Non-existence of time-periodic vacuum spacetimes

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(joint work with Spyros Alexakis and Arick Shao)

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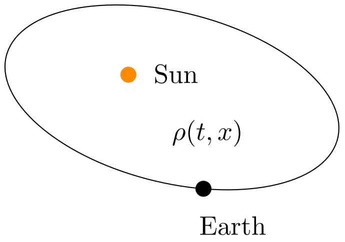
Outline

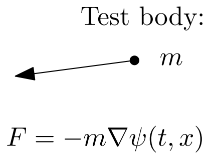
1. *2-body problem in general relativity*
   - Predictions of post-Newtonian theory
2. *Time-periodic vacuum spacetimes*
   - Statement of the main theorem
3. *Final state conjecture*
   - Dynamical non-radiating spacetimes
   - Relation to black hole uniqueness problem
4. *Proof of the non-existence result*
   - Uniqueness results for ill-posed hyperbolic p.d.e.'s
   - Null geodesics in spacetimes with positive mass

1. *2-body problem in general relativity*

Periodic motion in Newton's theory

Two-body problem in classical celestial mechanics: Kepler orbits.
Space: $\mathbb{R}^3$, Time: $\mathbb{R}$, Newtonian potential: $\psi(t,x)$





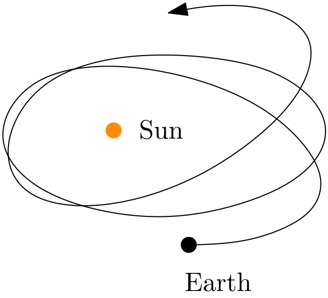
$$\triangle\psi(t,x) = 4\pi\rho(t,x) \quad : \text{mass density}$$

Note: In Newtonian theory the gravitational field can be periodic in time. (Action at a distance)

Post-Newtonian theory

Slow motion / post-Newtonian / weak field approximations:

Einstein-Infeld-Hoffmann, . . . , Damour-Deruelle, Blanchet, . . . , Will-Wiseman, Poisson-Will

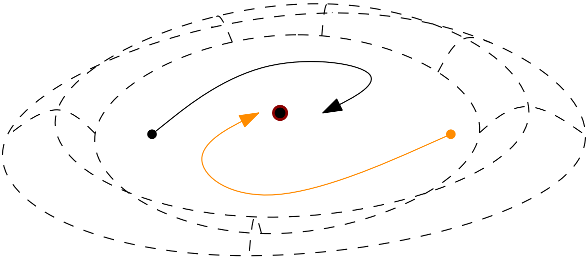


Correction to Newtonian potential: $\psi_{\text{post-Newtonian}} = \psi + \frac{1}{c^2}\omega$

In this approximation circular orbits still possible, but ruled out by *radiation reaction force* in higher orders of the expansion.

Dissipative dynamics in general relativity

Periodic motion should **not exist** in general relativity due to the emission of gravitational waves.

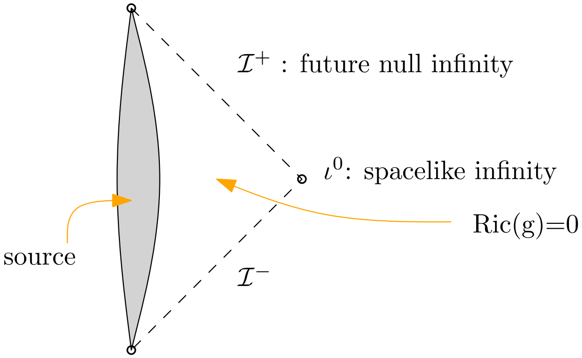


Our main result (joint with SPYROS ALEXAKIS) is that any time-periodic vacuum spacetime is in fact **time-independent**, at least far away from the sources.

2 *Time-periodic vacuum spacetimes*

Isolated self-gravitating systems in general relativity

$(\mathcal{M}^{3+1}, g)$ asymptotically flat, and solution to $\mathrm{Ric}(g) = 0$ outside a spatially compact set.

