Dynamics in asymptotically Anti-de-Sitter spacetimes

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Paris, November 19, 2015
A CELEBRATION OF THE 100th ANNIVERSARY

Introduction

- The most simple (maximally symmetric) solutions of the vacuum solutions of the Einstein equation

$$Ric(g) = \Lambda g$$

  are Minkowski space ($\Lambda = 0$), de Sitter space ($\Lambda > 0$) and Anti-de-Sitter space ($\Lambda < 0$, AdS).

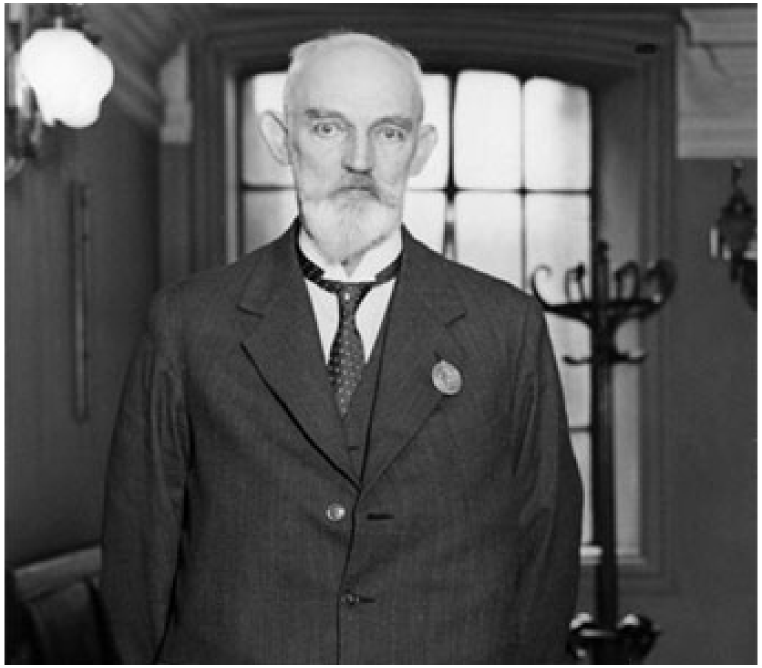
- (In the last 100 years) there has been a large amount of work trying to understand the linear and non-linear stability of asymptotically flat/de Sitter spacetimes.

- In particular, we know the stability of Minkowski space (Christodoulou-Klainerman 93, Lindblad-Rodnianski, Bieri,..) and that of de Sitter (Friedrich, Ringström,..)

- Beyond the trivial solutions: large amount of work on (linear) stability of asymptotically flat or de Sitter black hole spacetimes.

- In the theoretical physics literature, there is a huge research activity related to the study of asymptotically AdS spacetimes (cf AdS/CFT).
- Dynamical problems in AdS are not pure initial value problem but *initial-boundary problems*. Leads to many interesting analytical problems: singular boundary analysis, different boundary conditions $\rightarrow$ different dynamics, geometry of the boundary of AdS (in relation to dynamics), ...
- Fewer math results on this subject. In fact, even the stability (or instability!) of AdS is not known !
- Aim of the talk: present some of the rich dynamics of asymptotically AdS spacetimes.

de Sitter



Anti de Sitter

More precisely

- Given $l \neq 0$, we consider here AdS in (3+1) dimensions as the Lorentzian manifold $(\mathbb{R}^4, g_{AdS})$ where

$$g_{AdS} = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 \sigma_{\mathbb{S}^2},$$

  where $\sigma_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $r \in [0, +\infty)$ is a standard radial coordinate.

