Linear Stability of the Schwarzschild Solution under gravitational perturbations

joint work with M. Dafermos and I. Rodnianski

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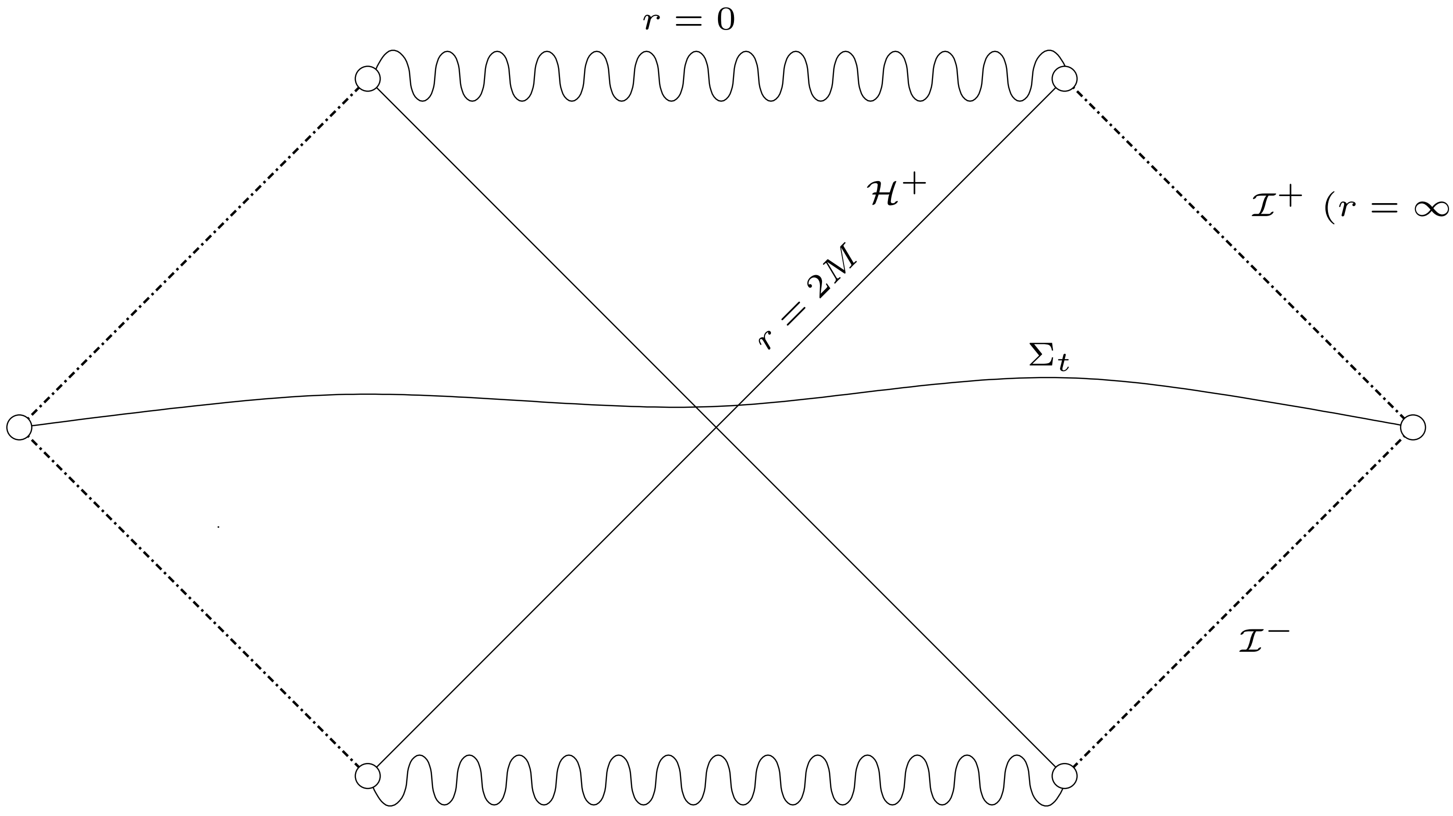
The **Schwarzschild spacetime (1915)** is the simplest non-trivial solution of the vacuum Einstein equations:

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



In this talk we shall prove the linear stability of the Schwarzschild solution. A first version of the theorem is the following:

Theorem. *Consider the system of equations obtained by linearizing the Einstein equations with respect to a fixed Schwarzschild metric of mass* M*. Given initial data for this system, bounded in suitable weighted norms, the evolution converges up to infinitesimal diffeomorphisms to a member of the linearized Kerr family of solutions.*

The purpose of this talk is to make all this precise.