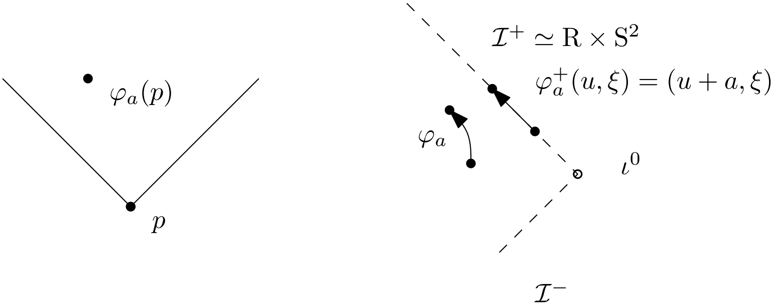


Future and past null infinities $\mathcal{I}^+, \mathcal{I}^- \simeq \mathbb{R} \times \mathbb{S}^2$ are complete.

Notion of time-periodicity

Definition

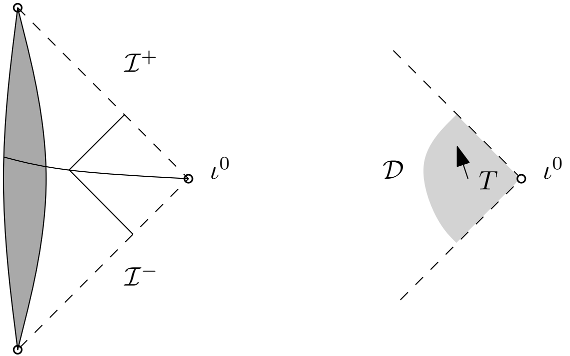
An asymptotically flat spacetime is called *time-periodic* if there exists a discrete isometry φ_a with time-like orbits.



Then in fact φ_a extends to a map φ_a^+ on future null infinity where it is an affine translation along the generating geodesics. (Similarly at past null infinity.)

Theorem (Alexakis-S. '15)

Any asymptotically flat solution $(\mathcal{M}, g)$ to the Einstein vacuum equations arising from a regular initial data set which is time-periodic (near infinity), must be **stationary** *near infinity.*



The theorem asserts that there exists a time-like vectorfield T on an **arbitrarily small** neighborhood $\mathcal{D}$ of infinity such that $\mathcal{L}_T g = 0$ on $\mathcal{D}$.

Previous results

Early work:

PAPAPETROU '57-'58 (weak field approximation, "non-singular" solutions, strong time-periodicity assumption)

Recent work:

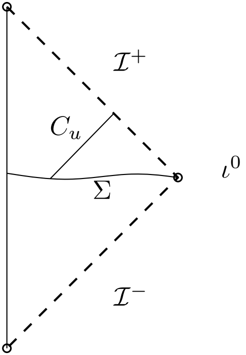
GIBBONS-STEWART '84, BICAK-SCHOLZ-TOD '10 (contains ideas how to exploit time-periodicity, stationarity inferred under much more restrictive **analyticity** assumption)

Cosmological setting:

TIPLER '79, GALLOWAY '84 (spatially closed case)

3. *Dynamical non-radiating spacetimes and the final state conjecture*

Gravitational radiation



Trautmann-Bondi energy:

The BONDI mass $M(u)$ signifies the amount of energy in the system at time u. It is known to be **positive**: $M(u) \geq 0$ (SCHOEN-YAU,..., CHRUŚCIEL-JEZIERKSI-LESKI, SAKOVICH), and dynamically **monotone decreasing**.

In CHRISTODOULOU-KLAINERMAN the *Bondi mass loss formula* is

$$\frac{\partial M(u)}{\partial u} = -\frac{1}{32\pi} \int_{\mathbb{S}^2} |\Xi|^2 \mathrm{d}\mu_{\overset{\circ}{\gamma}}$$

and it is shown that $\lim_{u \to -\infty} M(u) = M[\Sigma]$ $\lim_{u \to \infty} M(u) = 0$.

