Stable Big Crunches in General Relativity

Jared Speck

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The Einstein-scalar field equations

$$\boxed{\begin{aligned} \mathbf{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} &= \mathbf{T}_{\mu\nu}, \\ \square_{\mathbf{g}}\phi &= 0 \end{aligned}}$$

$$\mathbf{T}_{\mu\nu} := \mathbf{D}_\mu\phi\mathbf{D}_\nu\phi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{D}\phi\cdot\mathbf{D}\phi$$

- I. Rodnianski and I proved stable blow-up for near-FLRW solutions. Our FLRW had flat $\mathbb{T}^3$ spatial slices.
- Our work showed the stability of certain analytic singular solutions constructed by Andersson-Rendall.

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Data and the maximal development

- Data on $\Sigma_0 \simeq \mathbb{S}^3$ are: $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$
- Our data are close to FLRW data
- Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development $(\mathcal{M}, \mathbf{g}, \phi)$

We aim to understand the basic properties of $(\mathcal{M}, \mathbf{g}, \phi)$.

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FLRW solution

$$\boxed{\mathbf{g}_{FLRW} = -dt \otimes dt + \mathcal{A}^{2/3}(t)\gamma, \qquad \partial_t \phi_{FLRW} = \sqrt{\frac{2}{3}}\mathcal{A}^{-1}(t)}$$

- $\gamma :=$ round metric on $\mathbb{S}^3$ with scalar curvature $2/3$
- There exists an $\mathbb{S}^3-$global $\gamma-$orthonormal frame of $\gamma-$Killing vectorfields denoted by $\mathcal{Z} := \{Z_{(1)}, Z_{(2)}, Z_{(3)}\}$. Moreover, $\mathcal{L}_{Z_{(A)}} Z_{(B)} = \frac{2}{3}\epsilon_{ABC} Z_{(C)}$.
- $\gamma^{-1} = \sum_{A=1}^{3} Z_{(A)} \otimes Z_{(A)}$

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The role of the frame $\mathcal{Z}$

- Recall: $\mathcal{L}_Z \gamma = 0$ for $Z \in \mathcal{Z}$
- For data near FLRW, $\mathcal{L}_Z$ (data) is small; Z is approximately Killing for perturbed solutions

We will propagate a kind of smallness all the way to the Bang/Crunch by commuting the equations with $\mathcal{L}_Z$.

- Key difficulty: energies can blow-up near the Bang/Crunch

What else is different compared to the $\mathbb{T}^3$ case?

- Spatial curvature is initially order 1 rather than small
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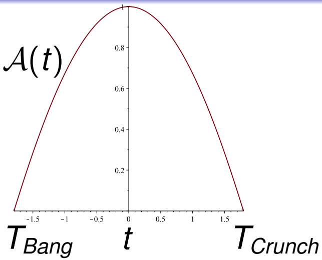
Scale factor and singularities



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- $|\mathbf{Riem}_{FLRW}|^2(t) \sim \mathcal{A}^{-4}(t)$

$\implies$ FLRW blows-up along $\Sigma_{T_{Bang}}, \Sigma_{T_{Crunch}}$

- Key estimate: $\mathcal{A}'(t) \sim -1$ near T_{Crunch}

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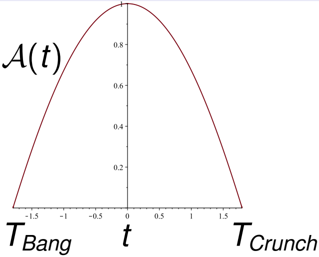

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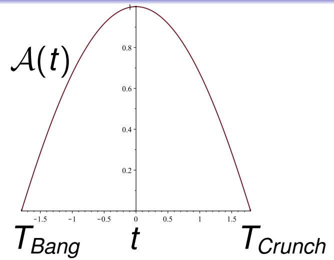

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Context and first statement of main result

Theorem (Hawking)

Assume

- $(\mathcal{M}, \mathbf{g}, \phi)$ *is the maximal globally hyperbolic development of data* $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$ *on* $\mathring{\Sigma} \simeq \mathbb{S}^3$
- $\mathbf{Ric}_{\alpha\beta}\mathbf{X}^{\alpha}\mathbf{X}^{\beta} \geq 0$ *for timelike* $\mathbf{X}$ *(strong energy)*
- $\mathrm{tr}\mathring{k} > C > 0$

Then no future-directed timelike geodesic emanating from $\mathring{\Sigma}$ *is longer than* $\frac{3}{C}$.

Main new result: For near-FLRW data on Σ_0, there is past and future incompleteness caused by uniform curvature blow-up along spacelike hypersurfaces.

