

General Relativity and Geometry: Interactions and Missed Opportunities

Jean-Pierre BOURGUIGNON

(CNRS-Institut des Hautes Études Scientifiques)

Institut Henri Poincaré, Paris

November 16, 2015

Outline of the Lecture

- All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way.
- Taking advantage of the historical dimension of this conference, this lecture presents such instances in the context of the Theory of General Relativity, some have been truly influential, others of the type “missed opportunities”.
- Actually, the extent to which this has happened is truly remarkable, and has, in my opinion, not been stressed enough.
- As several lectures at this conference will be dedicated to the latest mathematical developments, some very striking and representing remarkable technical achievements, I will concentrate more on the variety of the modes of interaction, stressing the differences in the points of view of physicists and mathematicians that they reveal from an epistemological point of view.

Outline of the Lecture

- All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way.
- Taking advantage of the historical dimension of this conference, this lecture presents such instances in the context of the Theory of General Relativity, some have been truly influential, others of the type “missed opportunities”.
- Actually, the extent to which this has happened is truly remarkable, and has, in my opinion, not been stressed enough.
- As several lectures at this conference will be dedicated to the latest mathematical developments, some very striking and representing remarkable technical achievements, I will concentrate more on the variety of the modes of interaction, stressing the differences in the points of view of physicists and mathematicians that they reveal from an epistemological point of view.

Outline of the Lecture

- All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way.
- Taking advantage of the historical dimension of this conference, this lecture presents such instances in the context of the Theory of General Relativity, some have been truly influential, others of the type “missed opportunities”.
- **Actually, the extent to which this has happened is truly remarkable, and has, in my opinion, not been stressed enough.**
- As several lectures at this conference will be dedicated to the latest mathematical developments, some very striking and representing remarkable technical achievements, I will concentrate more on the variety of the modes of interaction, stressing the differences in the points of view of physicists and mathematicians that they reveal from an epistemological point of view.

Outline of the Lecture

- All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way.
- Taking advantage of the historical dimension of this conference, this lecture presents such instances in the context of the Theory of General Relativity, some have been truly influential, others of the type “missed opportunities”.
- Actually, the extent to which this has happened is truly remarkable, and has, in my opinion, not been stressed enough.
- As several lectures at this conference will be dedicated to the latest mathematical developments, some very striking and representing remarkable technical achievements, I will concentrate more on the variety of the modes of interaction, stressing the differences in the points of view of physicists and mathematicians that they reveal from an epistemological point of view.

Outline of the Lecture

- All along history, and especially in the 19th and 20th century, Geometry and Physics have interacted in a very positive way.
- Taking advantage of the historical dimension of this conference, this lecture presents such instances in the context of the Theory of General Relativity, some have been truly influential, others of the type “missed opportunities”.
- Actually, the extent to which this has happened is truly remarkable, and has, in my opinion, not been stressed enough.
- As several lectures at this conference will be dedicated to the latest mathematical developments, some very striking and representing remarkable technical achievements, I will concentrate more on the variety of the modes of interaction, stressing the differences in the points of view of physicists and mathematicians that they reveal from an epistemological point of view.

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ❶ The Prehistory of General Relativity
- ❷ Gregorio RICCI-CURBASTRO, a Central Figure
- ❸ The Giant Step of General Relativity
- ❹ Hermann WEYL and Conformal Geometry
- ❺ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ❻ Cornelius LANCZOS and Generalized Lagrangians
- ❼ The ADM Approach to the Einstein Equations
- ❽ Supergravity and Killing Spinors
- ❾ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ❶ The Prehistory of General Relativity
- ❷ Gregorio RICCI-CURBASTRO, a Central Figure
- ❸ The Giant Step of General Relativity
- ❹ Hermann WEYL and Conformal Geometry
- ❺ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ❻ Cornelius LANCZOS and Generalized Lagrangians
- ❼ The ADM Approach to the Einstein Equations
- ❽ Supergravity and Killing Spinors
- ❾ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ❶ The Prehistory of General Relativity
- ❷ Gregorio RICCI-CURBASTRO, a Central Figure
- ❸ **The Giant Step of General Relativity**
- ❹ Hermann WEYL and Conformal Geometry
- ❺ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ❻ Cornelius LANCZOS and Generalized Lagrangians
- ❼ The ADM Approach to the Einstein Equations
- ❽ Supergravity and Killing Spinors
- ❾ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ➊ The Prehistory of General Relativity
- ➋ Gregorio RICCI-CURBASTRO, a Central Figure
- ➌ The Giant Step of General Relativity
- ➍ **Hermann WEYL and Conformal Geometry**
- ➎ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ➏ Cornelius LANCZOS and Generalized Lagrangians
- ➐ The ADM Approach to the Einstein Equations
- ➑ Supergravity and Killing Spinors
- ➒ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ➊ The Prehistory of General Relativity
- ➋ Gregorio RICCI-CURBASTRO, a Central Figure
- ➌ The Giant Step of General Relativity
- ➍ Hermann WEYL and Conformal Geometry
- ➎ **Kaluza-Klein Theory, an Anticipation of Bundle Geometry**
- ➏ Cornelius LANCZOS and Generalized Lagrangians
- ➐ The ADM Approach to the Einstein Equations
- ➑ Supergravity and Killing Spinors
- ➒ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ➊ The Prehistory of General Relativity
- ➋ Gregorio RICCI-CURBASTRO, a Central Figure
- ➌ The Giant Step of General Relativity
- ➍ Hermann WEYL and Conformal Geometry
- ➎ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ➏ Cornelius LANCZOS and Generalized Lagrangians
- ➐ The ADM Approach to the Einstein Equations
- ➑ Supergravity and Killing Spinors
- ➒ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ➊ The Prehistory of General Relativity
- ➋ Gregorio RICCI-CURBASTRO, a Central Figure
- ➌ The Giant Step of General Relativity
- ➍ Hermann WEYL and Conformal Geometry
- ➎ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ➏ Cornelius LANCZOS and Generalized Lagrangians
- ➐ **The ADM Approach to the Einstein Equations**
- ➑ Supergravity and Killing Spinors
- ➒ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ➊ The Prehistory of General Relativity
- ➋ Gregorio RICCI-CURBASTRO, a Central Figure
- ➌ The Giant Step of General Relativity
- ➍ Hermann WEYL and Conformal Geometry
- ➎ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ➏ Cornelius LANCZOS and Generalized Lagrangians
- ➐ The ADM Approach to the Einstein Equations
- ➑ **Supergravity and Killing Spinors**
- ➒ Some Concluding Remarks

