Transition conditions for isolated bodies

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Introduction

A paper by L. Andersson, T. Oliynyk and B.S. show the existence of dynamical solutions of Einstein's field equations in harmonic coordinates with matter sources described by elasticity.

If data are given which satisfy the constraints, the harmonicity conditions, the boundary conditions for the matter and further transition conditions, we have a full solution of the problem.

The plan of the talk:

1. Transition conditions.

2. Elastic bodies in General Relativity

3. Constraints, harmonicity , the boundary conditions for the matter and transition conditions.

1. Transition conditions.

Consider the wave equation on Minkowski space with a spatially compact source.

$$-\partial_t^2 u + \Delta u = \rho$$

(ρ positv for $0 \leq r \leq 1$ and ρ vanishes for $r > 1$)

If we take smooth data we must have a discontinuity of u_{tt} at $r = 1$.

However, if we take data with the right jump in u_{rr}

$$[\partial_r^2 u] = (\partial_r^2 u^+ - \partial_r^2 u^-)_{r=1} = \rho(1, \theta, \phi)$$

then u_{tt} is continuous at the boundary of the support.

Differentiating the equation:

$$-\partial_t^2 u + \Delta u = \rho$$

$$-\partial_t^3 u + \Delta \partial_t u = \partial_t \rho$$

$$-\partial_t^4 u + \Delta \partial_t^2 u = \partial_t^2 \rho$$

$$-\partial_t^4 u + \Delta\Delta u = \rho + \partial_t^2 \rho$$

$$\ldots\ldots$$

We obtain conditions on the data at $r = 1$ if we want $\partial_t^s u$ to be continuous at $r = 1$ for $s = 1, 2 \ldots n$.

If we want the solution to be have some differentiability inside and outside, we must satisfy the transition conditions up to some order.

In the paper

L. Andersson and T.A. Oliynyk, J. Differential Equations 256 (2014), 20232078.

A transmission problem for quasi-linear wave equations.

the above is generalized to systems of quasi-linear wave equations.

2. Elastic bodies in General Relativity

An elastic material in relativity is described as follows:

The basic field is the configuration, a map

$$\mathbf{f^A(x^\mu) : W \to \Omega}$$

from the material in space time to the abstract body manifold. The energy density

$$\rho(\mathbf{f^A}, \mathbf{H^{AB}}) \ , \quad \mathbf{H^{AB}} = \mathbf{f^A_\mu f^B_\nu g^{\mu\nu}}$$

defines a Lagrangian and the energy momentum tensor of the material

$$\mathbf{T}_{\mu\nu} = \rho \mathbf{u}_\mu \mathbf{u}_\nu + \mathbf{n}\tau_{\mathbf{AB}} \mathbf{f^A_\mu f^B_\nu}$$

(The matter flow $\mathbf{u}^\mu$ and the number density n are determined by the configuration $\mathbf{f^A}$.)

To deal with the free boundary value problem the deformation $\phi^{\mathbf{i}}(\mathbf{X^0}, \mathbf{X^B})$ is defined by

$$\mathbf{f^A}(\mathbf{X^0}, \phi^{\mathbf{i}}(\mathbf{X^0}, \mathbf{X^A})) = \mathbf{X^A}$$

and the matter equations are moved to the manifold $\mathbf{\Omega} \times \mathbf{R}$.

The field equation in harmonic coordinates:

$$-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{g}^{\mu\nu}\mathbf{g}_{\alpha\beta,\mu\nu} + \mathbf{H}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g}) = \mathbf{2}\kappa(\mathbf{T}_{\alpha\beta} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{T}\mathbf{g}_{\alpha\beta})$$

The field equation for the metric live in space time; the matter equation on the body!

We solve this difficulty (as in the static situation) by moving the field equations to the "extended body".

An extension operator defines the extended deformation $\tilde{\phi}$.

We use $\tilde{\phi}$ to move the field equations from space time to the extended body. We treat the components of the metric — in harmonic coordinates— as scalars:

$$\gamma_{\mu\nu}(\mathbf{X^0}, \mathbf{X^A}) = \mathbf{g}_{\mu\nu}(\mathbf{X^0}, \tilde{\phi}(\mathbf{X^0}, \mathbf{X^A})$$

Field equations: (unknown: $\gamma_{\mu\nu}, \phi^{\mathbf{i}}$)

$$\frac{\partial}{\partial \mathbf{X^K}} \left(\mathbf{a^{IK}}(\gamma, \partial\tilde{\phi}) \frac{\partial \gamma_{\mu\nu}}{\partial \mathbf{X^I}} \right) = \tilde{\mathbf{H}}_{\mu\nu}(\gamma, \partial\gamma, \tilde{\phi}) + \tilde{\mathbf{T}}_{\mu\nu}$$

Matter equations:

$$\frac{\partial}{\partial \mathbf{X^K}} (\mathbf{F^K_i})(\gamma, \partial\gamma, \phi) = \mathbf{w_i}(\gamma, \partial\gamma, \phi)$$

We have now a well defined PDE problem if we add boundary conditions for the material.