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Other contributors

Many researchers have studied gravitational singularities:

Partial list of contributors

Aizawa, Akhoury, Andersson, Anguige, Aninos, Antoniou, Barrow, Béguin, Belinskii, Berger, Beyer, Chitré, Christodoulou, Chrusciel, Claudel, Coley, Cornish, Damour, Dafermos, Eardley, Ellis, Elskens, van Elst, Fournodavlos, Garfinkle, Goode, Grubišić, Heinzle, Henneaux, Hsu, Isenberg, Kasner, Khalatnikov, Kichenassamy, Koguro, LeBlanc, LeFloch, Levin, Liang, Lifschitz, Lim, Luk, Misner, Moncrief, Newman, Nicolai, Reiterer, Rendall, Ringström, Röhr, Sachs, Saotome, Ståhl, Tod, Trubowitz, Uggla, Wainwright, Weaver, Woolgar $\cdots$

Previous work (except Dafermos, Luk) was heuristic, relied on symmetry assumptions, or proved existence (scattering-type problem) rather than stability.

First gauge choice

$$\mathbf{g} = -n^2 dt \otimes dt + \underbrace{g_{AB}\theta^{(A)} \otimes \theta^{(B)}}_{g}$$

- n is the "lapse"
- g is a Riemannian metric on Σ_t with connection ∇
- The $\theta^{(A)}$ are the global $\mathbb{S}^3$ one-forms initially verifying $\theta^{(A)}(Z_{(B)}) = \delta^A_B$
- We propagate $\theta^{(A)}$ and $Z_{(A)}$ by $\underline{\mathcal{L}}_{\hat{N}}\theta^{(A)} = \underline{\mathcal{L}}_{\hat{N}}Z_{(A)} = 0$
- $\hat{N} = n^{-1}\partial_t$ is the unit normal to Σ_t
- $\underline{\mathcal{L}} :=$ Lie derivative followed by projection onto Σ_t
- For FLRW: $n = 1$, $g_{AB} = A^{2/3}\delta_{AB}$, $\partial_t\phi = \sqrt{\frac{2}{3}}A^{-1}$

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CMC gauge and the elliptic lapse PDE

We impose $\mathrm{tr}_g k = -\mathcal{A}'\mathcal{A}^{-1}$ along Σ_t, where $k := -(1/2)\underline{\mathcal{L}}_{\hat{N}} g$. Using the Hamiltonian constraint, we find:

$$\mathcal{A}\Delta_g n - (\mathcal{A}')^2\mathcal{A}^{-1}(n-1) = \underbrace{2\sqrt{\frac{2}{3}}\mathcal{A}^{-1}\left[n^{-1}\mathcal{A}\partial_t\phi - \sqrt{\frac{2}{3}}\right]}_{\text{key linear term}} + \text{Error}$$

- Complementary linear term appears in ϕ equation
- ∞ propagation speed $\rightarrow$ synchronizes the singularity
- Could replace CMC gauge with a parabolic gauge in which for $t > 0$: $\mathcal{A}\Delta_g \rightarrow -(1/\lambda)\partial_t + \mathcal{A}\Delta_g$, $\lambda \gg 1$

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The complementary term

In our gauge, $\Box_{\mathbf{g}}\phi = 0$ is equivalent to

$$-\partial_t\left(n^{-1}\mathcal{A}\partial_t\phi - \sqrt{\frac{2}{3}}\right) + n\mathcal{A}\Delta\phi = \underbrace{\sqrt{\frac{2}{3}}\mathcal{A}'\mathcal{A}^{-1}(n-1)}_{\text{key linear term}} + \text{Error}$$

Detailed statement of main result

Theorem (JS; Stable blow-up (to appear))

Consider near-FLRW ($H^{12}-$close) Einstein-scalar field data on $\Sigma_0 = \mathbb{S}^3$ of small size ϵ.

- *Smallness of the metric data $(\mathring{g}, \mathring{k})$ is measured by* $\| |\mathcal{L}_{\mathcal{Z}}^{\leq 12}(\mathring{g} - \gamma)|_{\gamma} \|_{L^2_{\gamma}(\Sigma_0)} + \| |\mathcal{L}_{\mathcal{Z}}^{\leq 11} \hat{\mathring{k}}|_{\gamma} \|_{L^2_{\gamma}(\Sigma_0)}$
- $\hat{\mathring{k}} :=$ *trace-free part of second fundamental form of* Σ_0
- $\mathcal{M} = \cup_{t \in (T_{Bang}, T_{Crunch})} \Sigma_t$, Σ_t *has CMC* $-\frac{1}{3}\mathcal{A}'(t)\mathcal{A}^{-1}(t)$
- *Energy bound:* $\mathcal{E}_{(\leq 12)}(t) \lesssim \epsilon^2 \mathcal{A}^{-c\sqrt{\epsilon}}(t)$
- *Convergence and Stability:* $\mathcal{A}\partial_t \phi$, n, $\mathcal{A}k^{\#}$, $\mathcal{A}^{-1}\sqrt{g}$ *have finite, near-FLRW limits as* $t \to T_{Bang}, T_{Crunch}$
- *SCC:* $|\mathbf{Riem}|^2_{\mathbf{g}}$ *blows up uniformly like* $\mathcal{A}^{-4}(t)$ *near* $\Sigma_{T_{Bang}}, \Sigma_{T_{Crunch}}$
- *VTD: Spatial derivatives negligible near* $\Sigma_{T_{Bang}}, \Sigma_{T_{Crunch}}$