Outline of the Lecture (cont.)

Here is an outline of the topics to be covered:

- ① The Prehistory of General Relativity
- ② Gregorio RICCI-CURBASTRO, a Central Figure
- ③ The Giant Step of General Relativity
- ④ Hermann WEYL and Conformal Geometry
- ⑤ Kaluza-Klein Theory, an Anticipation of Bundle Geometry
- ⑥ Cornelius LANCZOS and Generalized Lagrangians
- ⑦ The ADM Approach to the Einstein Equations
- ⑧ Supergravity and Killing Spinors
- ⑨ Some Concluding Remarks

The Prehistory of General Relativity

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *“Hypothesen, welche der Geometrie zu Grunde liegen”*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *“Application on space”*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the “very large”, but also has a paragraph on the “very small”;
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *“Hypothesen, welche der Geometrie zu Grunde liegen”*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *“Application on space”*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the “very large”, but also has a paragraph on the “very small”;
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *"Hypothesen, welche der Geometrie zu Grunde liegen"*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *"Application on space"*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the "very large", but also has a paragraph on the "very small";
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *"Hypothesen, welche der Geometrie zu Grunde liegen"*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *"Application on space"*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the "very large", but also has a paragraph on the "very small";
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *"Hypothesen, welche der Geometrie zu Grunde liegen"*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *"Application on space"*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the "very large", but also has a paragraph on the "very small";
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *"Hypothesen, welche der Geometrie zu Grunde liegen"*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *"Application on space"*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the "very large", but also has a paragraph on the "very small";
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

The fundamental document by B. RIEMANN

Bernhard RIEMANN defended his habilitation on June 10, 1854, in Göttingen, and, for the accompanying presentation, the jury members, among whom Carl Friedrich GAUSS, picked the topic *"Hypothesen, welche der Geometrie zu Grunde liegen"*:

- This essay was produced in a very short period of time.
- In spite of the fact that it is dealing with Geometry, it does not contain any figure, and only one formula;
- In the last part entitled *"Application on space"*, B. Riemann discusses the relevance of his considerations to several parts of Physics: Astronomy, the "very large", but also has a paragraph on the "very small";
- He discusses explicitly the relevance of the concepts he introduces to discuss both continuous and discrete settings;
- He cares about whether the new concepts of a Riemannian manifold can be approached by actual measurements.

W.K. CLIFFORD's Visionary Note

In a note entitled “*On the Space-Theory of Matter*” published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

“I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

4) That in this physical world nothing else takes place but this variation subject to the law of continuity.”

W.K. CLIFFORD's Visionary Note

In a note entitled "*On the Space-Theory of Matter*" published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

- 1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.*
- 2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.*
- 3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.*
- 4) That in this physical world nothing else takes place but this variation subject to the law of continuity."*

W.K. CLIFFORD's Visionary Note

In a note entitled "*On the Space-Theory of Matter*" published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

4) That in this physical world nothing else takes place but this variation subject to the law of continuity."

W.K. CLIFFORD's Visionary Note

In a note entitled "*On the Space-Theory of Matter*" published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

4) That in this physical world nothing else takes place but this variation subject to the law of continuity."

W.K. CLIFFORD's Visionary Note

In a note entitled "*On the Space-Theory of Matter*" published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

4) That in this physical world nothing else takes place but this variation subject to the law of continuity."

W.K. CLIFFORD's Visionary Note

In a note entitled "*On the Space-Theory of Matter*" published in 1876, William Kingdon CLIFFORD develops a truly visionary anticipation of General Relativity. Here is what he says:

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold:

1) That small portions of space are in fact analogous to little hills on a surface which is on the average flat, namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

4) That in this physical world nothing else takes place but this variation subject to the law of continuity."