The Linearization Procedure and the System of Equations

The linearization procedure

Let us fix the Eddington-Finkelstein differential structure of the Schwarzschild metric and consider a one-parameter family of Lorentzian metrics in double null-coordinates

$$\boldsymbol{g}(\epsilon) \doteq -4\boldsymbol{\Omega}^2(\epsilon)\, du dv + \boldsymbol{\not{g}}_{CD}(\epsilon)\left(d\theta^C - \boldsymbol{b}^C(\epsilon)\, dv\right)\left(d\theta^D - \boldsymbol{b}^D(\epsilon)\, dv\right)$$

with $\boldsymbol{g}(0)$ being the Schwarzschild metric of mass M.

Remarks

1) Any metric can locally be put into this form.

2) Use of the double gauge for the analysis of EVE:

Christodoulou; Klainerman-Rodnianski, Luk-Rodnianski, Luk-Dafermos

$$\boldsymbol{g}(\epsilon) \doteq -4\boldsymbol{\Omega}^2(\epsilon)\, du dv + \not{\boldsymbol{g}}_{CD}(\epsilon)\left(d\theta^C - \boldsymbol{b}^C(\epsilon)\, dv\right)\left(d\theta^D - \boldsymbol{b}^D(\epsilon)\, dv\right)$$

Associated null-frame

$$\mathbf{e}_3 = \frac{1}{\boldsymbol{\Omega}}\partial_u \quad , \quad \mathbf{e}_4 = \frac{1}{\boldsymbol{\Omega}}\left(\partial_v + \boldsymbol{b}^A \partial_{\theta^A}\right) \quad , \quad \mathbf{e}_A = \frac{\partial}{\partial \theta^A} \, .$$

Compute the connection coefficients $\mathbf{\Gamma}$ in this frame. For instance,

$$\boldsymbol{\chi}_{AB} = g\left(\nabla_{e_A}\mathbf{e}_4, e_B\right) \quad \text{second fundamental form of } S^2_{u,v} \text{ in } C_u$$

Construct $\mathbf{tr}\boldsymbol{\chi}$ and $\widehat{\boldsymbol{\chi}}$, which is symmetric traceless.
In Schwarzschild, we have $\widehat{\chi} = 0$ and $\Omega tr\chi = \frac{2}{r}\left(1 - \frac{2M}{r}\right)$.
Write $\widehat{\chi}$ and $(\Omega tr\chi)^{(1)}$ for linearized part.

A full list of linearised metric and connection coefficients is

$$\sqrt{g}^{(1)}, \widehat{\not{g}}, \Omega^{(1)}, b$$

$$(\Omega tr\chi)^{(1)}, (\Omega tr\underline{\chi})^{(1)}, \widehat{\chi}, \widehat{\underline{\chi}}, \omega^{(1)}, \underline{\omega}^{(1)}, \eta, \underline{\eta}.$$

Note we allow ourselves to drop the superscript (1) if the background quantity vanishes.

Linearizing the (Weyl)-curvature components

The curvature tensor is determined by

$$\boldsymbol{\alpha}_{AB} = \boldsymbol{W}\left(\mathbf{e}_4, \mathbf{e}_A, \mathbf{e}_4, \mathbf{e}_B\right), \qquad \underline{\boldsymbol{\alpha}}_{AB} = \boldsymbol{W}\left(\mathbf{e}_3, \mathbf{e}_A, \mathbf{e}_3, \mathbf{e}_B\right)$$

$$\boldsymbol{\beta}_A = \frac{1}{2}\boldsymbol{W}(\mathbf{e}_A, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_3), \qquad \underline{\boldsymbol{\beta}}_A = \boldsymbol{W}(\mathbf{e}_A, \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_3)$$

$$\boldsymbol{\rho} = \boldsymbol{W}\left(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_4\right), \qquad \boldsymbol{\sigma} = \frac{1}{4}{}^*\boldsymbol{W}(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_4)$$

Only ρ is non-vanishing in Schwarzschild: $\rho = \rho_S = -\frac{2M}{r^3}$

Denote our linearised quantities by: $\alpha, \beta, \underline{\alpha}, \underline{\beta}, \sigma, \rho^{(1)}$

What do we estimate?

