Fuchsian polygonal surfaces in Minkowski space and the decorated Teichmüller space

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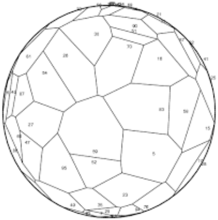
Workshop - DYNAMICS OF SELF-GRAVITATING MATTER
Emile Borel Centre of Henri Poincaré Institute, Paris

Convex polyhedra in Euclidean space



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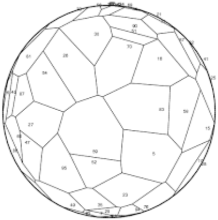

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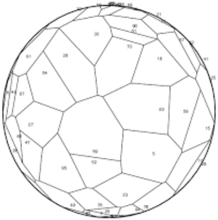

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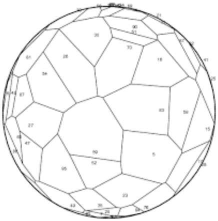

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Notion of sub-cellulation

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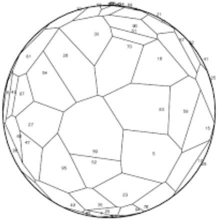

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Notion of sub-cellulation; any cellulation admits triangulations as sub-cellulations.

Alexandrov Theorem

Theorem (A.D. Alexandrov, 1942)

Any locally euclidean metric on $\mathbb{S}^2$ with cone singularities of angle $\leq 2\pi$ is isometric to the boundary of a polyhedron in $\mathbb{R}^3$.

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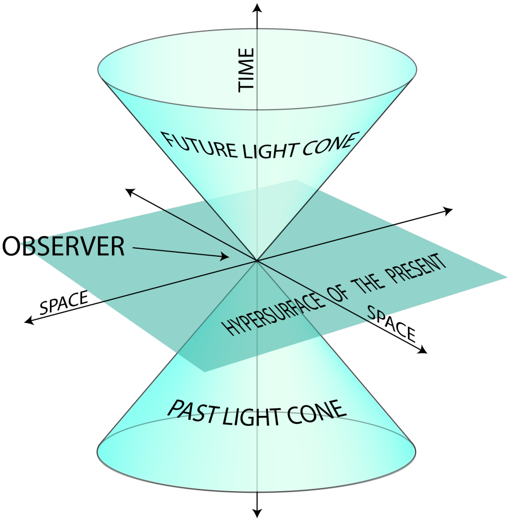
In particular, there is a particular geodesic cellulation associated to the locally euclidean metric. No known algorithm to find this cellulation (for example, starting from an arbitrary geodesic triangulation).

Analog problem in Minkowski space



Polyhedral surface in Minkowski space

Let Γ be the fundamental group of a closed surface S. A *polyhedral embedding* of S in $\mathbb{R}^{1,2}$ is the data of a representation $\rho : \Gamma \to \mathrm{Iso}_0(1,2)$ and a Γ-equivariant embedding $f : \tilde{S} \to \mathbb{R}^{1,2}$ such that the image is spacelike, and locally polyhedral.

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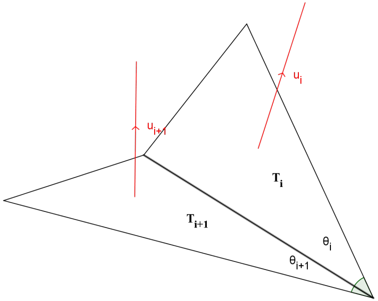
The surface S is then equipped with a locally flat metric with cone singularities.

Cone angles

Proposition

Cone angles are $\geq 2\pi$.