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature (1,3);
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature (1,3);
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature (1,3);
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature $(1,3)$;
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature (1,3);
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

The Geometry underlying Special Relativity

In 1905, Albert EINSTEIN introduces Special Relativity from a thorough reflection on the Michelson-Morley experiments giving grounds to the idea that the speed of light is absolute.

- He elaborated on hints given by Hendrik LORENTZ which led him to discuss simultaneity and synchronization of clocks;
- The mathematical consequences of the new approach were drawn by Henri POINCARÉ and Hermann MINKOWSKI;
- H. POINCARÉ's approach, presented in an article entitled "*Le mouvement de l'électron*" uses invariant theory to identify the geometric structure relevant, namely an indefinite metric of signature (1,3);
- H. MINKOWSKI's approach is much more in the spirit of EUCLIDE's Elements; he states very clearly that one has to think of a continuum unifying Space and Time, published in the famous article entitled "*Raum und Zeit*".

2. Gregorio RICCI-CURBASTRO, a Central Figure for the Development of Geometry

RICCI's Absolute Differential Calculus

G. RICCI-CURBASTRO gave a conceptual content to the objects identified by B. RIEMANN and E.B. CHRISTOFFEL as important to develop and understand RIEMANN's Geometry. He developed his ideas, which became standard tools in Differential Geometry at large, in four publications in the period 1888-1892:

- First in *"Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicati"*, published in *"Studi editi dall'Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna"* in 1888.
- He developed it in the 1892 issue of the Bulletin des Sciences Mathématiques;
- Later, his joint article with his student Tullio LEVI-CIVITA *"Méthodes de calcul différentiel absolu et leurs applications"* published in 1900 in the Mathematische Annalen became a reference on the subject.

RICCI's Absolute Differential Calculus

G. RICCI-CURBASTRO gave a conceptual content to the objects identified by B. RIEMANN and E.B. CHRISTOFFEL as important to develop and understand RIEMANN's Geometry. He developed his ideas, which became standard tools in Differential Geometry at large, in four publications in the period 1888-1892:

- First in *"Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicati*, published in *"Studi editi dall'Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna"* in 1888.
- He developed it in the 1892 issue of the Bulletin des Sciences Mathématiques;
- Later, his joint article with his student Tullio LEVI-CIVITA *"Méthodes de calcul différentiel absolu et leurs applications"* published in 1900 in the Mathematische Annalen became a reference on the subject.

RICCI's Absolute Differential Calculus

G. RICCI-CURBASTRO gave a conceptual content to the objects identified by B. RIEMANN and E.B. CHRISTOFFEL as important to develop and understand RIEMANN's Geometry. He developed his ideas, which became standard tools in Differential Geometry at large, in four publications in the period 1888-1892:

- First in *"Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicati"*, published in *"Studi editi dall'Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna"* in 1888.
- He developed it in the 1892 issue of the *Bulletin des Sciences Mathématiques*;
- Later, his joint article with his student Tullio LEVI-CIVITA *"Méthodes de calcul différentiel absolu et leurs applications"* published in 1900 in the *Mathematische Annalen* became a reference on the subject.

RICCI's Absolute Differential Calculus

G. RICCI-CURBASTRO gave a conceptual content to the objects identified by B. RIEMANN and E.B. CHRISTOFFEL as important to develop and understand RIEMANN's Geometry. He developed his ideas, which became standard tools in Differential Geometry at large, in four publications in the period 1888-1892:

- First in *"Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicati"*, published in *"Studi editi dall'Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna"* in 1888.
- He developed it in the 1892 issue of the Bulletin des Sciences Mathématiques;
- Later, his joint article with his student Tullio LEVI-CIVITA *"Méthodes de calcul différentiel absolu et leurs applications"* published in 1900 in the Mathematische Annalen became a reference on the subject.

RICCI's Absolute Differential Calculus (cont.)

MÉLANGES.

RÉSUMÉ DE QUELQUES TRAVAUX SUR LES SYSTÈMES VARIABLES DE FONCTIONS ASSOCIÉS A UNE FORME DIFFÉRENTIELLE QUADRATIQUE ^(*);

PAR M. G. RICCI,

Professeur à la Faculté des Sciences de Padoue.

1. *Notions et définitions générales sur les systèmes variables.*

— Dans l'Analyse pure aussi bien que dans ses applications à la Géométrie, à la Mécanique et à la Physique, il y a souvent lieu de considérer des systèmes de fonctions de n variables x_1, x_2, \dots, x_n , qui peuvent toutes être représentées par un symbole général $X_{rs\dots t}$, dans lequel on a un certain nombre d'indices, qui peuvent tous prendre les n valeurs $1, 2, \dots, n$. Si le nombre des indices est m , nous aurons là ce que nous appellerons un *système de fonctions à n variables m^{uple} ou d'ordre m* et les n^m fonctions, distinctes ou non, dont il résulte, seront les *éléments* du système.

RICCI's Absolute Differential Calculus (cont.)