The analytical content of $Ric\,[\boldsymbol{g}] = 0$ is contained in

$$\boldsymbol{\nabla}^{\mu}\boldsymbol{W}_{\mu\nu\sigma\tau} = 0 \qquad \text{Bianchi Equations}$$
$$\boldsymbol{\nabla\Gamma} + \boldsymbol{\Gamma\Gamma} = \boldsymbol{W} \qquad \text{Structure Equations}$$

Linearize around background with connection $\Gamma_\circ$ and curvature $W_\circ$:

$$\begin{aligned} (\partial + \Gamma_\circ)\, W^{(1)} + W_\circ \Gamma^{(1)} &= 0 \\ (\partial + \Gamma_\circ)\, \Gamma^{(1)} + \Gamma_\circ \Gamma^{(1)} &= W^{(1)} \end{aligned} \qquad (1)$$

In Minkowsi $W_\circ = 0$, the first equation decouples and keeps its variational structure. $\rightarrow$ conservations laws, decay estimates (Christodoulou-Klainerman 1990)

Around Schwarzschild $W_\circ \neq 0$, so **coupling even at linear level**.

$$\partial_u \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = \left(\Omega tr \underline{\chi}\right)^{(1)} \quad , \quad \partial_v \frac{\sqrt{g}^{(1)}}{\sqrt{g}_S} = \left(\Omega tr \chi\right)^{(1)} - d\!\!/iv\, b$$

$$\partial_u b^A = 2\Omega_S^2 \left[\left(\eta - \underline{\eta}\right)^\sharp\right]^A$$

$$\partial_v \left(\Omega tr \underline{\chi}\right)^{(1)} = \Omega_S^2 \left(2 d\!\!/iv \underline{\eta}^{(1)} + 2\rho^{(1)} - \frac{8M}{r^3} \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{\Omega_S^2}{r} \left(\left(\Omega tr \underline{\chi}\right)^{(1)} - \left(\Omega tr \chi\right)^{(1)}\right)$$

$$\partial_u \left(\Omega tr \chi\right)^{(1)} = \Omega_S^2 \left(2 d\!\!/iv \eta^{(1)} + 2\rho^{(1)} - \frac{8M}{r^3} \frac{\Omega^{(1)}}{\Omega_S}\right) - \frac{\Omega_S^2}{r} \left(\left(\Omega tr \underline{\chi}\right)^{(1)} - \left(\Omega tr \chi\right)^{(1)}\right)$$

$$\partial_v \left(\Omega tr \chi\right)^{(1)} = -\frac{2\Omega_S^2}{r} \left(\Omega tr \chi\right)^{(1)} + \frac{2M}{r^2} \left(\Omega tr \chi\right)^{(1)} + \frac{4\Omega_S^2}{r} \omega^{(1)}$$

$$\partial_u \left(\Omega tr \underline{\chi}\right)^{(1)} = +\frac{2\Omega_S^2}{r} \left(\Omega tr \underline{\chi}\right)^{(1)} - \frac{2M}{r^2} \left(\Omega tr \underline{\chi}\right)^{(1)} - \frac{4\Omega_S^2}{r} \underline{\omega}^{(1)}$$

$$\nabla\!\!\!/_3 \left(\frac{r^2}{\Omega_S} \underline{\widehat{\chi}}^{(1)} \right) = -\frac{r^2}{\Omega_S} \underline{\alpha}^{(1)} \quad , \quad \nabla\!\!\!/_4 \left(\frac{r^2}{\Omega_S} \widehat{\chi}^{(1)} \right) = -\frac{r^2}{\Omega_S} \alpha^{(1)}$$

$$\nabla\!\!\!/_3 \left(r\Omega_S \widehat{\chi}^{(1)} \right) + \frac{\Omega_S^2}{r} \left(\Omega_S^{-1} \underline{\widehat{\chi}}^{(1)} \right) = -2r\Omega_S \mathcal{D}\!\!\!/_2^{\star} \eta^{(1)}$$

$$\nabla\!\!\!/_4 \left(r\Omega_S \underline{\widehat{\chi}}^{(1)} \right) - \frac{\Omega_S^2}{r} \left(\Omega_S^{-1} \widehat{\chi}^{(1)} \right) = -2r\Omega_S \mathcal{D}\!\!\!/_2^{\star} \underline{\eta}^{(1)} \ .$$

$$d\!\!\!/iv \underline{\widehat{\chi}}^{(1)} = \frac{\Omega_S^2}{r} \eta^{(1)} + \underline{\beta}^{(1)} + \frac{1}{2\Omega_S} \nabla\!\!\!/_A \left(\Omega tr \underline{\chi} \right)^{(1)}$$