Proof:



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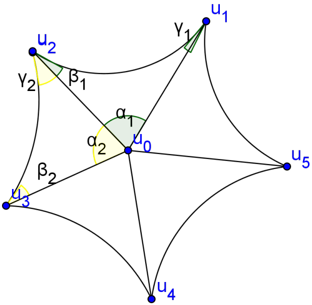

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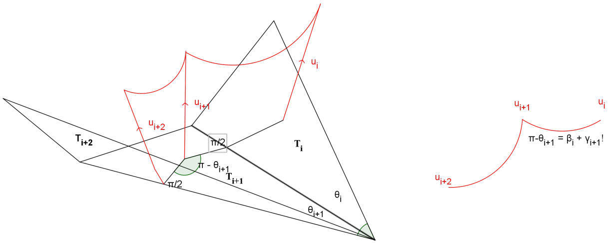

The angle $\beta_{i+1} + \gamma_i$ between two successive hyperbolic edges is $\pi - \theta_i$!

If n is the degree at the vertex, i.e. the number of hyperbolic triangles around u_0:

$$\begin{aligned} 0 < Area &= \sum[\pi - (\alpha_i + \beta_i + \gamma_i)] \\ &= n\pi - \sum \alpha_i - \sum(\beta_i + \gamma_i) \\ &= n\pi - \sum \alpha_i - \sum(\pi - \theta_i) \\ &= n\pi - \sum \alpha_i + \sum \theta_i - n\pi \\ &= \text{euclidean angle} - 2\pi \end{aligned}$$

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Once more, there is a particular geodesic cellulation associated to the locally euclidean metric.
This theorem provides a map between the space $\mathcal{M}^{\geq 2\pi}(S)$ of locally euclidean metric on S with cone angles $\geq 2\pi$ and Teich(S).

Why to go further?

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- Moreover, this result can be generalized to a statement dealing with space-times with particles.

Globally hyperbolic space-time

Let I_0^+ be the (strict) future in Minkowski of the origin 0. The quotient M_Γ of I_0^+ by a cocompact fuchsian group $\Gamma \subset SO_0(1,2)$ has the following properties:

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In other words, it is a *cauchy compact globally hyperbolic space-time*. Every Γ-invariant polyhedral surface in I_0^+ induces a compact spacelike cauchy surface in M_Γ.

Fuchsian space-times

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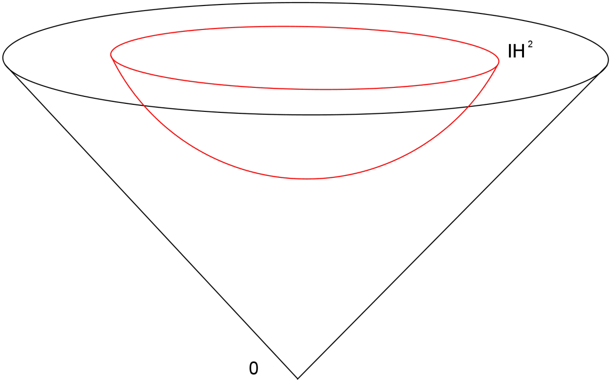
$$-d\tau^2+\tau^2\bar{g}$$

where $\bar{g}$ is a hyperbolic metric on S.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Particles

Massive particle

Let $(\mathbb{R}_\alpha, \bar{g}_\alpha)$ be the euclidean plane with a cone singularity of cone angle $\alpha > 0$. In polar coordinates, with $\lambda = \alpha/2\pi$.

$$\bar{g}_\alpha := \lambda^2 r^{2(\lambda-1)}(dr^2 + r^2 d\theta^2)$$

Then the product $\mathbb{R}_\alpha \times \mathbb{R}$ equipped with the metric $-dt^2 + \bar{g}_\alpha$ is a *space-time with a massive particle of mass* $2\pi - \alpha$. We denote it by $\mathbb{R}^{1,2}_\alpha$.

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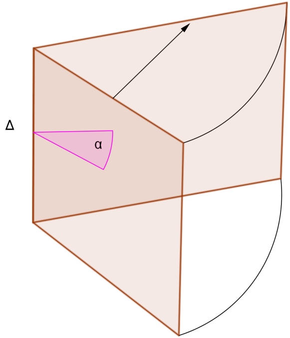
The singular locus $\{r = 0\}$ is the particle.