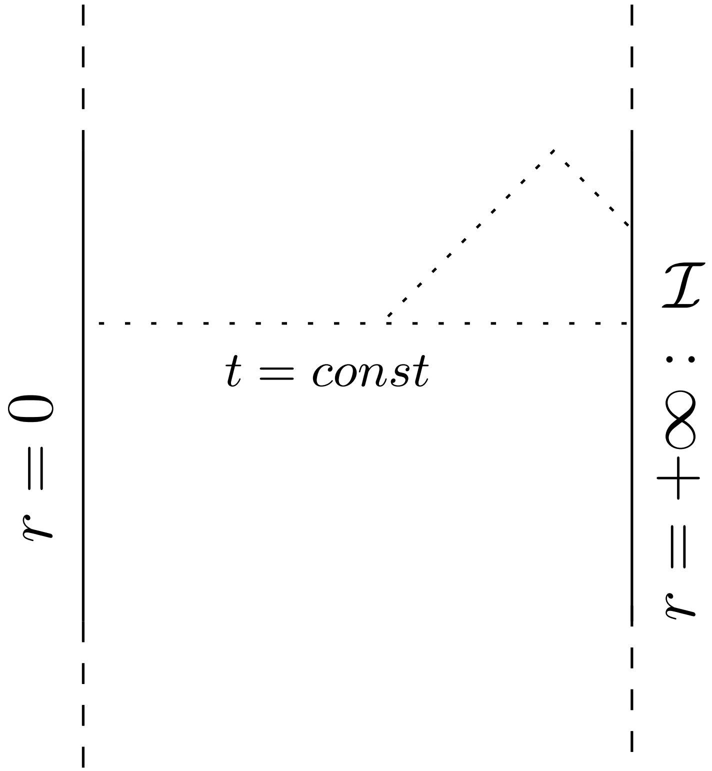
- g_{AdS} solves $Ric(g_{AdS}) = \Lambda g_{AdS}$ with $l^2 = -\frac{3}{\Lambda}$. We'll often take $l = 1$ in the rest of the talk.

- AdS is a geodesically complete Lorentzian manifold which is however *not globally hyperbolic.*

- In particular, cannot directly apply standard theory of linear wave equations of globally hyperbolic spacetimes to obtain well-posedness for $\Box_{g_{AdS}} \psi = 0$ in AdS.

Non-global hyperbolicity

- Non-global hyperbolicity of AdS is manifest in a Penrose diagram



Linear waves on AdS

- We consider for the moment the wave and Klein-Gordon equations

$$\Box_{g_{AdS}}\psi - \lambda\psi = 0, \quad (l = 1)$$

and try to construct solutions.

- Breitenlohner-Freedman (mode decomposition using separation of variables): Assuming $\lambda > -9/4$, find solutions of the form

$$\psi(t, r, \vartheta) = \frac{1}{r^{\lambda_+}}\psi^+(t, \vartheta) + \frac{1}{r^{\lambda_-}}\psi^-(t, \vartheta) + \mathcal{O}\left(\frac{1}{r^{5/2}}\right)$$

where $\lambda_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \lambda}$.

- Imposing that $\psi^- = 0$ corresponds to Dirichlet boundary conditions.

- Imposing $\psi^+ = 0$ corresponds to Neumann boundary conditions but only possible if $-9/4 < \lambda < -5/4$.

- $\lambda = -2$ corresponds to the conformal wave operator $\Box_g - \frac{1}{6}R$.

Wellposedness for linear hyp. equations in asymp. AdS spacetimes

- Pure AdS: Ishibashi-Wald (wave, Maxwell, Gravitational pertuburation), Bachelot (Dirac) : Rewrite the wave operator as $-{\partial_t}^2 + P$ where P is a (self-adjoint) elliptic operator.

- asymptotically AdS: Holzegel (energy methods, Dirichlet boundary conditions), Vasy (asymptotically AdS, microlocal analysis near the boundary), Warnick (asymptotically AdS, general boundary conditions, twisted derivatives)

Finite and infinite energies

- Let
$$T = d\psi \otimes d\psi - \frac{1}{2}\left(g(\nabla\psi, \nabla\psi) + \lambda\psi^2\right) g$$
be the energy momentum tensor for a Klein-Gordon field.

- Could try to construct solutions using energy estimates out of T.

