_{\odot}$
NGC3917	$4.0 \cdot 10^{10} M_{\odot}$
NGC3949	$3.5 \cdot 10^{10} M_{\odot}$
NGC4010	$2.2 \cdot 10^{10} M_{\odot}$

The masses agree well with the ones obtained by González, Plata & Ramos-Caro (2010).

Average tangential velocity

For the figures above we used the equation

$$v_c^2 = rU'(r)$$

to define the circular velocity $v_c(r)$ at radius r .

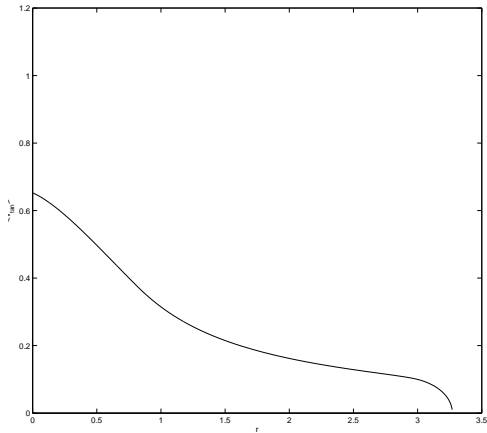
In the context of our model it is natural to compute the averaged tangential velocity of the stars from the phase space density f instead, i.e.,

$$\langle v_{\text{tan}}(r) \rangle = \frac{1}{\Sigma(r)} \int_{\mathbb{R}^2} v_{\text{tan}} f \, dv,$$

where $v_{\text{tan}} = (x_1 v_2 - x_2 v_1)/r = L_z/r$ is the tangential component of the velocity of a particle at x with velocity v .

It turns out that for the steady states we constructed this quantity behaves quite differently from v_c .

The average tangential velocity $\langle v_{\text{tan}} \rangle$ versus r



Open questions

1. It is an interesting problem to construct self-consistent steady states where two types of particles—stars and gas particles—are present and to compute rotation curves from the distribution of the gas particles.
2. An interesting issue is to construct a model of a galaxy with a spherically symmetric central bulge surrounded by a disk and to investigate the rotation curves.
3. A challenge is to find an ansatz function which gives rise to a steady state for which the average tangential velocity agrees with the observed rotation curves.

Thank you!