AN INTRODUCTION TO SELF-GRAVITATING MATTER

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General topic

- Einstein equations for self-gravitating matter $G_{\alpha\beta} = 8\pi \, T_{\alpha\beta}$
- Cauchy developments from prescribed initial data sets on a spacelike hypersurface
- Global dynamics of the matter content
- Global geometry of the spacetimes

SERIES of LECTURES given at the Institut Henri Poincaré in the Fall 2015

Three-Month Program on MATHEMATICAL GENERAL RELATIVITY

Celebration of the 100th Anniversary of the publication of Einstein's papers

VIDEOS available at /www.youtube.com/user/PoincareInstitute

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MAIN OBJECTIVES

Introduction to Matter Models

Three main results

Weakly Regular Vacuum Spacetimes with T2 Symmetry

impulsive gravitational waves

Weakly Regular Matter Spacetimes with Spherical Symmetry

shock waves in self-gravitating compressible fluids

Stability of Minkowski Space for Self-Gravitating Massive Fields

Einstein-Klein-Gordon equation, theory of modified gravity

Remarks.

- techniques of analysis
- matter models or/and weak solutions
- scope of the lectures
  - selected results with full proofs
  - overview of more advanced statements

CHAPTER I
Introduction to Matter Models

Outline of this chapter

- Section 1. **Field Equations**
  - Einstein equations, Euler equations
  - initial value problem
- Section 2. **The Theory of Modified Gravity**
  - the f(R)–gravity
  - Jordan coupling vs. Einstein coupling
- Section 3. **The Formulation in Wave Gauge**
  - augmented Formulation
  - wave coordinates
  - Einstein-Klein-Gordon system

Section 1. FIELD EQUATIONS

EINSTEIN EQUATIONS

Lorentzian metric g on a four-dimensional manifold M

- Inner product with signature $(-,+,+,+)$
- A vector $X \in T_pM$ at $p \in M$ is said to be *time-like, null,* or *space-like* iff $g(X,X)$ is negative, zero, or positive.
- *Time-orientation:* future or past timelike vectors

(observers, worldlines)

Differential gemetry

- Local coordinates (x^α) with Greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$
- $\frac{\partial}{\partial x^\alpha}$ defines a basis of the tangent space TM
- Vector fields $X = X^\alpha \frac{\partial}{\partial x^\alpha}$ and metric $g = g_{\alpha\beta} dx^\alpha dx^\beta$

(summation over repeated indices)

Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$

- Riemannian geometry (see below)
- Ricci curvature $R_{\alpha\gamma} := g^{\beta\delta} R_{\alpha\beta\gamma\delta}$
- scalar curvature $R = g^{\alpha\gamma} R_{\alpha\gamma}$

Minkowski spacetime $M = \mathbb{R}^4$ and $g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

- Vanishing curvature

VACUUM EINSTEIN EQUATIONS

$$R_{\alpha\beta} = 0 \qquad \text{(vacuum spacetime)}$$

- Vanishing scalar curvature

Decomposition of the Riemann curvature:

$$R_{\alpha\beta\gamma\delta} = \frac{R}{12}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) + \frac{1}{2}\left(g_{\alpha\gamma}S_{\beta\delta} - g_{\alpha\delta}S_{\beta\gamma} + g_{\beta\delta}S_{\alpha\gamma} - g_{\beta\gamma}S_{\alpha\delta}\right) + W_{\alpha\beta\gamma\delta}$$

- $S_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{4}Rg_{\alpha\beta}$ (traceless part)
- Only the Weyl curvature is non-vanishing. (gravitation radiation)

FIELD EQUATIONS

$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$$

- Einstein's gravitation tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2}\, g_{\alpha\beta}$
- $T_{\alpha\beta}$ stress-energy tensor / matter content of the spacetime
- 8π: gravitational constant after normalization

EULER EQUATIONS

Levi-Civita connection ∇ associated with the metric g

- Preserves the metric $\nabla g = 0$
- Torsion-free $\nabla_X Y - \nabla_Y X = [X, Y]$ (Lie bracket)

Riemann curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

(second contracted) **Bianchi identities**

$$\nabla^\beta R_{\alpha\beta} = \frac{1}{2} \nabla_\alpha R$$

EULER EQUATIONS

$$\nabla^\beta T_{\alpha\beta} = 0$$

Proof.

$$\begin{aligned} \nabla^\beta T_{\alpha\beta} = \frac{1}{8\pi} \nabla^\beta G_{\alpha\beta} &= \frac{1}{8\pi} \nabla^\beta \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \right) \\ &= \frac{1}{8\pi} \left(\nabla^\beta R_{\alpha\beta} - \frac{1}{2} \nabla^\beta R \, g_{\alpha\beta} \right) = 0 \end{aligned}$$

CAUCHY DEVELOPMENTS

Initial data set.