- Problem 1: if $\lambda < 0$ the pointwise energy density is not necessarily positive.

- Problem 2: Apply the standard energy method, 3 kind of boundary terms: two spacelikes ($t = const$ slices) and one timelike ($r = +\infty$).

- Boundary terms are finite only in the Dirichlet case (not enough decay in the Neumann case).

- Breitenlohner-Freedman: there exists a renormalized energy which is finite and conserved (usual energy plus an extra surface term).

The twisted derivative approach of Warnick (2012)

- Warnick twisted derivative : Instead of $\partial_{x^i}\phi$, consider $r^\alpha \partial_{x^i}(r^{-\alpha}\phi)$, where $\alpha = \alpha(\lambda, l)$.
- Renormalized energy can be written in terms of the twisted derivatives.
- In fact, a whole set of adapted *twisted* Sobolev spaces can be introduced, where all usual derivatives are replaced by their twisted versions.

Non-linear local wellposedness

- Friedrich (Full Einstein equations, using the conformal method)
- Holzegel-J.S. (spherically-symmetric Einstein-Klein-Gordon system with Dirichlet boundary conditions.)
- Holzegel-Warnick (spherically-symmetric Einstein-Klein-Gordon system, general boundary conditions)

Global dynamics in the neighbourhood of AdS with Dirichlet boundary conditions

- Let us fix AdS and consider the wave (or Klein-Gordon) equation.
- Because of the Dirichlet boundary conditions, there is no decay mechanism.
- At spectral level, spectrum of associated elliptic operator is discrete and solutions can be decomposed into modes.
- All stability proof for systems of (highly) non-linear wave equations such as the Einstein equations are based on decay.
- **Conjecture 1** (Holzegel-Dafermos, Anderson)**.** *The Anti-de-Sitter space is unstable as a solution of the Einstein equations with Dirichlet boundary conditions.*

Expectations

- A good toy model for this conjecture is the spherically-symmetric Einstein-Klein-Gordon system with Dirichlet boundary conditions.
- Instable behaviour confirmed by numerics by Bizoǹ-Rostworowski (2011).
- Moving energy from low modes to high modes. Bizoǹ-Rostworowski (2011): for small data of size ϵ, use a power series in ϵ, decompose the whole system on modes associated with the elliptic operator and try to understand heuristically the behaviour of each components.
- Interestingly, certain solutions seemed to stay close to AdS for a long time (ex: solutions arising from one mode)
- cf geons of Dias-Horowitz-Marolf-Santos, time periodic Maliborski and Rostworowski.
- Many other works (Balasubramanian-Buchel-Green-Lehner-Liebling, Craps-Evnin-Vanhoof, Maliborski, ..)

Adding a decay mechanism to the problem:

- Case 1: asymptotically AdS-black holes, keeping reflecting boundary conditions.

- Case 2: Replace the reflective boundary conditions by *dissipative boundary conditions.*

Asymptotically AdS-black holes

Schwarzschild-AdS

Let $M, l > 0$ and consider the metric

$$ds^2 = -(1-\mu)dt^2 + (1-\mu)^{-1}dr^2 + r^2\sigma^2_{\mathbb{S}^2}$$

where $(1-\mu) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$,

- $1-\mu$ has one real root denoted $r_+ > 0$, which depends on M and l.
- The black hole exterior+horizon is $\mathcal{R} = \mathbb{R}_t \times [r_+, \infty) \times \mathbb{S}^2$.

The Kerr-AdS black holes

Let $M > 0, l > 0$ and let a be a real number such that $|a| < l$. Then

$$g_{KAdS} = \frac{\Sigma}{\Delta_-}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\left(r^2+a^2\right)^2 - \Delta_- a^2 \sin^2\theta}{\Xi^2\Sigma}\sin^2\theta d\phi^2$$
$$-2\frac{\Delta_\theta\left(r^2+a^2\right) - \Delta_-}{\Xi\Sigma}a\sin^2\theta\ d\phi dt - \frac{\Delta_- - \Delta_\theta a^2\sin^2\theta}{\Sigma}dt^2$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta_\pm = \left(r^2+a^2\right)\left(1+\frac{r^2}{l^2}\right) \pm 2Mr$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}\,.$$

Moreover, r_+ is the largest real root of $\Delta_-(r)$.