$$d\!\!\!/iv \widehat{\chi}^{(1)} = -\frac{\Omega_S^2}{r} \underline{\eta}^{(1)} - \beta^{(1)} + \frac{1}{2\Omega_S} \nabla\!\!\!/_A \left(\Omega tr \chi \right)^{(1)} \ .$$

$$\nabla\!\!\!/_3 \underline{\eta}^{(1)} = -\frac{\Omega_S^2}{r} \left(\eta^{(1)} - \underline{\eta}^{(1)} \right) + \underline{\beta}^{(1)} \quad , \quad \nabla\!\!\!/_4 \eta^{(1)} = -\frac{\Omega_S^2}{r} \left(\eta^{(1)} - \underline{\eta}^{(1)} \right) - \beta^{(1)}$$

$$\partial_v \underline{\omega}^{(1)} = -\Omega_S^2 \left(\rho^{(1)} - \frac{4M}{r^3} \frac{\Omega^{(1)}}{\Omega_S} \right) \quad , \quad \partial_u \omega^{(1)} = -\Omega_S^2 \left(\rho^{(1)} - \frac{4M}{r^3} \frac{\Omega^{(1)}}{\Omega_S} \right)$$

$$\omega^{(1)} = \partial_v \left(\frac{\Omega^{(1)}}{\Omega_S} \right) \quad , \quad \underline{\omega}^{(1)} = \partial_u \left(\frac{\Omega^{(1)}}{\Omega_S} \right) \quad , \quad \left(\eta + \underline{\eta} \right)^{(1)} = 2 \nabla\!\!\!/_A \left(\frac{\Omega^{(1)}}{\Omega_S} \right)$$

$$\nabla\!\!\!/_3\alpha^{(1)} - \frac{\Omega_S^2}{r}\alpha^{(1)} - \frac{2M}{r^2}\Omega_S^{-1}\alpha^{(1)} = -2\mathcal{D}\!\!\!/_2^\star\beta^{(1)} + \frac{6M}{r^3}\widehat{\chi}^{(1)}$$

$$\nabla\!\!\!/_4\beta^{(1)} + 2tr\chi_S\beta^{(1)} - \widehat{\omega}_S\beta^{(1)} = d\!\!\!/iv\,\alpha^{(1)}$$

$$\nabla\!\!\!/_3\beta^{(1)} + tr\underline{\chi}_S\beta^{(1)} + \underline{\widehat{\omega}}_S\beta^{(1)} = \mathcal{D}\!\!\!/_1^\star\left(-\rho^{(1)},\sigma^{(1)}\right) + 3\rho_S\eta^{(1)}$$

$$\nabla\!\!\!/_4\rho^{(1)} + \frac{3}{2}tr\chi_S\rho^{(1)} = d\!\!\!/iv\,\beta^{(1)} - \frac{3}{2}\frac{\rho_S}{\Omega_S}\left(\Omega tr\chi\right)^{(1)}$$

$$\nabla\!\!\!/_3\rho^{(1)} + \frac{3}{2}tr\underline{\chi}_S\rho^{(1)} = -d\!\!\!/iv\,\underline{\beta}^{(1)} - \frac{3}{2}\frac{\rho_S}{\Omega_S}\left(\Omega tr\underline{\chi}\right)^{(1)}$$

$$\nabla\!\!\!/_4\sigma^{(1)} + \frac{3}{2}tr\chi_S\sigma^{(1)} = -cu\!\!\!/rl\,\beta^{(1)}$$

$$\nabla\!\!\!/_3\sigma^{(1)} + \frac{3}{2}tr\underline{\chi}_S\sigma^{(1)} = -cu\!\!\!/rl\,\underline{\beta}^{(1)}$$

$$\nabla\!\!\!/_4\underline{\beta}^{(1)} + tr\chi_S\underline{\beta}^{(1)} + \widehat{\omega}_S\underline{\beta}^{(1)} = \mathcal{D}\!\!\!/_1^\star\left(\rho^{(1)},\sigma^{(1)}\right) + 3\rho_S\underline{\eta}^{(1)}$$

$$\nabla\!\!\!/_3\underline{\beta}^{(1)} + 2tr\underline{\chi}_S\underline{\beta}^{(1)} - \underline{\widehat{\omega}}_S\underline{\beta}^{(1)} = -d\!\!\!/iv\,\underline{\alpha}^{(1)}$$

$$\nabla\!\!\!/_4\underline{\alpha}^{(1)} + \frac{1}{2}tr\chi_S\underline{\alpha}^{(1)} + 2\widehat{\omega}_S\underline{\alpha}^{(1)} = 2\mathcal{D}\!\!\!/_2^\star\underline{\beta}^{(1)} - 3\rho_S\underline{\widehat{\chi}}^{(1)}$$