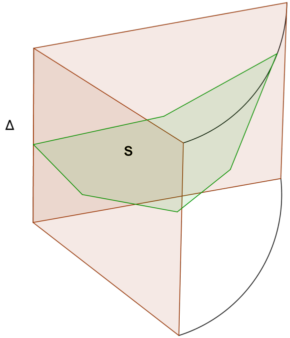
Massive particle

When $\alpha \leq 2\pi$ the space-time $(\mathbb{R}_\alpha, \bar{g}_\alpha)$ can also be described as the a wedge in $\mathbb{R}^{1,2}$ along a timelike line Δ with the two sides identified by a rotation in $SO_0(1,2)$ fixing Δ pointwise.



Space-like surface orthogonal to the surface

A space-like surface crossing the particle is obtained by taking a surface in the wedge, orthogonal to Δ, and whose intersections with the sides of the wedge are image one to the other by the rotation.



Flat globally hyperbolic space-time with particles

A *flat space-time with particle* is a manifold M equipped with a finite collection of 1-dimensional embedded submanifolds $\{\Delta_i\}$, a locally flat lorentzian metric outside $\{\Delta_i\}$, and such that every point x in each Δ_i admits a neighborhood isometric to a neighborhood of a singular point in $\mathbb{R}^{1,2}_{\alpha_i}$.

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It is *globally hyperbolic* if furthermore there is a time function $t : M \to]0, +\infty[$ such that every level set of t is a compact spacelike surface (orthogonal to every Δ_i).

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M is *fuchsian* if the holonomy of its regular part preserves a point in $\mathbb{R}^{1,2}$, called the *origin*.

Fuchsian globally hyperbolic space-time with particles

M is *fuchsian* if the holonomy of its regular part preserves a point in $\mathbb{R}^{1,2}$, called the *origin*. Then, every singular line Δ_i has a lift whose "development" is a line in $\mathbb{R}^{1,2}$ containing the origin. Such a space-time can be described as the product $]0, +\infty[\times S$ equipped with the metric:

$$-d\tau^2 + \tau^2 \bar{g}$$

where $\bar{g}$ is a locally hyperbolic metric on S with cone singularities (of any cone angle, as long as the Gauss-Bonnet equation is satisfied).

Construction of fuchsian space-time with particles via triangulations

From now one, we fix the locally euclidian surface $(S, \bar{g})$ with (arbitrary) cone angles θ_i. We aim to realize it as a fuchsian polyhedral surface in a fuchsian space-time with particles of masses $2\pi - \alpha_i$, with $\theta_i \geq \alpha_i$.

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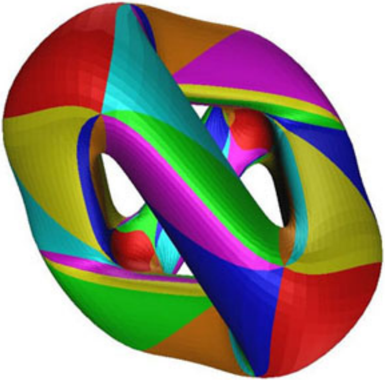
Let $\mathcal{T}$ be the space of triangulations of S, with vertices precisely at the cone singularities. Actually, one can think that $(S, \bar{g})$ is defined by an element T of $\mathcal{T}$, and a realization of every triangle of T as an euclidean triangle. Equivalently, as the data of a positive real number on every edge of T, such that the edges of every triangle satisfies the triangular inequality.

A triangulated surface



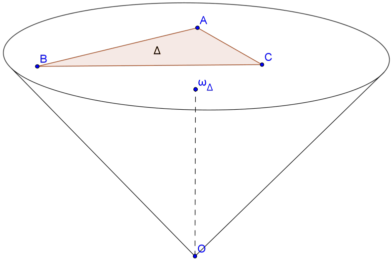
A triangle in the polyhedral surface

A triangle $\Delta = ABC$ in the lift of $\widetilde{S}$ embedded in $\mathbb{R}^{1,2}$ is defined up to isometry by:

– the euclidean triangle in the plane (ABC) containing the image,

– the orthogonal projection ω_Δ of 0 in (ABC),

– the radius R_Δ of the circle $(ABC) \cap \partial I_0^+$ (it is also the cosmological time of ω_Δ).