γ is defined on the extended body and ϕ on the body.

Andersson, Oliynyk, Schmidt show:

(gr–qc, to appear in "Archive for Rational Mechanics and Analysis")

For given initial data this time dependent boundary initial value problem has a solution — if the data satisfy the compatibility conditions.

If the data satisfy the constraints, the transition conditions and the boundary conditions for the matter we have a solution of Einstein's field equations.

It remains to show that such data exist.

3. Constraints and Compatibility conditions

I will give an outline how this problem could be solved. This was recently developed with Todd.

I will just consider the conditions of lowest order and assume that the metric and all first derivatives are continuous at the boundary. However, the second derivatives normal to the boundary may have finite jumps.

We consider these condition first in space time and use variables:

$$\mathbf{u}^{\alpha\beta} = \sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta} - \eta^{\alpha\beta} \ , \ |\mathbf{g}| = -\mathbf{det}(\mathbf{g}^{\alpha\beta})$$

$$\tilde{\mathbf{g}}^{\alpha\beta} = \eta^{\alpha\beta} + \mathbf{u}^{\alpha\beta}$$

$$\tilde{\mathbf{g}}^{\mu\nu}\partial_\mu\partial_\nu\mathbf{u}^{\alpha\beta} + \mathbf{H}^{\alpha\beta}(\mathbf{u}, \partial\mathbf{u}) = 4|\mathbf{g}|\kappa\mathbf{T}^{\alpha\beta}$$

The constraints: $\alpha = \mathbf{0}, \beta$

$$\tilde{\mathbf{g}}^{\mu\nu}\partial_\mu\partial_\nu\mathbf{u}^{\mathbf{0}\beta} + \mathbf{H}^{\mathbf{0}\beta}(\mathbf{u}, \partial\mathbf{u}) = 4|\mathbf{g}|\kappa\mathbf{T}^{\mathbf{0}\beta}$$

We put

$$\mathbf{u}^{\mathbf{00}} = \mathbf{U} \ , \mathbf{u}^{\mathbf{0i}} = \mathbf{W}^{\mathbf{i}} \ , \mathbf{u}^{\mathbf{ik}} = \mathbf{Z}^{\mathbf{ik}}$$

The harmonicity condition and their time derivative are: $\dot{} = \partial_0$

$$\dot{\mathbf{U}} = -\partial_k \mathbf{W}^k \ , \dot{\mathbf{W}}^i = -\partial_k \mathbf{Z}^{ik}$$

$$\ddot{\mathbf{U}} = \partial_i \partial_k \mathbf{Z}^{ik} \ , \ddot{\mathbf{W}}^i = -\partial_k \dot{\mathbf{Z}}^{ik}$$

For simplicity we consider the case $\mathbf{W}_i = 0, \dot{\mathbf{f}}^A = 0$. Then the only constraint is

$$\tilde{\mathbf{g}}^{\mu\nu} \partial_\mu \partial_\nu \mathbf{u}^{00} + \mathbf{H}^{00}(\mathbf{u}, \partial \mathbf{u}) = 4|\mathbf{g}|\kappa \mathbf{T}^{00}$$

or using the harmonicity conditions:

$$-\partial_i \partial_k \mathbf{Z}^{ik} + \Delta \mathbf{U} - \mathbf{U} \partial_i \partial_k \mathbf{Z}^{ik} + \mathbf{Z}^{ik} \partial_i \partial_k \mathbf{U} +$$

$$+\mathbf{H}^{00}(\mathbf{u}, \partial \mathbf{u}) = 4|\mathbf{g}|\kappa \mathbf{T}^{00}$$

For $\kappa = 0$ we have a solution

$$\bar{\mathbf{U}} = 1, \ \bar{\mathbf{Z}}^{IK} = \delta^{ik}, \ \bar{\mathbf{f}}^A = \mathrm{id}$$

which satisfies the constraint above. This is the trivial, stress–free configuration without gravity.