- Riemannian 3-manifold $(\overline{M}, \overline{g})$
- Symmetric 2-covariant tensor field $\overline{k}$
  (second fundamental form, embedding)
- Notation
  - $\overline{g}_{ij}$, $\overline{k}_{ij}$ (with $i, j = 1, 2, 3$)
  - $\overline{\nabla}$: connection associated with $\overline{g}$ $\overline{R}$: scalar curvature of $\overline{g}$
  - Trace $\mathrm{Tr}\,\overline{k} = \overline{k}_j^j = \overline{g}^{ij}\overline{k}_{ij}$ norm $|\overline{k}|^2 = \overline{k}_{ij}\overline{k}^{ij} = g^{ii'}g^{jj'}\overline{k}_{i'j'}\overline{k}_{ij}$
- Einstein's constraint equations

$$\overline{R} + \left(\mathrm{Tr}\,\overline{k}\right)^2 - |\overline{k}|^2 = 16\pi\, T_{00} \qquad (\text{Hamiltonian } G_{00} = 8\pi\, T_{00})$$

$$\overline{\nabla}_j \overline{k}_i^j - \overline{\nabla}_i(\mathrm{Tr}\,\overline{k}) = 8\pi\, T_{0j} \qquad (\text{Momentum } T_{0j} = 8\pi\, T_{0j})$$

(Gauss-Codazzi equations, nonlinear elliptic equations)

- Matter fields (components T_{00}, T_{0j})
  (scalar field, perfect fluid, modified gravity, etc.)

INITIAL VALUE PROBLEM. Future development of the initial data set

- Lorentzian manifold satisfying the Einstein equations (M, g)

- Embedding $\psi : \overline{M} \to \mathcal{H} \subset M$
- Induced metric $\psi^\star g = \overline{g}$
- Second fundamental form k (extrinsic curvature) $\psi^\star k = \overline{k}$

  In coordinates $g_{\alpha\beta} \frac{\partial \psi^\alpha}{\partial x^i} \frac{\partial \psi^\beta}{\partial x^j} = \overline{g}_{ij}$
- Matter fields (scalar field, perfect fluid, modified gravity, etc.)

FORMULATION AS PARTIAL DIFFERENTIAL EQUATIONS.

- Einstein equation as a hyperbolic-elliptic system of PDE's
- choice of coordinates / diffeomorphism invariance
- **wave gauge**
  - coordinate functions such that
    $$\Box_g x^\alpha := \nabla^\alpha \nabla_\alpha x^\alpha = 0$$
  - system of coupled nonlinear wave equations for the metric
    $$\Box_g g_{\alpha\beta} = Q_{\alpha\beta}(g, \partial g)$$

SELF-GRAVITATING MASSIVE FIELDS

Massive scalar field with potential $V(\phi)$ (minimally coupled)

$$T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2}\nabla_\gamma \phi \nabla^\gamma \phi + V(\phi)\Big) g_{\alpha\beta}$$

Write the Euler equations $\nabla_\alpha T^{\alpha\beta} = 0$ for ϕ

Einstein-Klein-Gordon system

$$\Box_g \phi - V'(\phi) = 0$$
$$R_{\alpha\beta} - 8\pi\big(\nabla_\alpha \phi \nabla_\beta \phi + V(\phi)\, g_{\alpha\beta}\big) = 0$$

for instance with the quadratic potential $V(\phi) = \frac{c^2}{2}\phi^2$ with mass $c > 0$

The theory of modified gravity. (next section)

OUR OBJECTIVE: *Long-time behavior of perturbations of Minkowski spacetime by a self-gravitating massive field*

- system of coupled wave-Klein-Gordon equations
- global nonlinear stability problem
- future geodesically complete spacetime
- Hyperboloidal Foliation Method