$|\Xi|(u, \xi)$: **power** of gravitational waves radiated in direction $\xi \in \mathbb{S}^2$, at time $u \in \mathbb{R}$.

Aside: Gravitational wave experiments

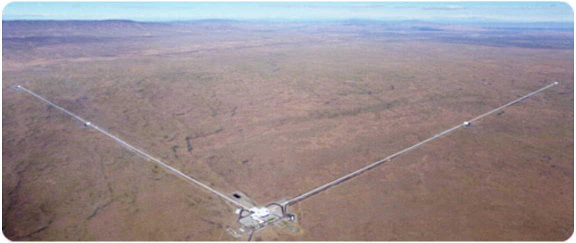
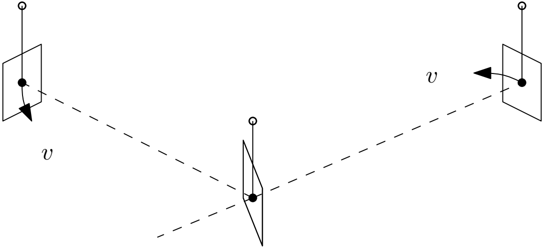


Figure: LIGO, Washington



$$v^{B}_{(A)} = \frac{d_0}{r}\Xi_{AB}(t)$$

Non-radiating spacetimes

Definition

An asymptotically flat spacetime is called *non-radiating* if the Bondi mass $M(u)$ is constant along future (and past) null infinity.

Theorem (Alexakis-S. '15)

Any asymptotically flat solution $(\mathcal{M}^{3+1}, g)$ *to the vacuum equations arising from regular initial data which is assumed to be* **non-radiating**, *and in addition* **smooth at null infinity**, *must be* **stationary** *near infinity.*

Remark:

Here *smooth at null infinity* means in particular that the curvature components ρ admit a full asymptotic expansion near null infinity which is well behaved towards spacelike infinity:

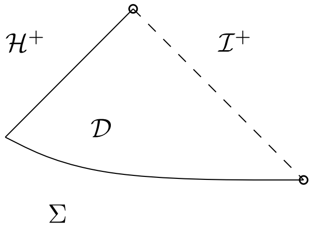
$$\rho \sim \sum_{l=0}^{\infty} \kappa_l(u) r^{k-l} \qquad \lim_{u \to -\infty} |\kappa_l(u)| < \infty \,.$$

Conjectures

The **final state conjecture** gives a *characterisation of all possible end states* of the dynamical evolution in general relativity, as a result of a scenario due to PENROSE invoking both **weak cosmic censorship** and **black hole uniqueness**.

Conjecture

Any smooth asymptotically flat black hole exterior solution to the Einstein vacuum equations which is assumed to be **non-radiating** is *isometric* to the exterior of a KERR solution $(\mathcal{M}, g_{M,a})$.



$\mathcal{I}^+$: future null infinity

$\mathcal{H}^+$: future event horizon

$\mathcal{D}$: black hole exterior

Aside: Soliton resolution conjecture

The energy-critical focusing non-linear wave equation

$$\Box\phi = -\phi^5$$

has soliton solutions

$$\phi_\lambda(t,x) = \frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right) \qquad W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} \qquad \lambda > 0$$

It is expected that for global solutions

$$\|\phi - \left(\phi_L \pm \sum_j \phi_{\lambda_j}\right)\| \longrightarrow 0 \quad (t \to \infty)$$

Established for radial solutions by Duyckaerts-Kenig-Merle '13.