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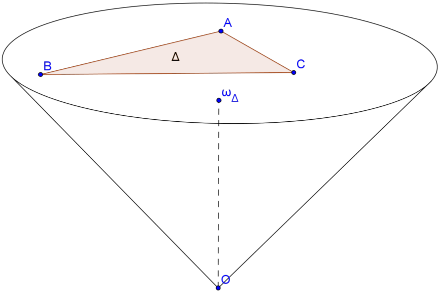

The restriction of $h = \tilde{\tau}^2$ to Δ is then the function:

$$h(.) = d_0^2(., \omega_\Delta) - R_\Delta^2$$

Lemma

Let $\Delta = ABC$ be an euclidean triangle, and let h_A, h_B, h_C be three positive real numbers. There is a unique function $h : \Delta \to]0, +\infty[$ such that:

– its values at A, B and C are h_A, h_B and h_C,

– it is the (square of the) cosmological time restricted to a geodesic embedding of Δ in $\mathbb{R}^{1,2}$.

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Therefore, if N is the number of singular points of $(S, \bar{g})$, we have a well-defined map

$$\mathcal{H} = (\mathbb{R}_+^*)^N \to \mathcal{P}$$

where $\mathcal{P}$ is the space of fuchsian polyhedral embedding of S.

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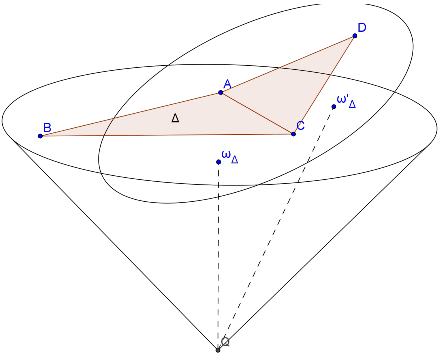
When is the surface convex?



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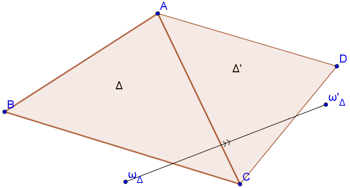

To be convex, for every pair Δ, Δ' of triangles adjacent in $\mathcal{T}$, and if we "rotate" (ACD) so that it coincides with (ABC), the basis $(\overrightarrow{\omega_\Delta \omega_{\Delta'}}, \overrightarrow{CA})$ must be direct (for the orientation of $(ABC) = (ACD)$ defined by the orientation and temporal orientation of $\mathbb{R}^{1,2}$).

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Moreover, this condition only depends on the positions of ω_Δ, hence is invariant up to an additive constant on h.

Lemma

The subset $\mathcal{C}_T$ is a closed convex subset of $\mathcal{H}$, invariant by translation along the direction $(1, 1, ..., 1)$.

Changing the triangulation

Definition

The union of all $\mathcal{C}_T$ over $T \in \mathcal{T}$ is denoted by $\mathcal{C}$.

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Let $h \in \mathcal{C}$. There is a unique cellulation σ of S such that h lies in $\mathcal{C}_T$ if and only if T is a sub-cellulation of σ.

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Polygons of σ will be the faces of the polyhedral embedding of $(S, \bar{g})$ with the given cosmological time h at the vertices.

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which map every admissible triangulation the hyperbolic cone angles of the associated polyhedral surface is a bijection. Fillastre's Theorem states that, if every θ_i is bigger than 2π, there is one and only one element of $\mathcal{C}$ whose image is $(2\pi, ..., 2\pi)$.