Log decay of Klein-Gordon waves in Kerr-AdS

We prove

Theorem 1 (Holzegel-J.S., 2011-2013)**.** *Let* ψ *be a solution of* $\Box_g \psi + \alpha\psi = 0$ *with Dirichlet boundary conditions in the exterior of a Kerr-AdS spacetime such that* $|a|l < r_+^2$, $\lambda > -\frac{9}{4l^2}$. *Then, for all* $t^\star \geq t_0^\star$,

$$E_1[\psi](t^\star) \leq \frac{C}{\log(2+t^\star)} E_2(\psi)(t_0^\star),$$

where $C > 0$ *is some universal constant. Moreover, the estimate is* ***sharp***.

Remark 1:

- Uniform boundedness (for non-degenerate energies up to the horizon) follows from Holzegel and Holzegel-Warnick.
- The condition $|a|l < r_+^2$ implies that there is globally timelike vector field (the Hawking-Real vector-field)

- If $|a|l > r_+^2$, then one can construct growing modes (Dominic Dold, see also Cardoso-Dias).

Remark 2: toric AdS black holes, much stronger decay (Dunn-Warnick)

Remark 3:

- Obstacle problem: log decay in the presence of stable trapping (general result of Burq)
- Asymptotically flat spacetimes: recent general result of Moschidis

Remark 4: Log decay is again very slow decay in view of any standard stability type results...

Remark 5: several works on quasinormal modes for Schwarzschild-AdS and Kerr-AdS (Gannot, Warnick, Numerics by Festuccia-Liu).

Remark 6: scattering for Dirac in Schwarzschild-AdS : Idelon-Riton

The trapping: geodesic flow on Schwarzschild-AdS and Kerr-AdS

- In Schwarzschild-AdS, there exists null geodesics orbiting around $r = 3M$.
- In asymptotically flat Schwarzschild, this is all the trapping, but in the asymptotically AdS case with Dirichlet boundary, there is also a trapping at infinity.
- Similar situation in Kerr-AdS.
- This trapping at infinity is a *stable trapping* and thus responsible for the very slow decay.
- Interestingly, (at least in the Schwarzschild case), after decomposition into spherical harmonics, each mode have exponential decay, but the constant deteriorates so much that summing loses almost everything !

The spherically-symmetric case

Theorem 2 (Holzegel, J.S. 2011)**.** *The (black hole exterior of) Schwarzschild-AdS is both orbitally and asymptotically stable within the spherically-symmetric Einstein-Klein-Gordon system with Dirichlet boundary conditions (and mass $\lambda \geq -2$)*

Moreover, we have convergence to the original Schwarzschild-AdS at exponential rate.

Adding dissipative boundary conditions

(Joint Work with Holzegel, Luk and Warnick)

3 equations on a fixed AdS spacetime.

1. The conformal wave equation $\Box_{g_{AdS}} u + 2u = 0$.
2. The maxwell equation for a two form F: $dF = 0$, $d \star_{AdS} F = 0$.
3. The Bianchi equation for a Weyl field $\nabla^a_{AdS} W_{abcd} = 0$.

The boundary conditions

- Wave equation, we impose

$$\partial_t(ru) + r^2\partial_r(ru) = 0, \quad \text{as } r \to +\infty.$$

- Maxwell case:

$$r^2(E_A + \epsilon_A^{\ B} H_B) = 0, \quad \text{as } r \to +\infty,$$

for E and H, electric and magnetic field.