Choose $\delta \mathbf{Z}^{\mathbf{ik}}$, $\delta \mathbf{f}^{\mathbf{A}}$ small and put $\bar{\mathbf{Z}}^{\mathbf{ik}} + \delta \mathbf{Z}^{\mathbf{ik}}$, $\bar{\mathbf{f}}^{\mathbf{A}} + \delta \mathbf{f}^{\mathbf{A}}$ in the constraint. Then we can solve for small κ the constraint for $\bar{\mathbf{U}} + \delta \mathbf{U}$ by the implicit function theorem. The relevant linearized map is the flat Laplacian.

Hence there is a functional

$$\mathbf{U} = \mathbf{F}[\mathbf{Z}^{\mathbf{ik}},\ \mathbf{f}^{\mathbf{A}}, \kappa]$$

which gives solutions of the constraint provided we choose $\mathbf{Z}^{\mathbf{ik}}$, $\mathbf{f}^{\mathbf{A}}$ near trivial stress free solution.

Next we try to impose further conditions on $\mathbf{Z^{ik}}$, $\mathbf{f^A}$ such that the compatibility conditions and the material boundary conditions hold.

The goal — in lowest order — is to make the second time derivatives of the metric continuous at the boundary of the body.

From the harmonicity condition $\ddot{\mathbf{U}} = \partial_\mathbf{i}\partial_\mathbf{k}\mathbf{Z^{ik}}$ we see that $\partial_\mathbf{i}\partial_\mathbf{k}\mathbf{Z^{ik}}$ must be continuous at the boundary.

Next we consider $\ddot{\mathbf{Z}}^\mathbf{ik}$ given by the field equations

$$\tilde{\mathbf{g}}^{\alpha\beta}\partial_\mu\partial_\nu\mathbf{Z^{ik}} + \mathbf{H^{ik}}(\mathbf{u},\partial\mathbf{u}) = 4|\mathbf{g}|\kappa\mathbf{T^{ik}}$$

or

$$(-\mathbf{1}+\mathbf{U})\ddot{\mathbf{Z}}^\mathbf{ik} + \boldsymbol{\Delta}\mathbf{Z^{ik}} + \mathbf{Z^{ab}}\partial_\mathbf{a}\partial_\mathbf{b}\mathbf{Z^{ik}} + \mathbf{H^{ik}}(\mathbf{u},\partial\mathbf{u}) =$$

$$= 4|\mathbf{g}|\kappa\mathbf{T^{ik}}$$

$\ddot{\mathbf{Z}}^\mathbf{ik}$ is only continuous at the boundary if the jump of the matter is absorbed by $\partial^\mathbf{2}_\mathbf{n}\mathbf{Z^{ik}}$, the second derivatives normal to the boundary.

To simplify the notation assume that we have coordinates such that $\mathbf{x^3} = \mathbf{1}$ is the boundary and $\mathbf{n_i} = (\mathbf{0},\mathbf{0},\mathbf{1})$ the normal.

The matter boundary conditions $\mathbf{T^{ik} n_k = 0}$ imply that the second derivatives of $\mathbf{Z^{3i}}$ must be continuous in order that $\mathbf{\ddot{Z}^{3i}}$ is continuous. (Then $\mathbf{\ddot{U}}$ is also ok.)

$\mathbf{Z^{11}, Z^{12}, Z^{22}}$ must have jumps in their second normal derivatives at the boundary because the matter terms do not vanish.

Consider:

$$(-\mathbf{1} + \mathbf{U})\ddot{\mathbf{Z}}^{\mathbf{11}} + \mathbf{\Delta Z^{11}} + \mathbf{Z^{ab}}\partial_{\mathbf{a}}\partial_{\mathbf{b}}\mathbf{Z^{11}} + \mathbf{H^{11}}(\mathbf{u}, \partial\mathbf{u}) =$$

$$= \mathbf{4|g|}\kappa\mathbf{T^{11}}$$

This implies for the jump

$$(\mathbf{1} + \mathbf{Z^{33}})[\partial_{\mathbf{3}}^{\mathbf{2}}\mathbf{Z^{11}}] = \mathbf{4|g|}\kappa\mathbf{T^{11}}|_{\mathbf{b}}$$

similarly

$$(\mathbf{1} + \mathbf{Z^{33}})[\partial_{\mathbf{3}}^{\mathbf{2}}\mathbf{Z^{12}}] = \mathbf{4|g|}\kappa\mathbf{T^{12}}|_{\mathbf{b}}$$

$$(\mathbf{1} + \mathbf{Z^{33}})[\partial_{\mathbf{3}}^{\mathbf{2}}\mathbf{Z^{22}}] = \mathbf{4|g|}\kappa\mathbf{T^{22}}|_{\mathbf{b}}$$