In the above $\nabla\!\!\!/_3$ and $\nabla\!\!\!/_4$ are appropriate covariant differential operators in the e_3 and e_4 null directions, $\nabla\!\!\!/^A$ denotes the angular covariant derivative, $d\!\!/iv$ denotes the divergence on the spheres, and $\mathcal{D}\!\!\!/_1^\star$ and $\mathcal{D}\!\!\!/_2^\star$ denote the angular differential operators:

$$\mathcal{D}\!\!\!/_1^\star\left(-\rho^{(1)}, \sigma^{(1)}\right) = \nabla\!\!\!/_A \rho^{(1)} + \epsilon_{AB} \nabla\!\!\!/^B \sigma^{(1)}$$

$$\left(\mathcal{D}\!\!\!/_2^\star \xi\right)_{AB} = \nabla\!\!\!/_A \xi_B + \nabla\!\!\!/_B \xi_A - g\!\!\!/_{AB}\left(\nabla\!\!\!/^C \xi_C\right)$$

which are adjoints of $d\!\!/iv$.

All operators are now defined with respect to the background Schwarzschild metric!

The above linearized system is *well-known in the physics literature*, at least at the level of mode decomposed solutions. It is usually expressed in the Newman-Penrose ('62) formalism.

However, the theory can be formulated completely in physical space, admits a well-posed initial value problem with initial data satisfying constraints.

Remarks

Of course this is by far not the only way to study the problem!

We should mention the pioneering works of (Regge-Wheeler ’57, Vishveshwara ’70, Zerilli ’70) studying *metric perturbations* at the level of mode decomposed solutions.

A gauge invariant approach in this context is due to Moncrief ’75.

Analysis of the System of Gravitational Perturbations

Special Solutions I: Pure gauge solutions

There are infinitesimal coordinate transformations *preserving the double null-form* of the metric. These can be worked out explicitly. For instance,

Lemma 1. *For any smooth function $f = f(v, \theta, \phi)$, the following is a (pure gauge) solution of the system of gravitational perturbations*

$$\widehat{\not g}^{(1)} = -\frac{4}{r} r^2 \not{\mathcal{D}}_2^\star \not\nabla_A f \quad , \quad \frac{\sqrt{g}^{(1)}}{\sqrt{\not g_S}} = \frac{2\Omega_S^2 f}{r} + \frac{2}{r} r^2 \not\Delta f$$

$$2\frac{\Omega^{(1)}}{\Omega_S} = \frac{1}{\Omega_S^2}\partial_v\left(f\Omega_S^2\right) \quad , \quad b = -2r^2 \not\nabla_A \left[\partial_v\left(\frac{f}{r}\right)\right]$$

$$\eta^{(1)} = \ldots \quad , etc.$$

Special Solutions II: Linearized Kerr solutions

Lemma 2. *The following is a solution of the system of gravitational perturbations:*

$$\widehat{\not{g}}^{(1)} = 0 \quad , \quad \frac{\sqrt{g}^{(1)}}{\sqrt{\not{g}_S}} = 0 \quad , \quad \frac{\Omega^{(1)}}{\Omega_S} = 0 \quad , \quad b_\phi = \frac{4Ma}{r} \sin^2\theta$$

$$\sigma^{Kerr} = \frac{6}{r^4} aM \cos\theta \sim Y_0^1(\theta, \phi) \quad , ... \; etc.$$

Non-trivial Kerr solutions can be detected at the initial data.

Gauge invariant quantities which decouple

It has long been known that the gauge invariant null-curvature components α and $\underline{\alpha}$ satisfy decoupled wave equations: The Teukolsky (or Bardeen-Press '73) equation:

$$\Box_{g_M}\alpha + \left(1 - \frac{3M}{r}\right)\partial_t\alpha + V\alpha = 0.$$

.

Only mode stability but not even uniform boundedness was known.