SPACETIMES WITH SYMMETRY

Symmetry assumptions

- spherical symmetry (SO(3) isometry group action)
- T2 symmetry (T2 isometry group action)
- Gowdy/plane symmetry (vanishing twists/polarization)
- 1+1 nonlinear wave systems *arbitrary large data*

  rich global dynamics

Long-time issues

- maximal hyperbolic Cauchy developments
- property of the future boundary
- late-time asymptotics, geodesic completeness
- formation of trapped surfaces, (critical) collapse of matter, censorship conjectures (generic data)

Remarks. Bianchi models (even further symmetry assumptions and vacuum)

- techniques of dynamical systems, bifurcation theory
- still many open problems

Compressible matter

- describe, for instance, the interior of a star
- discontinuity hypersurfaces (propagation at about the sound speed)
- implying curvature discontinuities, shock interactions
- scalar fields included as a special case

irrotational null fluids (see below)

OUR OBJECTIVES

- *Weakly regular Ricci-flat spacetimes with T^2 symmetry*

impulsive gravitational waves

- *Weakly regular matter spacetimes with spherical symmetry*

shock waves

PERFECT FLUIDS

Energy-momentum tensor. $T^{\alpha\beta} = (\mu + p(\mu))\, u^\alpha\, u^\beta + p(\mu)\, g^{\alpha\beta}$

- Proper mass-energy density $\mu > 0$
- Unit time-like velocity vector u^α $u_\alpha u^\alpha = -1$ and $u^0 > 0$
- Perfect fluid with given pressure-law $p = p(\mu)$

Hyperbolicity condition

$$0 < \frac{dp}{d\mu} \leqslant 1$$

FORMULATION OF THE CAUCHY PROBEM

- **Initial data set:** matter fields (energy density, current) $\overline{\mu}, \overline{J}$
- **Cauchy development**: matter fields μ, J

$$\mu := T_{\alpha\beta} n_\alpha n_\beta \qquad\qquad J_\alpha = (g_{\alpha\beta} + n_\alpha n_\beta)\, T^{\beta\gamma} n_\gamma$$

(measured orthogonally to the time foliation)
n normal 1-form

$\overline{\mu}, \overline{J}$ coincide with the restriction of μ, J to the initial hypersurface $\mathcal{H}$

Dominant energy condition:

- The mass-energy density $T_{\alpha\beta}X^\alpha X^\beta \geqslant 0$ for every timelike vector X (or observer).
- For every future-oriented, timelike vector, the energy flux $g^{\alpha\gamma}T_{\beta\gamma}X^\beta$ is causal and future-oriented

Proposition. Dominant energy condition for perfect fluids

$$|p(\mu)| \leqslant \mu$$

Sketch of the proof.

- By using that $\mu \geqslant 0$, we have $T_{\alpha\beta}X^\alpha X^\beta = (\mu + p)(u_\alpha X^\alpha)^2 - p \geqslant 0$ since $(u_\alpha X^\alpha)^2 \geqslant 1$ for all unit timelike vectors X, u.
- By using that $p^2 \leqslant \mu^2$, we have $g^{\beta\gamma}T_{\alpha\beta}X^\alpha\, T^{\gamma\delta}X^\delta = (-\mu^2 + p^2)(u_\alpha X^\alpha)^2 - p^2 \leqslant 0$ for all unit timelike vectors X, u.

EXAMPLES

- **Isothermal fluids** $p = k^2\mu$ with constant "sound speed" $k \in (0, 1)$
- **Dust fluid** $p = 0$ (not hyperbolic) ▸ **Radiation dominated** $p = \frac{1}{3}\mu$
- **Fluids with constant pressure** $p(\mu) = -\mu$ with $\mu = \Lambda/(8\pi)$

$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta} = -\Lambda g_{\alpha\beta}$$

where Λ is the "cosmological constant".

IRROTATIONAL NULL FLUID

- **Null fluid:** pressure equal to its mass-energy density $p = \mu$

$$T_{\alpha\beta} = 2\mu\, u^\alpha u^\beta + \mu\, g_{\alpha\beta}$$

sound speed $=$ light speed normalized to 1.