2. *Les systèmes variables associés à une forme différentielle quadratique. Dérivations covariante et contrevariante selon cette forme.*

Dans ce qui va suivre nous allons associer les systèmes variables de fonctions de n variables indépendantes à une expression homogène et du deuxième degré par rapport aux différentielles de ces variables. Nous représenterons par

$$\varphi = \sum_{rs} a_{rs} dx_r dx_s$$

cette expression, qu'on pourra appeler une *forme différentielle quadratique* à n variables, et nous supposons qu'elle soit *irréductible*, c'est-à-dire qu'il n'existe pas une forme différentielle quadratique à m variables y_1, y_2, \dots, y_m , m étant moindre que n , qui devienne identique à φ , en substituant à y_1, y_2, \dots, y_m certaines fonctions de x_1, x_2, \dots, x_n . Comme je l'ai démontré (*), pour que la forme φ soit irréductible, il est nécessaire et il suffit que son discriminant, que nous indiquerons par α , ne soit pas nul.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

The Introduction of the Ricci Curvature

In his 1854 essay B. RIEMANN attaches to a metric g what is now called the *Riemann curvature tensor* R^g , that is a 4-tensor measuring the deviation of a space from being Euclidean.

- In “*Direzioni et invarianti principali in una varieta qualunque*” (Atti del Real Inst. Veneto) published in 1904, G. RICCI-CURBASTRO introduces the *Ricci curvature* r^g , that he defines as

$$r^g(X, Y) = \text{Trace}(Z \mapsto R_{Z, X}^g Y) .$$

- His motivation is purely geometric, namely that of introducing some privileged directions at every point of a Riemannian manifold.
- This motivation proved to be totally useless... but the Ricci curvature proved itself to be a very important concept for completely other reasons.

5. The Giant Leap Forward of General Relativity

The Geometry behind General Relativity

In 1913, Albert EINSTEIN and Marcel GROSSMANN made a first attempt to define a *theory of General Relativity* based on a Lorentzian metric.

- The shift is considerable as it consists in replacing the scalar field that generated the gravitational force in Newton's theory by a 2-tensor field, measuring *generalized lengths*, a very considerable mathematical sophistication;
- This requires passing from Minkowski Geometry (the analog of Euclidean Geometry in a purely algebraic setting and the framework of Special Relativity) to *Lorentzian Geometry*, the analog of Riemannian Geometry.
- If the field equations proposed in their back-to-back articles published in 1913 in the *Annalen der Physik* involve the Ricci curvature, it does it in a non covariant way, when the theory needs to be fully covariant.

The Geometry behind General Relativity

In 1913, Albert EINSTEIN and Marcel GROSSMANN made a first attempt to define a *theory of General Relativity* based on a Lorentzian metric.

- The shift is considerable as it consists in replacing the scalar field that generated the gravitational force in Newton's theory by a 2-tensor field, measuring *generalized lengths*, a very considerable mathematical sophistication;
- This requires passing from Minkowski Geometry (the analog of Euclidean Geometry in a purely algebraic setting and the framework of Special Relativity) to *Lorentzian Geometry*, the analog of Riemannian Geometry.
- If the field equations proposed in their back-to-back articles published in 1913 in the *Annalen der Physik* involve the Ricci curvature, it does it in a non covariant way, when the theory needs to be fully covariant.

The Geometry behind General Relativity

In 1913, Albert EINSTEIN and Marcel GROSSMANN made a first attempt to define a *theory of General Relativity* based on a Lorentzian metric.

- The shift is considerable as it consists in replacing the scalar field that generated the gravitational force in Newton's theory by a 2-tensor field, measuring *generalized lengths*, a very considerable mathematical sophistication;
- This requires passing from Minkowski Geometry (the analog of Euclidean Geometry in a purely algebraic setting and the framework of Special Relativity) to *Lorentzian Geometry*, the analog of Riemannian Geometry.
- If the field equations proposed in their back-to-back articles published in 1913 in the *Annalen der Physik* involve the Ricci curvature, it does it in a non covariant way, when the theory needs to be fully covariant.

The Geometry behind General Relativity

In 1913, Albert EINSTEIN and Marcel GROSSMANN made a first attempt to define a *theory of General Relativity* based on a Lorentzian metric.

- The shift is considerable as it consists in replacing the scalar field that generated the gravitational force in Newton's theory by a 2-tensor field, measuring *generalized lengths*, a very considerable mathematical sophistication;
- This requires passing from Minkowski Geometry (the analog of Euclidean Geometry in a purely algebraic setting and the framework of Special Relativity) to *Lorentzian Geometry*, the analog of Riemannian Geometry.
- If the field equations proposed in their back-to-back articles published in 1913 in the *Annalen der Physik* involve the Ricci curvature, it does it in a non covariant way, when the theory needs to be fully covariant.

The Geometry behind General Relativity

In 1913, Albert EINSTEIN and Marcel GROSSMANN made a first attempt to define a *theory of General Relativity* based on a Lorentzian metric.

- The shift is considerable as it consists in replacing the scalar field that generated the gravitational force in Newton's theory by a 2-tensor field, measuring *generalized lengths*, a very considerable mathematical sophistication;
- This requires passing from Minkowski Geometry (the analog of Euclidean Geometry in a purely algebraic setting and the framework of Special Relativity) to *Lorentzian Geometry*, the analog of Riemannian Geometry.
- If the field equations proposed in their back-to-back articles published in 1913 in the *Annalen der Physik* involve the Ricci curvature, it does it in a non covariant way, when the theory needs to be fully covariant.