- Bianchi case:

$$r^3\left(E_{AB} - \frac{1}{2}\delta_{AB}E_C^C + \epsilon_{(A}^{\ \ C} H_{B)}C\right) = 0, \quad \text{as } r \to +\infty,$$

for $E_{AB} = W(e_0, e_A, e_0, e_B)$ and $H_{AB} = \star_{AdS}W(e_0, e_A, e_0, e_B)$, electric and magnetic part of the Weyl tensor.

- The point of the dissipative boundary conditions is that running an energy estimates, we want to see a negative contribution coming from the boundary term at $r = +\infty$, so that the energy decays.
- In case of the wave or Maxwell equations, this is immediate.
- In the case of the Bianchi equations, (using the Bel-Robinson tensor) the term appearing on the boundary at infinity does not have a sign unless additional boundary conditions are imposed. However, our boundary conditions already determined a unique solution, so with additional conditions the problem will not be well-posed.
- To see that the dissipative boundary conditions are effective, we use a reduced set of equations (which combined evolution and conststraint equations cf work of Friedrich), and we prove the energy estimate and an integrated decay estimate at the same time.

Let Ψ be

1. a solution to the wave equation in AdS with dissipative boundary condition and $\varepsilon[\Psi]$ be

$$\varepsilon[\Psi] := \sqrt{1+r^2}\left(\frac{(\partial_t\Psi)^2+\Psi^2}{1+r^2}+\partial_r\left(\sqrt{1+r^2}\Psi\right)^2+\left|\nabla\!\!\!/\Psi\right|^2\right)$$

2. a solution to the Maxwell equations in AdS with dissipative boundary conditions and $\varepsilon[\Psi]$ be

$$\varepsilon[\Psi] = \sqrt{1+r^2}\left(|E|^2+|H|^2\right)$$

3. a solution to the Bianchi equations with dissipative boundary conditions and $\varepsilon[\Psi]$ be

$$\varepsilon[\Psi] = \left(1+r^2\right)^{\frac{3}{2}}\left(|E|^2+|H|^2\right)$$

Decay of linear waves in AdS with dissipative boundary conditions

Theorem 3 (Holzegel, Luk, Warnick, J.S). *Let* Ψ *and* $\varepsilon[\Psi]$ *be as the cases 1,2 and 3. Then we have the following estimates*

1. *Uniform Boundedness: For any* $t \geq 0$*, we have*

$$\int_{\Sigma_t} \frac{\varepsilon[\Psi]}{\sqrt{1+r^2}} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon[\Psi]}{\sqrt{1+r^2}} r^2 dr d\omega$$

2. *Degenerate (near infinity) integrated decay without derivative loss:*

$$\int_0^\infty dt \int_{\Sigma_t} \frac{\varepsilon[\Psi]}{1+r^2} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon[\Psi]}{\sqrt{1+r^2}} r^2 dr d\omega \,.$$

3. *Non-degenerate (near infinity) integrated decay with derivative loss:*

$$\int_0^\infty dt \int_{\Sigma_t} \frac{\varepsilon[\Psi]}{\sqrt{1+r^2}} r^2 dr d\omega \lesssim \int_{\Sigma_0} \frac{\varepsilon[\Psi] + \varepsilon[\partial_t \Psi]}{\sqrt{1+r^2}} r^2 dr d\omega \,.$$

Moreover,

- we can get polynomial pointwise decay.
- The loss of derivatives in the non-degenerate integrated decay is sharp. This means there is yet another source of trapping !

- The conformal wave equation in AdS can be mapped via conformal transformation into a wave equation on 1/2 of the Einstein cylinder $(\mathbb{R} \times \mathbb{S}^3, g_E)$, where

$$g_E = -dt^2 + d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2).$$

- The conformal transformation then map AdS into the Einstein cylinder restricted to $\psi \leq \frac{\pi}{2}$.
- The trapping then originates from gliding rays near the boundary $\psi = \frac{\pi}{2}$.
- The proof of the sharpness uses a Gaussian beam approaches based in particular on recent work of Sbierski.