- Second contracted Bianchi identities: Euler equations

$$(u^\alpha \nabla_\alpha \mu)\, u^\beta + \mu\, (\nabla_\alpha u^\alpha)\, u^\beta + \mu\, u^\alpha \nabla_\alpha u^\beta - \frac{1}{2} \nabla^\beta \mu = 0$$

- **Irrotational fluid:** u^α is a (normalized) gradient of a scalar potential ψ:

$$u^\alpha = \frac{\nabla^\alpha \psi}{\sqrt{-\nabla_\beta \psi\, \nabla^\beta \psi}}$$

Reformulation of the Euler equations

- Multiply by u_β and obtain

$$u^\alpha \nabla_\alpha \mu + \mu \nabla_\alpha u^\alpha + \mu\, u^\alpha u_\beta \nabla_\alpha u^\beta - \frac{1}{2} u^\alpha \nabla_\alpha \mu = 0$$

which, in view of $u_\beta \nabla_\alpha u^\beta = 0$, simplifies into

$$\frac{1}{2} u^\alpha \nabla_\alpha \mu + \mu \nabla_\alpha u^\alpha = 0.$$

Setting $M := \frac{1}{2} \log \mu$

$$u^\alpha \nabla_\alpha M + \nabla_\alpha u^\alpha = 0$$

- Multiply by the projection operator $H_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$:

$$\mu\, u^\alpha \nabla_\alpha u^\beta - \frac{1}{2} H^{\alpha\beta} \nabla_\alpha \mu = 0$$

$$H^{\alpha\beta} \nabla_\alpha M - u^\alpha \nabla_\alpha u^\beta = 0$$

Reduced equations for irrotational null fluids

Impose that $u^\alpha = \frac{\nabla^\alpha \psi}{\sqrt{-\nabla_\beta \psi \, \nabla^\beta \psi}}$ for some potential ψ.

After some elementary computations:

$$\nabla_\alpha \left(\mu^{1/2} \frac{\nabla^\alpha \psi}{\sqrt{-\nabla_\beta \psi \, \nabla^\beta \psi}} \right) = 0$$

$$\nabla_\beta \left(M - \log \sqrt{-\nabla^\alpha \psi \nabla_\alpha \psi} \right) = \Omega \, \nabla_\beta \psi$$

where the scalar factor Ω reads $\Omega = \frac{\nabla^\alpha \psi \nabla_\alpha M}{\nabla^\alpha \psi \nabla_\alpha \psi} - \frac{\nabla^\alpha \psi \nabla^\beta \psi \nabla_{\alpha\beta} \psi}{(\nabla^\alpha \psi \nabla_\alpha \psi)^2}$

- The gradient of $M - \log \sqrt{-\nabla^\alpha \psi \nabla_\alpha \psi}$ is parallel to $\nabla \psi$,
- so that $M - \log \sqrt{-\nabla^\alpha \psi \nabla_\alpha \psi}$ is simply a function $F(\psi)$ for some F.
- Replace the chosen potential ψ by some function $G(\psi)$ (if necessary), so that

$$M - \log \sqrt{-\nabla^\alpha \psi \nabla_\alpha \psi} = 0.$$

$$\mu = -\nabla^\alpha \psi \nabla_\alpha \psi$$

Proposition. Einstein-Euler system for irrotational null fluids

The fluid potential satisfies the **wave equation** on the curved space

$$\Box_g \psi = 0$$

coupled to the **Einstein equations** for the metric

$$R_{\alpha\beta} - 8\pi \nabla_\alpha \phi \nabla_\beta \phi = 0$$

Recover the fluid unknowns

- Velocity vector field $u^\alpha = \frac{\nabla^\alpha \psi}{\sqrt{-\nabla_\beta \psi \, \nabla^\beta \psi}}$
- Mass -energy density function $\mu = -\nabla^\alpha \psi \nabla_\alpha \psi$
- Relativistic analogue of Bernoulli's law:
  - irrotational flows in classical fluid dynamics
  - determines μ explicitly, once we know the velocity

OUTLINE OF THIS COURSE

Introduction to Matter Models

Einstein equations, matter models, modified gravity

Weakly Regular Ricci-Flat Spacetimes with T2 Symmetry

- weak solutions with singular curvature / impulsive gravitational waves
- weakly regular geodesics, late-time asymptotics
- future geodesic completeness (Gowdy case, polarized T2)

Weakly Regular Matter Spacetimes with Spherical Symmetry

- spacetimes with bounded variation / shock waves in self-gravitating compressible fluids
- Riemann problem, random choice method
- global geometry of weakly regular matter spacetimes

Stability of Minkowski Space for Self-Gravitating Massive Fields

- Einstein-massive scalar field system & theory of modified gravity
- wave gauge: system of coupled wave-Klein-Gordon equations
- hyperboloidal foliation method
- future geodesic completeness

SELECTED REFERENCE

Textbooks

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- G. Galloway, http://www.math.miami.edu/~galloway/paris.pdf, 2015
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- R.M. Wald, University of Chicago Press, 1984

Long-time dynamics (Bianchi spacetimes)

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- A.D. Rendall, Oxford University Press, 2008

Section 2. THE THEORY OF MODIFIED GRAVITY

FIELD EQUATIONS

$f(R)$–**Theory of modified gravity**.