The Einstein-Hilbert Equations

Consistent field equations were found in 1915 as the result of a joint effort by A. EINSTEIN and David HILBERT.

- As you all know, they obtain these equations as Euler-Lagrange equations of a variational principle for the gravitational potential

$$g \mapsto \int_{\text{space-time}} s^g \, v_g ,$$

where $s^g = \text{Trace}_g r^g$ is the *scalar curvature* of g .

- The Einstein equations are

$$r^g - \frac{1}{2} s^g g = T ,$$

where T is the stress-energy tensor (in the vacuum $T \equiv 0$).

- Several approaches to discuss the Einstein equations have been used since their inception; we will discuss some of them.

The Einstein-Hilbert Equations

Consistent field equations were found in 1915 as the result of a joint effort by A. EINSTEIN and David HILBERT.

- As you all know, they obtain these equations as Euler-Lagrange equations of a variational principle for the gravitational potential

$$g \mapsto \int_{\text{space-time}} s^g v_g ,$$

where $s^g = \text{Trace}_g r^g$ is the *scalar curvature* of g .

- The Einstein equations are

$$r^g - \frac{1}{2} s^g g = T ,$$

where T is the stress-energy tensor (in the vacuum $T \equiv 0$).

- Several approaches to discuss the Einstein equations have been used since their inception; we will discuss some of them.

The Einstein-Hilbert Equations

Consistent field equations were found in 1915 as the result of a joint effort by A. EINSTEIN and David HILBERT.

- As you all know, they obtain these equations as Euler-Lagrange equations of a variational principle for the gravitational potential

$$g \mapsto \int_{\text{space-time}} s^g v_g ,$$

where $s^g = \text{Trace}_g r^g$ is the *scalar curvature* of g .

- The Einstein equations are

$$r^g - \frac{1}{2} s^g g = T ,$$

where T is the stress-energy tensor (in the vacuum $T \equiv 0$).

- Several approaches to discuss the Einstein equations have been used since their inception; we will discuss some of them.

The Einstein-Hilbert Equations

Consistent field equations were found in 1915 as the result of a joint effort by A. EINSTEIN and David HILBERT.

- As you all know, they obtain these equations as Euler-Lagrange equations of a variational principle for the gravitational potential

$$g \mapsto \int_{\text{space-time}} s^g v_g ,$$

where $s^g = \text{Trace}_g r^g$ is the *scalar curvature* of g .

- The Einstein equations are

$$r^g - \frac{1}{2} s^g g = T ,$$

where T is the stress-energy tensor (in the vacuum $T \equiv 0$).

- Several approaches to discuss the Einstein equations have been used since their inception; we will discuss some of them.

The Einstein-Hilbert Equations

Consistent field equations were found in 1915 as the result of a joint effort by A. EINSTEIN and David HILBERT.

- As you all know, they obtain these equations as Euler-Lagrange equations of a variational principle for the gravitational potential

$$g \mapsto \int_{\text{space-time}} s^g v_g ,$$

where $s^g = \text{Trace}_g r^g$ is the *scalar curvature* of g .

- The Einstein equations are

$$r^g - \frac{1}{2} s^g g = T ,$$

where T is the stress-energy tensor (in the vacuum $T \equiv 0$).

- Several approaches to discuss the Einstein equations have been used since their inception; we will discuss some of them.

4. Hermann WEYL and Conformal Geometry

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

An attempt for a Further Unification

In 1917, Hermann WEYL made a first attempt to unify further Physics in proposing that a scalar factor in front of the metric could allow a unification of Gravitation with Electromagnetic theory.

- This was actually a first serious consideration of a *gauge theory*, and led to further understanding of the general concept of a covariant derivative, not necessarily a metric one.
- By the very way he introduced the scalar field, he was in fact considering *conformal classes of metrics*, i.e. metrics $\tilde{g} = e^u g$, where u is a smooth function and g a Riemannian or Lorentzian metric.
- This led him to identify the part of the curvature that is not affected by a conformal change of metric, now called the *Weyl curvature tensor*.
- It is a remarkable fact that this part of the curvature tensor is precisely the part that is complementary to the Ricci curvature, when lifted to the space of curvature tensors.

5. Kaluza-Klein Theory, an Anticipation of Bundle Geometry

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

The Kaluza Ansatz

In 1918, Theodor KALUZA sent to A. EINSTEIN another approach to unifying Gravitation and Electromagnetism.