Hierarchy of gauge invariant quantities

Define the symmetric traceless tensors

$$\psi := -\frac{1}{2r\Omega_S^2}\nabla\!\!\!/_3\left(r\Omega_S^2\alpha\right) = \mathcal{D}\!\!\!/_2^\star\beta - \frac{3M}{r^3}\widehat{\chi}\,,$$

$$\underline{\psi} := \frac{1}{2r\Omega_S^2}\nabla\!\!\!/_4\left(r\Omega_S^2\underline{\alpha}\right) = \mathcal{D}\!\!\!/_2^\star\underline{\beta} + \frac{3M}{r^3}\widehat{\underline{\chi}}\,.$$

We also define

$$P := \frac{1}{r^3\Omega_S}\nabla\!\!\!/_3\left(\psi r^3\Omega_S\right) \qquad \text{and} \qquad \underline{P} := -\frac{1}{r^3\Omega_S}\nabla\!\!\!/_4\left(\underline{\psi} r^3\Omega_S\right)\,. \quad (2)$$

A quick computation shows that we can write

$$P = \mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star\left(-\rho^{(1)},\sigma\right) - \frac{3M\Omega_S}{r^4}\left(\widehat{\chi} - \widehat{\underline{\chi}}\right)\,, \quad (3)$$

$$\underline{P} = \mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star\left(-\rho^{(1)},-\sigma\right) - \frac{3M\Omega_S}{r^4}\left(\widehat{\chi} - \widehat{\underline{\chi}}\right)\,. \quad (4)$$

These transformations appear at the level of mode solutions in the work of Chandrasekhar.

The point is:

- the quantities P and $\underline{P}$ satisfy the Regge-Wheeler equation, a wave equation which *does* admit a positive conserved energy and an integrated decay estimate.
- P and $\underline{P}$ are symmetric traceless tensors: they don't see the Kerr modes
- the quantities P and $\underline{P}$ control ψ and $\underline{\psi}$ and then α, $\underline{\alpha}$ respectively

Analysis of the Regge-Wheeler equation

We have that $\phi = r^3 P_{AB}$ satisfies

$$\frac{1}{1-\frac{2M}{r}}\partial_u\partial_v\phi - \left(\Delta\!\!\!/ - \frac{4}{r^2}\right)\phi - \frac{6M}{r^3}\phi = 0\,. \tag{5}$$

All insights for the linear wave equation can be used:

1. the existence of a positive conserved energy is immediate.
2. the redshift estimate holds near $\mathcal{H}^+$.
3. an integrated decay estimate was shown by [Blue-Soffer, GH]. Trapping enters!
4. one can apply the black-box (we actually reprove them) results of [DafRod, Schlue, Moschidis] to go from integrated decay to polynomial decay rates for the energy.

Intermezzo: The linear wave equation on black holes

The following definite result has been obtained:

Theorem. *Solutions of the linear wave equation*

$$\Box_{g_{M,a}}\psi = 0 \tag{6}$$

for $g_{M,a}$ a subextremal member of the Kerr-family decay polynomially in time on the black hole exterior.

- $a = 0$ case: Kay–Wald, Dafermos–Rodnianski, Blue–Sterbenz
- $|a| \ll M$ case: Dafermos and Rodnianski, Tataru-Tohaneanu, Andersson–Blue
- $|a| < M$ case: Dafermos–Rodnianski–Shlapentokh-Rothman

Remarks

- Theorem 1 fails in the extremal case $a = M$ (Aretakis).
- A version of Theorem 1 has been proven for Maxwell's equation in the $a \ll M$ case by Andersson–Blue and Blue ($a = 0$).
- Generalizations to Kerr de Sitter (Dyatlov, Vasy, Hintz) and Kerr-anti de Sitter (Holzegel–Smulevici, Gannot, Dold)

Why is the linear problem $\Box_{g_M}\psi = 0$ hard?

Recall Minkowski $\Box_\eta \psi = 0$. Two key estimates

$$\int_{\Sigma_t} (\partial_t\psi)^2 + |\nabla\psi|^2 = \int_{\Sigma_0} (\partial_t\psi)^2 + |\nabla\psi|^2 \quad \text{energy conservation}$$

$$\int_0^T dt \int_{\Sigma_t \cap \{r \le R\}} (\partial_t\psi)^2 + |\nabla\psi|^2 \le C_R \int_{\Sigma_0} (\partial_t\psi)^2 + |\nabla\psi|^2 \quad \text{ILED}$$

Already in the Schwarzschild case deriving analogues of these two estimates requires

- understanding of the *redshift* near $\mathcal{H}^+$ to prove boundedness
- understanding of *trapping* at the photon sphere to prove decay

The Kerr case is much more complicated (superradiance)!