Limit case $h = \infty$

The bigger all the h_i's, the closer is α_i to θ_i. At the ideal limit $h_i = \infty$ we obtain the static space-time

$$\begin{array}{ccc} S & \times & \mathbb{R} \\ \bar{g} & - & dt^2 \end{array}$$

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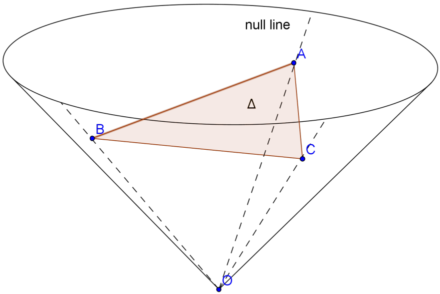
When all the h_i's go to 0, the hyperbolic angle α_i converges to 0. When $h_i = 0$ for all vertices, every triangle has a realization in $\mathbb{R}^{1,2}$ with all vertices in the null cone ∂I_0^+.



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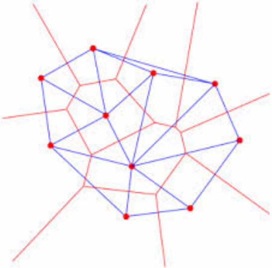
It is equivalent to the fact that T is a sub-triangulation of the *Delaunay cellulation*, for which the convexity condition is automatically satisfied.



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In the sequel, we will show that the limit case $h = 0$ corresponds to a flat singular space-time without particles, but with *extreme BTZ white holes*.

Link with the decorated Teichmüller space

Let Γ be a **non-uniform** lattice in $SO_0(2,1)$. The quotient $\Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface with n cusps.

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There is a natural stratification of $\widetilde{Teich}_0(S)$, where each stratum corresponds to a cellulation of S with vertices at the cusps, and where top-dimensional strata correspond to triangulations.

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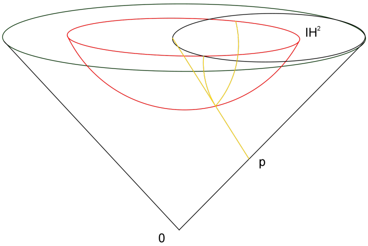
Geometric interpretation of $\widetilde{Teich_0}(S)$

Horocycle in $\mathbb{H}^2 \Longleftrightarrow$ point in the null cone ∂H^2.



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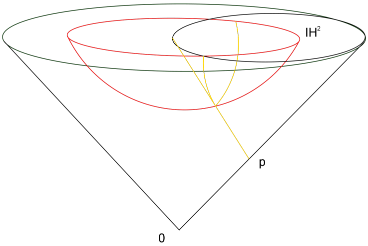

A point in $\widetilde{Teich_0}(S)$ is the data of a non-uniform lattice $\Gamma \subset SO_0(1,2)$ and a Γ-equivariant choice of a γ-fixed point for every parabolic element γ of Γ.

An element of $\widetilde{Teich_0}(S)$ determines a polyhedral surface

Let $\Gamma \subset \mathrm{SO}_0(1,2)$ be a non uniform lattice and $E \subset \partial I_0^+$ a Γ-invariant choice of points in lines fixed by parabolic elements. Penner proved that the boundary of the convex hull of E is a polyhedral surface.

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Proposition

The geometric cellulation of $\partial Conv(E)$ is the Delaunay cellulation for the induced metric.

Extreme BTZ white hole

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Description of fuchsian space-times with BTZ extreme white holes.

In summary

There is a one-to-one correspondence between the space of locally euclidean metric on S with any cone angle and globally hyperbolic space-time with extreme BTZ white hole.

Reformulation of the main Question

For any collection α_1, ... , α_N of non negative real numbers, we define the overdecorated Teichmüller space $\widetilde{Teich}_{\alpha_1,\ldots,\alpha_N}(S)$ as the space of fuchsian space-times with particles of cone angles α_1, ... , α_N, equipped with a polyhedral cauchy surface.

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The main question is to prove that the map $\mathcal{C} \to \widetilde{Teich}_{\alpha_1,\ldots,\alpha_N}$ is a bijection.
Last remark: in the case $N = 1$: the main statement is obviously true!