- Starting from a Lorentzian metric g on the space-time M and the electromagnetic potential ξ on M , he introduced an extended 5-dimensional Lorentzian space-time $\tilde{M} = M \times \mathbb{R}$ with metric \tilde{g} obtained using the *Ansatz*: if $\pi : M \times \mathbb{R} \rightarrow M$, then $\tilde{g} = \phi(\pi^*g + (du + \pi^*\xi) \otimes (du + \pi^*\xi))$ (u extra-variable).
- The key formula is the one that describes the 5-dimensional Ricci tensor $r^{\tilde{g}}$ in terms of the data g , ϕ and ξ .
- The striking fact is that claiming that $r^{\tilde{g}} = 0$ gives back the Maxwell equations for ξ and the right coupled Einstein-Maxwell equations for g et ξ .
- What lies behind is the formalism of *Riemannian submersions* and the remarkable formulas for computing the curvature of the bundle metric from other geometric elements, developed 40 years later by Barrett O'NEILL.

6. Cornelius LANCZOS and Generalized Lagrangians

What is Special about the Total Scalar Curvature

It was known since the middle of the 19th century that in dimension 2 the Total Scalar Curvature functional on the space $\mathcal{Riem} M$ of Riemannian metrics on M is constant. Hence:

- It is therefore a differential invariant (actually a topological one) giving rise to the GAUSS-BONNET Formula:

$$\int_M s^g v_g = 2\pi \chi(M) ,$$

where $\chi(M)$ denotes the *Euler-Poincaré characteristic* of M ;

- This fact is reflected in all dimensions in the property that, although the functional involves second order derivatives of the field, the terms of fourth order in the First Variation Formula disappear by integration by parts, giving only boundary terms if M has a boundary.

What is Special about the Total Scalar Curvature

It was known since the middle of the 19th century that in dimension 2 the Total Scalar Curvature functional on the space $\mathcal{Riem} M$ of Riemannian metrics on M is constant. Hence:

- It is therefore a differential invariant (actually a topological one) giving rise to the GAUSS-BONNET Formula:

$$\int_M s^g v_g = 2\pi \chi(M) ,$$

where $\chi(M)$ denotes the *Euler-Poincaré characteristic* of M ;

- This fact is reflected in all dimensions in the property that, although the functional involves second order derivatives of the field, the terms of fourth order in the First Variation Formula disappear by integration by parts, giving only boundary terms if M has a boundary.

What is Special about the Total Scalar Curvature

It was known since the middle of the 19th century that in dimension 2 the Total Scalar Curvature functional on the space $\mathcal{Riem} M$ of Riemannian metrics on M is constant. Hence:

- It is therefore a differential invariant (actually a topological one) giving rise to the GAUSS-BONNET Formula:

$$\int_M s^g v_g = 2\pi \chi(M) ,$$

where $\chi(M)$ denotes the *Euler-Poincaré characteristic* of M ;

- This fact is reflected in all dimensions in the property that, although the functional involves second order derivatives of the field, the terms of fourth order in the First Variation Formula disappear by integration by parts, giving only boundary terms if M has a boundary.

What is Special about the Total Scalar Curvature

It was known since the middle of the 19th century that in dimension 2 the Total Scalar Curvature functional on the space $\mathcal{Riem} M$ of Riemannian metrics on M is constant. Hence:

- It is therefore a differential invariant (actually a topological one) giving rise to the GAUSS-BONNET Formula:

$$\int_M s^g v_g = 2\pi \chi(M) ,$$

where $\chi(M)$ denotes the *Euler-Poincaré characteristic* of M ;

- This fact is reflected in all dimensions in the property that, although the functional involves second order derivatives of the field, the terms of fourth order in the First Variation Formula disappear by integration by parts, giving only boundary terms if M has a boundary.

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- **Indeed, one has:**

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

Cornelius LANCZOS' Generalized Lagrangians

In the late 30s, while looking for other Lagrangians for General Relativity, Cornelius LANCZOS got interested in quadratic functionals in the curvature of the following type on space-times of dimension 4: n

$$\int_M (\alpha |R^g|^2 + \beta |r^g|^2 + \gamma (s^g)^2) v_g .$$

He noticed that:

- For a good choice of coefficients α , β and γ , the functional does not generate any field equation;
- He then deduced that this Lagrangian was not interesting;
- From the point of view of a mathematician, this once more means that one has caught an invariant of M , a major feat;
- Indeed, one has:

$$\int_M (|R^g|^2 - \frac{1}{4} |r^g|^2 + (s^g)^2) v_g = 8\pi^2 \chi(M) .$$

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on "*A New Intrinsic Proof of the Gauss-Bonnet Theorem*", that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on “*A New Intrinsic Proof of the Gauss-Bonnet Theorem*”, that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on “*A New Intrinsic Proof of the Gauss-Bonnet Theorem*”, that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on “*A New Intrinsic Proof of the Gauss-Bonnet Theorem*”, that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on “*A New Intrinsic Proof of the Gauss-Bonnet Theorem*”, that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

The Missed Opportunity

The article by Cornelius LANCZOS was published in 1938 in the Annals of Mathematics (actually in a volume that contained one of the first articles by CHERN Shiing Shen):

- It is only in 1944 that S.S. CHERN published again in Annals of Mathematics his ground breaking article on “*A New Intrinsic Proof of the Gauss-Bonnet Theorem*”, that lead to the theory of characteristic classes.
- The whole new idea is of Kaluza-Klein type, namely lifting objects on a manifold to its frame bundle.
- The formula for the Euler characteristic in terms of curvature in terms of polynomials in the curvature belongs to these developments.
- I was very surprised to realize that actually S.S. CHERN had never heard of the Lanczos Generalized Lagrangians.