The redshift

Schwarzschild has a globally causal Killing field ∂_t
$\implies$ coercive energy identity degenerating on $\mathcal{H}^+$:

$$\int_{\Sigma_{t^\star}} |D\psi|^2 \left(1 - \frac{2M}{r}\right) \lesssim \int_{\Sigma_0} |D\psi|^2 \left(1 - \frac{2M}{r}\right)$$

with $|D\psi|^2$ denoting (the sum of) all derivatives of ψ.

The redshift effect is about removing the degeneracy at $r = 2M$ to get a non-degenerate boundedness statement.

Commutation, elliptic estimates and Sobolev embedding lead to pointwise estimates.

Trapping

Restricting again to $a = 0$, one easily checks the existence of null-geodesics contained in the timelike hypersurface $r = 3M$

$$\gamma(s) = (t(s), r(s), \theta(s), \phi(s)) = \left(s, 3M, \frac{\pi}{2}, \frac{1}{3\sqrt{3}M} \cdot s\right)$$

In the high frequency approximation, solutions to the wave equation travel along null-geodesics!

$\Longrightarrow$ Non-degenerate decay estimates for Schwarzschild are necessarily associated with a loss of derivatives! (Sbierski 2013)

$$\int_0^T dt^\star \int_{\Sigma_{t^\star} \cap \{r \leq R\}} |D\psi|^2 \leq C_R \int_{\Sigma_0} |D\psi|^2 + |D^2\psi|^2$$

Back to the estimates for (P, ψ, α) and $\left(\underline{P}, \underline{\psi}, \underline{\alpha}\right)$

Theorem (gauge invariant quantities)**.** *Consider a solution of the system of gravitational perturbations arising from a smooth asymp. flat seed initial data set. Then the gauge invariant quantites* (P, ψ, α) *and* $(\underline{P}, \underline{\psi}, \underline{\alpha})$ *satisfy boundedness and integrated decay estimates. Moreover, the quantities decay both in energy and pointwise at polynomial rates.*

Proof. For P and $\underline{P}$ this statement follows from previous work. From the evolution equation for ψ we derive

$$\partial_u \left[|\psi|^2 r^6 \Omega_S^2 \right] = 2r^6 \Omega_S^3 P \psi$$

or, multiplying by r^n and using $r_u = -\Omega_S^2$,

$$\partial_u \left(|\psi|^2 r^6 \Omega_S^2 \cdot r^n \right) + |\psi|^2 r^6 \Omega_S^4 n r^{n-1} = 2r^{6+n} \Omega_S^3 P \psi$$

$$\partial_u \left(|\psi|^2 r^6 \Omega_S^2 \cdot r^n \right) + \frac{1}{2} |\psi|^2 r^6 \Omega_S^4 n r^{n-1} \leq \frac{2}{n} r^{7+n} |P|^2 \Omega_S^2 \,.$$

The argument for α (and then $(\underline{P}, \underline{\psi}, \underline{\alpha})$) is similar. □

Corollary. *Solutions to the Teukolsky equation decay inverse polynomially in time.*

Estimating the (gauge-dependent) part of the solution.

How do we obtain bounds on the other quantities?

Can't estimate $\widehat{\chi}$ directly as it is blue-shifted.

$$\Omega_S \nabla\!\!\!\!/_4 \left(r^2 \Omega_S \widehat{\chi}\right) - \frac{2M}{r^2}\left(r^2 \Omega_S \widehat{\chi}\right) = -\alpha \Omega_S^2 r^2 \,. \tag{7}$$

Need to commute with the redshift vectorfield $\Omega_S^{-1} \nabla\!\!\!\!/_3$ (twice) before doing estimates. This requires additional fluxes on the horizon which are in turn obtained from the gauge invariant quantities.

For $\underline{\widehat{\chi}}$ (and certain other quantites) we can only show boundedness but not decay. Why is that?

Before giving you the answer we state an informal version of the boundedness statement:

Theorem. *Consider a solution $\mathcal{S}$ of the system of gravitational perturbations arising from a smooth seed initial data set. Then all geometric quantities satisfy (weighted) boundedness estimates, e.g.*

$$\left|\frac{\Omega^{(1)}}{\Omega_S}\right| \leq C \quad , \quad |r\Omega_S^{-1}\underline{\widehat{\chi}}| \leq C \quad , \quad |r^2\underline{\beta}| \leq C \quad , etc. \tag{8}$$

Recall that we defined the notion of pure gauge solutions above.