4. *Proof of the theorems:*
extension of a time translation symmetry from infinity

Strategy of Proof

1. Construction of time-like candidate vectorfield T, such that **by time-periodicity**

$$\lim_{r\to\infty} r^k \mathcal{L}_T R = 0 \qquad \forall k \in \mathbb{N}$$

2. Use that by virtue of the vacuum Einstein equations,

$$\Box_g R = R * R$$

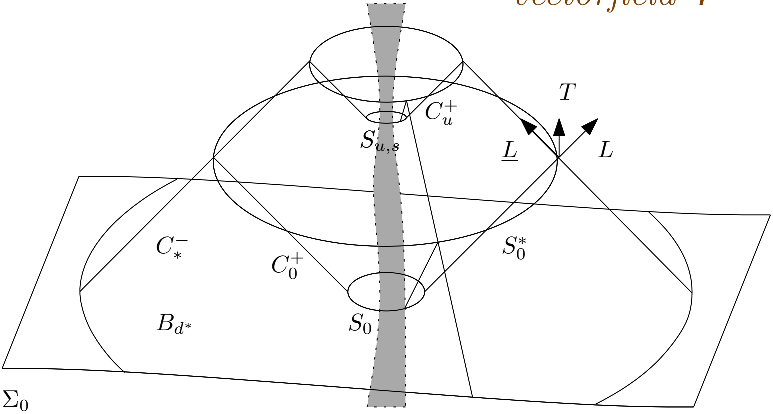
thus

$$\text{“} \Box_g \mathcal{L}_T R = R * \mathcal{L}_T R \text{”} \,.$$

Then use our **unique continuation theorem** which asserts that solutions to wave equations on asymptotically flat spacetimes are uniquely determined if all higher order radiation fields are known, to show that

$$\mathcal{L}_T R = 0 \,.$$

Construction of "candidate" Killing vectorfield T



Define $T = \frac{\partial}{\partial u}$: **binormal** to spheres S_u^*. Then extend inwards by Lie transport along **geodesics**: $[\underline{L}, T] = 0$, $\nabla_{\underline{L}} \underline{L} = 0$, $g(\underline{L}, \underline{L}) = 0$.

Time-periodicity

and time-independence to leading order

The components of the curvature fall off at different rates in the distance (**peeling**):

$$R_{\underline{L}\underline{L}} = \mathcal{O}(r^{-1}) \quad R_{\underline{L}L} = \mathcal{O}(r^{-2}) \quad R_{LL} = \mathcal{O}(r^{-3})$$

$$\lim_{u;r\to\infty} rR_{\underline{L}\underline{L}} = \underline{A} \quad \lim_{u;r\to\infty} r^2 R_{\underline{L}L} = P \quad \lim_{u;r\to\infty} r^3 R_{LL} = A$$

Using the asymptotic laws obtained in CHRISTODOULOU-KLAINERMAN it follows

$$\Xi = 0 \Longrightarrow \underline{A} = -\partial_u \Xi = 0,\ \partial_u P = -\underline{A} = 0$$

however, in general,

$$\partial_u A = \nabla_\xi P + \underline{A}\,.$$

The **idea** is to differentiate a second time,

$$\partial_u^2 A = \nabla_\xi \partial_u P = 0$$

which implies that A is a *linear* function in u,

$$A(u_2, \xi) - A(u_1, \xi) = A_0(\xi)(u_2 - u_1)\,.$$

But **by time-periodicity** $A_0(\xi) = 0$, therefore

$$\partial_u A = 0\,.$$

Note, same conclusion if *instead of* time-periodicity we assume

$$\lim_{u_1 \to -\infty} |A(u_1, \xi)| < \infty$$

This also yields $\nabla_\xi P = 0$, namely that P is spherically symmetric.