7. The Arnowitt-Deser-Misner Approach to the Einstein Equations

The Einstein Equations in the ADM Approach

- In the ADM approach, first suggested by Yvonne BRUHAT, the 4-dimensional Lorentzian metric g on M is translated into that of a curve $t \mapsto (g_t, k_t, \phi_t)$ where g_t is the induced metric on a space hypersurface Σ , hence a 3-dimensional Riemannian manifold, k_t its second fundamental form and ϕ_t the lapse function.
- The Einstein equations take the form of *evolution equations*

$$\begin{cases} \frac{\partial g_t}{\partial t} &= -2 \phi_t k_t \\ \frac{\partial k_t}{\partial t} &= -D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_{t \cdot g_t} k_t)) ; \end{cases}$$

and constraint equations

$$D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_{t \cdot g_t} k_t)) = 0$$

$$D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_{t \cdot g_t} k_t)) = 0$$

The Einstein Equations in the ADM Approach

- In the ADM approach, first suggested by Yvonne BRUHAT, the 4-dimensional Lorentzian metric g on M is translated into that of a curve $t \mapsto (g_t, k_t, \phi_t)$ where g_t is the induced metric on a space hypersurface Σ , hence a 3-dimensional Riemannian manifold, k_t its second fundamental form and ϕ_t the lapse function.
- The Einstein equations take the form of *evolution equations*

$$\begin{cases} \frac{\partial g_t}{\partial t} &= -2 \phi_t k_t \\ \frac{\partial k_t}{\partial t} &= -D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_t \cdot_{g_t} k_t)) ; \end{cases}$$

and *constraint equations*

$$\begin{cases} \delta_{g_t} k_t - d(c_{g_t}(k_t)) &= 0 \\ s_{g_t} - |k_t|_{g_t}^2 + (c_{g_t}(k_t))^2 &= 0 . \end{cases}$$

The Einstein Equations in the ADM Approach

- In the ADM approach, first suggested by Yvonne BRUHAT, the 4-dimensional Lorentzian metric g on M is translated into that of a curve $t \mapsto (g_t, k_t, \phi_t)$ where g_t is the induced metric on a space hypersurface Σ , hence a 3-dimensional Riemannian manifold, k_t its second fundamental form and ϕ_t the lapse function.
- The Einstein equations take the form of *evolution equations*

$$\begin{cases} \frac{\partial g_t}{\partial t} &= -2 \phi_t k_t \\ \frac{\partial k_t}{\partial t} &= -D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_t \cdot_{g_t} k_t)) ; \end{cases}$$

and *constraint equations*

$$\begin{cases} \delta_{g_t} k_t - d(c_{g_t}(k_t)) &= 0 \\ s_{g_t} - |k_t|_{g_t}^2 + (c_{g_t}(k_t))^2 &= 0 . \end{cases}$$

The Einstein Equations in the ADM Approach

- In the ADM approach, first suggested by Yvonne BRUHAT, the 4-dimensional Lorentzian metric g on M is translated into that of a curve $t \mapsto (g_t, k_t, \phi_t)$ where g_t is the induced metric on a space hypersurface Σ , hence a 3-dimensional Riemannian manifold, k_t its second fundamental form and ϕ_t the lapse function.
- The Einstein equations take the form of *evolution equations*

$$\begin{cases} \frac{\partial g_t}{\partial t} &= -2 \phi_t k_t \\ \frac{\partial k_t}{\partial t} &= -D^{g_t} d\phi_t + \phi_t (r_{g_t} + (c_{g_t}(k_t)k_t - 2 k_t \cdot_{g_t} k_t)) ; \end{cases}$$

and *constraint equations*

$$\begin{cases} \delta_{g_t} k_t - d(c_{g_t}(k_t)) &= 0 \\ s_{g_t} - |k_t|_{g_t}^2 + (c_{g_t}(k_t))^2 &= 0 . \end{cases}$$

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem} \Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2} ds^g$ expresses that $r^g - \frac{1}{2} s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem}\Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2}ds^g$ expresses that $r^g - \frac{1}{2}s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem}\Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2}ds^g$ expresses that $r^g - \frac{1}{2}s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem} \Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2} ds^g$ expresses that $r^g - \frac{1}{2} s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem}\Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2}ds^g$ expresses that $r^g - \frac{1}{2}s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

Working in Superspace

- This suggests to consider as a primary object the space of Riemannian metrics on a manifold $\mathcal{Riem} \Sigma$, that relativists called for some time *Superspace*.
- Actually, the Total Scalar Curvature is a function on the space $\mathcal{Lor} M$ of Lorentzian metrics on M .
- The Einstein equations, the Euler-Lagrange equations of this Lagrangian, express the vanishing of its differential.
- There are usually expressed as the vanishing of the gradient of the function using a metric on $\mathcal{Lor} M$, a space which has several natural Riemannian or pseudo-Riemannian metrics, one of them introduced by Bryce DE WITT.
- Using this approach, the fundamental conservation law $\delta r^g = -\frac{1}{2} ds^g$ expresses that $r^g - \frac{1}{2} s^g g$ lies perpendicularly to the orbit of g under diffeomorphisms of M , an obvious fact as the Total Scalar Curvature function is invariant under the action of diffeomorphisms.