Let us write $\mathcal{S}$ as representing all geometric quantities $\big(\alpha, \beta, ..., \widehat{\chi}, \underline{\widehat{\chi}}, \eta, ...\big)$ of the solution in the initial data gauge and $\mathcal{S}_{new} = \mathcal{S} + \mathcal{S}_f$ for the geometric quantities corrected by a pure gauge solution f.

We determine the gauge function f dynamically from the solution $\mathcal{S}$ (by solving an ODE along the event horizon).

The point is that for this suitably defined f all geometric quantities of S_{new} will actually decay. Moreover, the gauge function f is controlled by the initial data in $\mathcal{S}$!

Theorem 1 (summary version). *Consider a solution of the equations of linearised gravity around Schwarzschild arising from general asymptotically flat characteristic initial data on a double null cone.*

1. *Quantitative boundedness and inverse polynomial decay holds for the gauge invariant quantities, in fact for general solutions of Regge–Wheeler and Teukolsky equations.*

2. *In a gauge determined by initial data, <u>all</u> quantities of the system of linearised gravity remain bounded by a constant times their initial values.*

3. *In a gauge determined by the future, i.e. after addition of a pure gauge solution normalised to the event horizon behaviour, <u>all</u> quantities of the system decay inverse polynomially to a member of the 4-dimensional family of standard linearised Kerr solutions.*

The final linearised Kerr solution can be read off from initial data and the pure gauge solution normalised to the future <u>is bounded by initial data</u>.

Epilogue: The connection with Wald's canonical energy

Epilogue: Conservation laws for gravitational perturbations of Schwarzschild

While we do not use them in our argument, let us mention that there exist certain conservation laws for the system of linearized gravitational perturbations on Schwarzschild.

$$F_v\left[\Gamma\right](u_0,u_1) = \int_{u_0}^{u_1} dud\theta d\phi\ r^2 \sin\theta \Bigg[-2\underline{\omega}^{(1)}\left(\Omega tr\chi\right)^{(1)} - \frac{1}{2}\left(\left(\Omega tr\underline{\chi}\right)^{(1)}\right)^2$$

$$-\frac{4M}{r^2}\left(\frac{\Omega^{(1)}}{\Omega_S}\right)\left(\Omega tr\underline{\chi}\right)^{(1)} + 2\Omega_S^2|\eta|^2 + \Omega_S^2|\widehat{\underline{\chi}}|^2\Bigg]$$

$$F_u\left[\Gamma\right](v_0,v_1) = \int_{v_0}^{v_1} dvd\theta d\phi\ r^2 \sin\theta \Bigg[-2\omega^{(1)}\left(\Omega tr\underline{\chi}\right)^{(1)} - \frac{1}{2}\left(\left(\Omega tr\chi\right)^{(1)}\right)^2$$

$$+\frac{4M}{r^2}\left(\frac{\Omega^{(1)}}{\Omega_S}\right)\left(\Omega tr\chi\right)^{(1)} + 2\Omega_S^2|\underline{\eta}|^2 + \Omega_S^2|\widehat{\chi}|^2\Bigg]$$

The following important conservation law holds:

Proposition 1. *For any* $u_0 < u_1 < u_2 < \infty$ *and* $v_0 < v_1 < v_2 < \infty$ *we have the conservation law*

$$F_v\left[\Gamma\right](u_0, u_1) + F_u\left[\Gamma\right](v_0, v_1) = F_{v_0}\left[\Gamma\right](u_0, u_1) + F_{u_0}\left[\Gamma\right](v_0, v_1)$$

A key property is that these fluxes are gauge invariant up to boundary terms!

This conservation law can be exploited in connection with a gauge transformation to produce a priori control on the flux of $\widehat{\chi}$ through the event horizon as well as that of $\underline{\widehat{\chi}}$ through null-infinity.

Theorem 2 (GH). *Given a solution to the system of gravitational perturbations on Schwarzschild, the L^2-energy flux of the linearized shear $\widehat{\chi}$ along the event horizon and the L^2-energy flux of the (appropriately weighted) linearized shear $\underline{\widehat{\chi}}$ along null-infinity are uniformly bounded a priori by an L^2-norm on initial data which involves only first derivatives of the metric perturbation.*

There are further conservations laws at the level of curvature, which allow to strengthen the above theorem.