Schematically, this was the first step of an **induction** which proves

$$\lim_{u;r\to\infty} r^k \partial_u R = 0 \qquad \forall k \in \mathbb{N}\,.$$

In fact, we show at the same time using the **propagation equations** along outgoing null geodesics

$$\lim_{u;r\to\infty} r^k \partial_u g = 0 \qquad \lim_{u;r\to\infty} r^k \partial_u \Gamma = 0$$

For example, consider $\Gamma = \hat{\chi}$. (Recall we already saw $\Xi = \lim r\underline{\hat{\chi}}$.) Schematically, since by construction $[L, T] = 0$, $T = \partial_u$:

$$L\hat{\chi} = -\alpha \qquad \alpha = R_{LL}$$
$$L\partial_u\hat{\chi} = -\partial_u\alpha$$

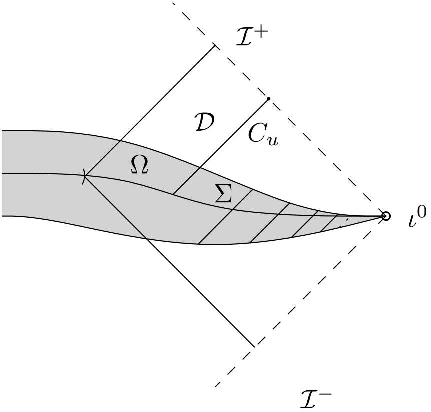
Time-periodicity

and required regularity

In the time-periodic setting the required regularity can be *deduced* from a corresponding regularity assumption on the **initial data**:

$$g|_\Sigma = g_{\mathsf{Kerr}}|_{t=0} + g^\infty$$

$$g^\infty_{\alpha\beta} \sim \sum_l g^l_{\alpha\beta}(\vartheta) r^{k-l}$$



Recall strategy of the proof

1. We have now constructed a time-like candidate vectorfield T, such that **by time-periodicity**, or *alternatively* by assuming a regular expansion,

$$\lim_{r \to \infty} r^k \mathcal{L}_T R = 0 \qquad \forall k \in \mathbb{N}$$

2. Use that by virtue of the vacuum Einstein equations

$$\Box_g R = R * R$$

the Lie derivative of the curvature satisfies a wave equation

$$\text{“ } \Box_g \mathcal{L}_T R = R * \mathcal{L}_T R \text{ ” .}$$

Then apply our **unique continuation theorem** to show that

$$\mathcal{L}_T R = 0 \,.$$

Unique continuation from infinity

Theorem (ALEXAKIS-S.-SHAO '14)

Let $(\mathcal{M}, g)$ *be an asymptotically flat spacetime with* **positive mass**, *and* L_g *a linear wave operator*

$$L_g = \Box_g + a \cdot \nabla + V$$

with suitably fast decaying coefficients a, *and* V. *If* ϕ *is a solution to* $L_g \phi = 0$ *which in addition satisfies*

$$\int_{\mathcal{D}} r^k \phi^2 + r^k |\partial \phi|^2 < \infty$$

where $\mathcal{D}$ *is an arbitrarily small neighborhood of infinity* ι^0, *then*

$$\phi \equiv 0 \qquad : \text{ on } \mathcal{D}' \subset \mathcal{D} \, .$$

Application of the theorem

to the Einstein equations

The application of the theorem is **not** immediate because

$$\Box_g R = R * R$$

is not a scalar equation, but a **covariant equation** for the Riemann curvature tensor. Moreover, $[\Box_g, \mathcal{L}_T] \neq 0$ and differentiating the equation produces **additional terms** which are not in the scope of the theorem:

$$\Box_g \mathcal{L}_T R - [\Box_g, \mathcal{L}_T] R = R * \mathcal{L}_T R + \mathcal{L}_T g * R^2$$

These obstacles can be overcome in the general framework of IONESCU-KLAINERMAN '13 for the extension of Killing vectorfields in Ricci-flat manifolds.

Application of the theorem

IONESCU-KLAINERMAN *approach*

Define *modified* Lie-derivative

$$W = \mathcal{L}_T R - B \cdot R \qquad B = \mathcal{L}_T g + \omega \qquad \nabla_L \omega = \mathcal{M}(\mathcal{L}_T g)$$

which then satisfies a covariant equation

$$\Box W = R \cdot W + \nabla R \cdot \nabla B + R^2 \cdot B + R \cdot \nabla P$$

coupled to o.d.e.'s

$$\nabla_L B = \mathcal{M}(P, B) \qquad \nabla_L P = \mathcal{M}(W, B, P)$$

This is **only** true if $[L, T] = 0$, which we have by construction.