For elastic matter we have $\mathbf{T^{ik} = \tilde{T}^{ik}(U, Z^{ab}, f^A)}$ or using the functional solving the constraint

$$\mathbf{T^{ik}} = \mathbf{\tilde{T}^{ik}}(\mathbf{F}[\mathbf{Z^{ik}},\ \mathbf{f^A}, \kappa], \mathbf{Z^{ab}}, \mathbf{f^A})$$

On the left side of the jump conditions we have functions at the boundary. To calculate the right side, however, we need $\mathbf{Z^{ik}}$ as functions on the Cauchy surface to evaluate the constraint functional.

To get around this we choose some fixed map

$$(\mathbf{i}, \mathbf{k} = \mathbf{1}, \mathbf{2})\ \mathbf{E} : (\mathbf{Z^{ik}}, \partial \mathbf{Z^{ik}}, \partial^2 \mathbf{Z^{ik}})|_{\mathbf{Bound}} \rightarrow \mathbf{Z^{ik}}|_{\mathbf{R^3}}$$

(We have to choose E such that the constraint can be solved.)

Now we can write the jump conditions as equations just for $\mathbf{z} = (\mathbf{Z^{ik}}, \partial \mathbf{Z^{ik}}, \partial^2 \mathbf{Z^{ik}})|_{\mathbf{Bound}}$ (i,k=1,2)

$$\mathbf{T^{ik}} = \tilde{\mathbf{T}}^{\mathbf{ik}}(\mathbf{F}[\mathbf{E}(\mathbf{z}), \mathbf{f^A}, \kappa], \mathbf{Z^{ab}}, \mathbf{f^A})$$

This way we finally obtain a system of three equations.

We have a solution of these equations, determined by our stress free, $\kappa = \mathbf{0}$, flat solution.

We can solve for small κ by the implicit function theorem, because we can solve for ∂_3^2. In the relevant linear operator the complicated left side of the equations does not enter because $\kappa = \mathbf{0}$. (We still have to arrange the function spaces properly.)

Finally we have to consider the boundary conditions for the matter.

These are analised in the thesis of Wernig–Pichler.

$\mathbf{T^{ik}}$ **is a function of** $(\mathbf{g}_{\mu\nu}, \partial_\mu \mathbf{f^A})$**. For materials with the stored energy we consider, it turns out that the equations** $\mathbf{T^{ik} n_k}$ **can at the boundary be solved for** $\partial_3 \mathbf{f^A}$ **if the configuration is given on the boundary.**

Because the geometry enters the condition we have to solve the matter condition together with the jump conditions.

(May be the best is if we solve for U, the matter conditions and the jumps together.)

We looked at the compatibility conditions in space. The analysis must however be done in the material setting. $\mathbf{X^\Delta} = (\mathbf{X^0}, \mathbf{X^A})$

$$\bar{\mathbf{u}}^{\alpha\beta}(\mathbf{X^\Gamma}) = \mathbf{u}^{\alpha\beta}(\tilde{\phi}^\sigma(\mathbf{X^\Gamma}))$$

Or

$$\mathbf{u}^{\alpha\beta}(\mathbf{x}^\sigma) = \bar{\mathbf{u}}^{\alpha\beta}(\tilde{\mathbf{f}}^{\mathbf{\Gamma}}(\mathbf{x}^\sigma))$$

The principle part of the equations on the extended body is

$$\tilde{\mathbf{g}}^{\mu\nu}\partial_\mu\tilde{\mathbf{f}}^{\mathbf{\Gamma}}\partial_\nu\tilde{\mathbf{f}}^{\mathbf{\Delta}}\partial_{\mathbf{\Delta}}\partial_{\mathbf{\Lambda}}\bar{\mathbf{u}}^{\alpha\beta}(\mathbf{X^\Gamma})$$

For the solution on which the implicit function theorem is based we have $\tilde{\phi}^\mu = \mathbf{id}$**. Hence the relevant linear operator in the material picture will be the one we considered in space time.**

Up to now we looked just at the conditions in lowest order.

However, we need the conditions at least up to the fourth oder in time to obtain a reasonable solution.

We have to differentiate the equations several times and check that all works.