Proof overview

Two main steps:

1. High-order energy estimates $\mathcal{E}_{(\leq 12)}(t) \leq \epsilon^2 \mathcal{A}^{-c\sqrt{\epsilon}}(t)$ near the Bang and Crunch
2. Improved L^∞ estimates at the low orders showing that spatial derivatives become negligible near the Bang and Crunch. "**V**elocity **T**erm **D**ominated"

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The key monotonicity identity near T_{Crunch}

- $\mathcal{E}[\phi](t) := \int_{\Sigma_t} (n^{-1}\mathcal{A}\partial_t\phi - \sqrt{2/3})^2 + \mathcal{A}^2|\nabla\phi|_g^2$
- $Q(t) := 2\sqrt{\frac{2}{3}}\frac{\mathcal{A}'(t)}{\mathcal{A}(t)}\int_{\Sigma_t}\left(n^{-1}\mathcal{A}\partial_t\phi - \sqrt{\frac{2}{3}}\right)(n-1)$

Energy estimate for ϕ:

$$\frac{d}{dt}\mathcal{E}[\phi](t) = -Q(t) + \underbrace{C\mathcal{A}'(t)\int_{\Sigma_t}\mathcal{A}|\nabla\phi|_g^2}_{\text{Negative near } T_{Crunch}} + \text{Error}$$

Elliptic estimate for n:

$$\int_{\Sigma_t}\mathcal{A}|\nabla n|_g^2 + \mathcal{A}^{-1}(n-1)^2 = Q(t) + \text{Error}$$

- Now just use the lapse estimate to substitute for $Q(t)$ $\Longrightarrow$ quantitative monotonicity up to errors!

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Borderline error terms

Bound for data of size ϵ (can be saturated):

- $\|\hat{k}\|_{L^\infty(\Sigma_t)} \approx \epsilon \mathcal{A}^{-1}(t) \sim \epsilon (T_{Crunch} - t)^{-1}$

Thus, there are borderline energy terms such as $\|\hat{k}\|_{L^\infty(\Sigma_t)} \mathcal{E}[\underline{\mathcal{L}}_Z g](t) \sim c\epsilon (T_{Crunch} - t)^{-1} \mathcal{E}[\underline{\mathcal{L}}_Z g](t)$:

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Less singular behavior for lower derivatives

By allowing derivative loss, we can derive better estimates at the lower derivative levels.

The Einstein evolution equation for $\mathcal{A}\hat{k}^{\#}$:

$$\left|\partial_t \left|\mathcal{A}\hat{k}^{\#}\right|\right| \leq \left|\mathcal{A}^{1/3}\left\{\mathcal{A}^{2/3}Ric^{\#} - \frac{2}{9}I\right\}\right| + \cdots$$

Bootstrapping $\mathcal{E}[\mathcal{L}_{\mathcal{Z}}^{\leq 4}g](t) \leq \epsilon^2\mathcal{A}^{-\sigma}(t)$ for a small parameter σ, we easily deduce

$$\left\|\mathcal{A}^{2/3}Ric^{\#} - \frac{2}{9}I\right\|_{L^\infty(\Sigma_t)} \lesssim \epsilon\mathcal{A}^{-2/3-c\sigma} \sim \epsilon(T_{Crunch} - t)^{-2/3-c\sigma}$$

Thus, $\left|\partial_t \left|\mathcal{A}\hat{k}^{\#}\right|(t)\right| \lesssim \underbrace{\epsilon(T_{Crunch} - t)^{-1/3-c\sigma}}_{\text{Integrable!}} + \cdots$

Integrating, we obtain (for σ small and $0 \leq t < T_{Crunch}$):

$$\left|\mathcal{A}\hat{k}^{\#}\right|(t) \lesssim \left|\mathcal{A}\hat{k}^{\#}\right|(0) + \epsilon \lesssim \epsilon.$$

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Proof comments

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Why are scalar fields special?

- CMC lapse PDE with a scalar field:

$$\begin{aligned}\Delta_g n - \left\{ (\mathcal{A}')^2 \mathcal{A}^{-2} + \frac{2}{3}\mathcal{A}^{-2/3} \right\} (n-1) \\ = n \left\{ R_g - \frac{2}{3}\mathcal{A}^{-2/3}) \right\} - n\nabla\phi \cdot \nabla\phi \\ = \text{only spatial derivatives}\end{aligned}$$

- The absence of time derivatives is $\simeq \exists!$ characteristic cone in the Einstein-scalar field system.

- For some other matter models, time derivatives appear. Consequently, we are not able to show that spatial derivatives effectively decouple and become negligible.

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Thank you