Now choose *Cartesian* coordinates near infinity, such that

$$\Gamma = \mathcal{O}(r^{-2})\,.$$

and apply our **Carleman estimates**.

4. *Proof of the theorems: unique continuation from infinity for linear waves*

Linear theory on Minkowski space

Consider the linear wave equation on $\mathbb{R}^{3+1}$:

$$\Box\phi = 0$$

The **radiation field** is defined by

$$\Psi(u,\xi) = \lim_{r\to\infty}(r\phi)(t = u + r, x = r\xi)\,.$$

Theorem (Friedlander '61)

For solution $\Box\phi = 0$ there is a 1-1 correspondence

$$(\phi|_{t=0}, \partial_t\phi|_{t=0}) \in \dot{\mathrm{H}}^1 \times \mathrm{L}^2 \longleftrightarrow \Psi \in \mathrm{L}^2\,.$$

However, **no generalisations** to perturbations of Minkowski space are known, i.e. for equations

$$\Box_g\phi + a\cdot\nabla\phi + V\phi = 0$$

Counterexamples in linear theory

Question:

Without the finite energy condition, does the vanishing of the radiation field imply the vanishing of the solution?

$$\Psi = 0 \Longrightarrow \phi \equiv 0$$

Answer:

No, because $\phi = \frac{1}{r}$ is a solution, and thus also

$$\phi_i = \partial_{x^i} \frac{1}{r} \sim \frac{1}{r^2}$$

which is a non-trivial solution with $\Psi = 0$.

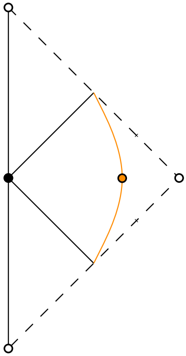
This shows the necessity of the *infinite order vanishing assumption*:

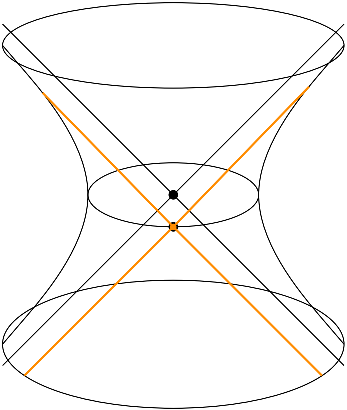
$$\lim_{r \to \infty} (r^k \phi)(u + r, r\xi) = 0 \qquad \forall k \in \mathbb{N} .$$

Obstructions to unique continuation

from infinity

There is an obstruction related to the behavior of light rays.





ALINHAC-BAOUENDI '83: Unique continuation fails across surfaces which are not **pseudo-convex**, in general.

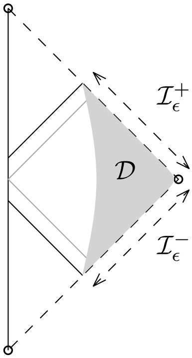
Unique continuation from infinity for linear waves

Theorem (Alexakis-S.-Shao '13)

Let $(\mathcal{M}, g)$ be a perturbation of Minkowski space, and L_g a linear wave operator with decaying coefficients. If ϕ is a solution to $L_g \phi = 0$ which in addition satisfies an infinite order vanishing condition on "at least half" of future and past null infinity, then

$$\phi \equiv 0$$

in a neighborhood of infinity.



Pseudo-convexity

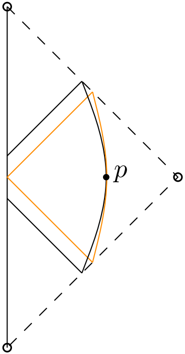
The proof crucially relies on the construction of a family of **pseudo-convex** time-like hypersurfaces.