- Long history in physics: Weyl 1918, Pauli 1919, Eddington 1924, ...
- A function $f(R) \simeq R$ of the scalar curvature
- Fourth-order derivatives (additional gravitational degrees of freedom)
- Motivations from cosmology / broad literature in physics
    - accelerated expansion of the Universe, structure formation
    - observed 1998, Nobel Prize 2011
    - without adding unknown forms of dark matter / dark energy
    - formation of structures in the Universe (galaxies, etc.)

Examples. $f(R) = R + \kappa R^2, \qquad f(R) = R + \kappa R^n, \qquad f(R) = R + \frac{\kappa R^n}{R^n + R_*^n}$

Other alternative theories of gravity. *The gravitational field is mediated by one of more scalar fields in addition to the metric.*

- Brans-Dicke theory
- scalar-tensor theories of gravity
- less/more general, depending on the choice of the coupling

Our objective in this introductory section

- structure of the equations of modified gravity
- augmented conformal formulation
- Einstein-Klein-Gordon system

Mathematically rigorous validation

- a 'correction' to Einstein's theory
- initial data set in modified gravity
- initial value formulation

EINSTEIN'S THEORY

$(3+1)$–dimensional spacetime (M, g)
Lorentzian signature $(-,+,+,+)$

Hilbert-Einstein's action

$$\mathcal{A}_{\mathsf{HE}}[\phi, g] := \int_M \Big(R_g + 16\pi L[\phi, g] \Big)\, dV_g$$

- massless scalar field (minimally coupled) $L[\phi, g] = -\frac{1}{2}\nabla_\gamma \phi \nabla^\gamma \phi$
- *stress-energy tensor* $T_{\alpha\beta}[\phi, g] = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2}\big(\nabla_\gamma \phi \nabla^\gamma \phi\big) g_{\alpha\beta}$

Principle of least action.

Critical metrics $\delta \mathcal{A}_{\mathsf{HE}}[\phi, g] \equiv 0$

Einstein equations

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R_g\, g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}[\phi, g]$$

Example. In the vacuum case $T_{\alpha\beta} \equiv 0$, we obtain the Ricci-flat condition $R_{\alpha\beta} = 0$.

$f(R)$-MODIFIED GRAVITY THEORY

$$\mathcal{A}_{\mathrm{MG}}[\phi, g] := \int_M \Big(f(R_g) + 16\pi L[\phi, g] \Big)\, dV_g$$

- Prescribed function $f : \mathbb{R} \to \mathbb{R}$
  - $f(R) = R + \kappa\Big(\frac{R^2}{2} + \mathcal{O}(\kappa R^3)\Big)$
  - sign of $\kappa := f''(0) > 0$ essential for global stability
- Critical points $\delta\mathcal{A}_{\mathrm{MG}}[\phi, g] \equiv 0$

Curvature tensor of modified gravity

$$\begin{aligned} N_{\alpha\beta} :=& f'(R_g)\, R_{\alpha\beta} - \frac{1}{2} f(R_g) g_{\alpha\beta} + \Big(g_{\alpha\beta} \Box_g - \nabla_\alpha \nabla_\beta \Big) \big(f'(R_g)\big) \\ =& f'(R_g)\, G_{\alpha\beta} - \frac{1}{2} \Big(f(R_g) - R_g f'(R_g) \Big) g_{\alpha\beta} + \Big(g_{\alpha\beta} \Box_g - \nabla_\alpha \nabla_\beta \Big) \big(f'(R_g)\big) \end{aligned}$$

Proposition. Field equations of modified gravity

$$N_{\alpha\beta} = 8\pi\, T_{\alpha\beta}[\phi, g]$$

- Fourth-order derivatives of the unknown metric
- When f is linear, $N_{\alpha\beta}$ reduces to $G_{\alpha\beta}$.