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right) .$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t} .$$

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right).$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t}.$$

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right).$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t}.$$

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right).$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t}.$$

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right).$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t}.$$

On the Way to the Ricci Flow

- If, in the ADM setting, we focus on the case where ϕ is a constant, say 1, the evolution equations take the form:

$$\frac{\partial^2 g_t}{\partial t^2} = -r^{g_t} + \left(c_{g_t} \left(\frac{\partial g_t}{\partial t} \right) \frac{\partial g_t}{\partial t} - 2 \frac{\partial g_t}{\partial t} \cdot_{g_t} \frac{\partial g_t}{\partial t} \right) .$$

- Hence, this suggests to consider differential equations involving the Ricci curvature in $\mathcal{Riem} \Sigma$, here the second time derivative of the curve of metrics.
- Considering the first derivative is *a priori* simpler. This is what Thierry AUBIN did, in 1970: he deformed a metric in the direction of its Ricci curvature to improve the curvature.
- The key formula is $\frac{ds^{g-tr^g}}{dt} = |r^g|^2 - \frac{1}{2} \Delta_g s^g$, showing for example that, at a constant scalar curvature metric g , this deformation increases the value of the scalar curvature.
- This then suggests to consider the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -r^{g_t} .$$

Asking about the Ricci Flow

1979 Berlin Conference on Global Analysis

RICCI CURVATURE and EINSTEIN METRICS

by

Jean Pierre BOURGUIGNON

Centre de Mathématiques*
Ecole Polytechnique

91128 PALAISEAU CEDEX (France)

0.0 After some historical comments, this survey talk intends to summarize what is known about the Ricci curvature of a Riemannian manifold and why it is today especially interesting to consider problems connected with it. Emphasis is put on the relation between the study of the Ricci curvature and the study of Einstein metrics. Since presently we are far from a complete understanding of the situation, we state some problems which, we think, could serve as useful steps towards a better knowledge of this important geometric quantity.

3.24 QUESTION .- Does the local flow theorem hold for the vector fields $g \mapsto \text{Ric}^g - k \text{ Scal}^g g$ on the space of metrics ? What is the global behaviour of their integral curves if they exist ?

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

The Ricci Flow

- Showing that the Ricci flow has a local solution in the space of metrics requires to resolve a system of non-linear PDEs that is degenerate from the point of view of analysis as

$$r_{ij}^g = -\frac{1}{2} g^{kl} \left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^l \partial x^i} \right) + Q(g, \frac{\partial g}{\partial x})$$

- This was done by Richard HAMILTON and Dennis DETURCK in the early 1980s.
- To show that global solutions exist requires that one makes geometric assumptions, and this was done in a series of remarkable papers by R. HAMILTON.
- The rest is History: this has been pushed much further by Grisha PERELMAN, who showed how to go beyond singularities, opening the way to solving the Poincaré Conjecture, a purely topological question.
- Many other applications of the Ricci flow have been made.

8. Supergravity and Killing Spinors

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

Supergravity

- Among generalizations of General Relativity, *Supergravity* stands out as it reappeared a few times after being discarded.
- It combined *Supersymmetry* with General Relativity and fostered the study of very specific dimensions, mostly 7 and 11, with rich geometries such as exceptional *holonomy* G_2 .
- A precise definition of an infinitesimal supersymmetry emerged from these considerations in the form of *Killing Spinors*, i.e. spinor fields ψ satisfying, for any tangent vector X , the equation

$$\nabla_X \psi + \lambda X.\psi = 0,$$

where $.$ denotes Clifford multiplication, and λ is a scalar. The concept was introduced by Martin WALKER and Roger PENROSE studying geodesics of the Kerr metric.

- Spaces admitting non trivial Killing spinors are automatically Einstein. Actually, a cone over them admits a parallel spinor field, as was shown by Christian BÄR.

9. Concluding Remarks

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
 - or changed the point of view on others (Ricci curvature),
 - or else suggested new approaches (the Ricci flow),
 - or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

Concluding Remarks

The purpose of the lecture was to highlight several instances where the Theory of General Relativity

- either triggered the study of new objects, which turned to lead to important developments in Mathematics (bundle geometry, Killing spinors),
- or changed the point of view on others (Ricci curvature),
- or else suggested new approaches (the Ricci flow),
- or even led to missed opportunities.

More deeply, as a system of non-linear PDEs with an analytical degeneracy due to its natural geometric nature, the Einstein equations of General Relativity have posed tremendous challenges as far as getting local, and later global solutions, and also for the study of its singularities.

The presentations of recent outstanding results of this nature will be the topic of several lectures during this conference.

I thank you for your attention.

Jean-Pierre BOURGUIGNON
Institut des Hautes Études Scientifiques
35, route de Chartres
F-91440 BURES-SUR-YVETTE
(France)
JPB@ihes.fr