Definition

A time-like hypersurface $H = \{f = c\}$ is *pseudo-convex* at a point p, if

$$(\nabla^2 f)_p(X, X) < 0$$

for all vectors $X \in \mathrm{T}_p\mathcal{M}$ which
(i) are *null*, $g(X, X) = 0$,
(ii) are tangential to H, $g(X, \nabla f) = 0$.



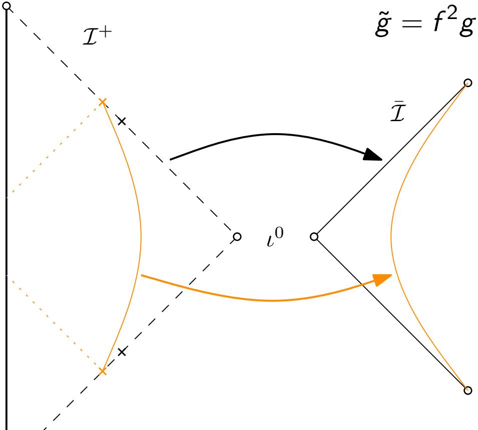
We find a family of pseudo-convex hypersurfaces that **foliate** a neighborhood of infinity and derive a CARLEMAN inequality to prove the uniqueness result.

In fact, we choose

$$f = \frac{1}{(-u+\epsilon)(v+\epsilon)} \qquad u = \frac{1}{2}(t-r)\,, v = \frac{1}{2}(t+r)$$

and consider the conformally inverted metric

$$\tilde{g} = f^2 g$$



Note this is **not** the standard Penrose compactification $\Omega^2 g$ where

$$\Omega = \frac{1}{\sqrt{(1+u^2)(1+v^2)}}$$

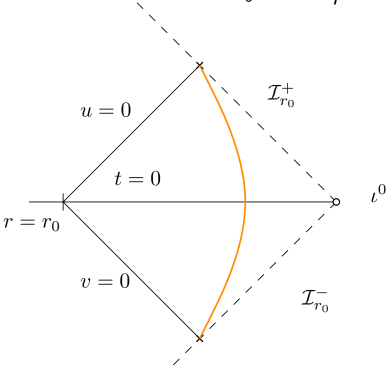
In fact $\tilde{g}$ is *singular*.

In Schwarzschild with $m > 0$,

$$g = -4\left(1 - \frac{2m}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\gamma$$

where, for arbitrary $r_0 > 2m$,

$$v - u = r^* = \int_{r_0}^{r} \frac{1}{1 - \frac{2m}{r}}\mathrm{d}r = r + 2m\log|r - 2m| - r_0^*$$



We set

$$f = \frac{1}{(-u)(v)}$$

and consider the conformally inverted metric

$$\tilde{g} = f^2 g\,.$$

While in Minkowski space

$$-\nabla^2 f(X,X) \sim \frac{\epsilon}{r} \qquad \forall X : g(X,X) = g(X,\nabla f) = 0$$

we find that in Schwarzschild

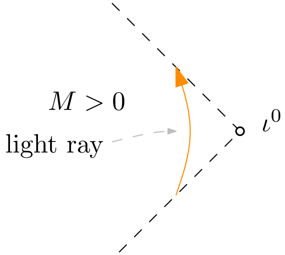
$$-\nabla^2 f(X,X) \sim \frac{2m}{r}\log r - \frac{r_0}{r} > 0$$

for *arbitrarily large* $r_0 > 2m$.

This is the reason unique continuation from infinity holds in an **arbitrarily small neighborhood of infinity** whenever the spacetime has a **positive mass**.

Positive mass

and the behaviour of light rays



This is related to ideas of PENROSE, ASHTEKAR-PENROSE '90, and CHRUSCIEL-GALLOWAY '04 to **characterise** the positivity of mass by the behaviour of null geodesics near infinity.
(See also PENROSE-SORKIN-WOOLGAR '93)

Thank you for your attention!