Derivation of the field equations.

- Variation with respect to $g^{\alpha\beta}$ with $dV_g = \sqrt{-g}dx$ and $g = \det(g_{\alpha\beta})$
- Variation of the action

$$\delta\mathcal{A} = \int_M \left(\tfrac{\delta}{\delta g^{\alpha\beta}} \left(\sqrt{-g} f(R) \right) + 16\pi \tfrac{\delta}{\delta g^{\alpha\beta}} \left(\sqrt{-g} L[\phi, g] \right) \right) \delta g^{\alpha\beta} dx$$

- Field equations

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} \left(\sqrt{-g} f(R) \right) = -16\pi \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} \left(\sqrt{-g} L[\phi, g] \right) =: 8\pi T_{\alpha\beta}$$

Observe

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} \left(\sqrt{-g} f(R) \right) = f(R) \frac{\delta}{\delta g^{\alpha\beta}} \left(\ln(\sqrt{-g}) \right) + f'(R) \frac{\delta R}{\delta g^{\alpha\beta}}$$

and it thus suffices to check that

$$\frac{\delta R}{\delta g^{\alpha\beta}} = R_{\alpha\beta} \qquad\qquad -2 \frac{\delta}{\delta g^{\alpha\beta}} \left(\ln(\sqrt{-g}) \right) = g_{\alpha\beta}$$

Variation of the volume form.

$$\delta g = \delta \det(g_{\alpha\beta}) = g\, g^{\alpha\beta} \delta g_{\alpha\beta}$$

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\delta g_{\alpha\beta} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\delta g^{\alpha\beta}$$

by differentating $g_{\alpha\gamma} g^{\gamma\beta} = \mathrm{Kronecker}_{\alpha}^{\beta}$

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} g_{\alpha\beta}$$

$$-2\frac{\delta}{\delta g^{\alpha\beta}}\left(\ln(\sqrt{-g}\,)\right) = g_{\alpha\beta}$$

Variation of the scalar curvature.

$$R^\rho{}_{\sigma\alpha\beta} = \partial_\alpha \Gamma^\rho_{\beta\sigma} - \partial_\beta \Gamma^\rho_{\alpha\sigma} + \Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma} - \Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma}$$

$$\delta R^\rho{}_{\sigma\alpha\beta} = \partial_\alpha \delta\Gamma^\rho_{\beta\sigma} - \partial_\beta \delta\Gamma^\rho_{\alpha\sigma} + \delta\Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma} + \Gamma^\rho_{\alpha\lambda}\delta\Gamma^\lambda_{\beta\sigma} - \delta\Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma} - \Gamma^\rho_{\alpha\lambda}\delta\Gamma^\lambda_{\beta\sigma}$$

Covariant derivative of the Christoffel symbols:

$$\nabla_\lambda\left(\delta\Gamma^\rho_{\beta\alpha}\right) = \partial_\lambda\left(\delta\Gamma^\rho_{\beta\alpha}\right) + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\beta\alpha} - \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\beta\alpha},$$

hence

$$\delta R^\rho{}_{\sigma\alpha\beta} = \nabla_\alpha \delta\Gamma^\rho_{\beta\sigma} - \nabla_\beta \delta\Gamma^\rho_{\alpha\sigma}$$

and, after contraction of two indices,

$$\delta R_{\alpha\beta} = \nabla_\alpha \delta\Gamma^\rho_{\beta\rho} - \nabla_\beta \delta\Gamma^\rho_{\alpha\rho}$$

For the scalar curvature $R = g^{\alpha\beta}R_{\alpha\beta}$

$$\delta R = g^{\alpha\beta}\delta R_{\alpha\beta} + \delta g^{\alpha\beta}R_{\alpha\beta} = \nabla_\alpha\left(g^{\alpha\beta}\delta\Gamma^\rho_{\beta\rho} - g^{\alpha\beta}\delta\Gamma^\rho_{\beta\rho}\right) + R_{\alpha\beta}\delta g^{\alpha\beta}$$

The divergence term does not contribute the variation the action.

$$\frac{\delta R}{\delta g^{\alpha\beta}} = R_{\alpha\beta}$$

Lemma

The contracted Bianchi identities $\nabla^\alpha R_{\alpha\beta} = \frac{1}{2}\nabla_\beta R$ or $\nabla^\alpha G_{\alpha\beta} = 0$ imply the divergence equation

$$\nabla^\alpha N_{\alpha\beta} = 0.$$

Proof. We compute the three relevant terms

$$\begin{aligned}\nabla^\alpha(f'(R)\,R_{\alpha\beta}) &= R_{\alpha\beta}\nabla^\alpha(f'(R)) + f'(R)\,\nabla^\alpha R_{\alpha\beta} \\ &= R_{\alpha\beta}\nabla^\alpha(f'(R)) + \frac{1}{2}f'(R)\,\nabla_\beta R\end{aligned}$$

$$\frac{1}{2}\,\nabla^\alpha\Big(f(R)\,g_{\alpha\beta}\Big) = \frac{1}{2}\,\nabla_\beta(f(R)) = \frac{1}{2}\,f'(R)\,\nabla_\beta R$$

$$\begin{aligned}\nabla^\alpha\Big(\nabla_\alpha\nabla_\beta f'(R) - g_{\alpha\beta}\Box_g f'(R)\Big) &= \Big(\nabla^\alpha\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla^\lambda\nabla_\lambda\Big)f'(R) \\ &= \Big(\nabla_\alpha\nabla_\beta\nabla^\alpha - \nabla_\beta\nabla_\alpha\nabla^\alpha\Big)f'(R) \\ &= R_{\alpha\beta}\nabla^\alpha(f'(R))\end{aligned}$$

Matter model.

- *Coupling between the gravity field and the matter fields*
- From a physical standpoint, need to choose the frame in which measurements are made: *ongoing debate*
  - "Jordan frame": original metric $g_{\alpha\beta}$ considered as physically relevant
  - "Einstein frame": **conformally-transformed metric**

$$g^{\dagger}{}_{\alpha\beta} := f'(R_g) g_{\alpha\beta}$$

- Lead to different systems of PDE's

Massless scalar field.

- Jordan's coupling (minimal in the Jordan metric)

$$T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \phi \nabla^\gamma \phi$$

- Einstein's coupling (non-minimal in the Jordan metric)

$$T^{\dagger}_{\alpha\beta} := f'(R_g) \Big(\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \phi \nabla^\gamma \phi \Big)$$

Jordan's matter coupling

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \phi \nabla^\gamma \phi$$

With the field equations $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and the contracted Bianchi identities $\nabla^\alpha G_{\alpha\beta} = 0$:

$$0 = \nabla^\alpha T_{\alpha\beta} = \nabla_\beta \phi \Box_g \phi + \nabla_\alpha \phi \nabla^\alpha \nabla_\beta \phi - g_{\alpha\beta} \nabla_\gamma \phi \nabla^\alpha \nabla^\gamma \phi$$

so that

$$\partial_\beta \phi \Box_g \phi = 0.$$

The scalar field ϕ satisfies the wave equation

$$\Box_g \phi = 0$$

associated the unknown metric g.

Einstein's matter coupling

$$
\begin{aligned}
T^{\dagger}_{\alpha\beta} &= f'(R_g)\Big(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}\nabla_\gamma\phi\nabla^\gamma\phi\Big)\\
&= f'(R_g)T_{\alpha\beta}
\end{aligned}
$$

$\nabla^\alpha T^{\dagger}_{\alpha\beta} = 0$ reads

$$
f''(R_g)\nabla^\alpha R_g T_{\alpha\beta} + f'(R_g)\nabla_\beta\phi\Box_g\phi = 0
$$

equivalent to

$$
\Box_g\phi\,\nabla_\beta\phi = -\frac{f''(R_g)}{f'(R_g)}\Big(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}\nabla_\gamma\phi\nabla^\gamma\phi\Big)\nabla^\alpha R_g
$$

$$
\left(\frac{f''(R_g)}{f'(R_g)}\nabla_\alpha\phi\nabla^\alpha R_g - \Box_g\phi\right)\nabla_\beta\phi = \frac{f''(R_g)}{2f'(R_g)}\,(\nabla_\gamma\phi\nabla^\gamma\phi)\nabla_\beta R_g
$$

- The unknown being a scalar field
- Over-determined partial differential system
- Einstein's coupling not mathematically (nor physically ?!) well-behaved

MATHEMATICAL VALIDITY OF THE MODIFIED GRAVITY

CAUCHY DEVELOPMENTS.

Field equations of modified gravity $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$

- based on $f(R) \simeq R$, assumed to satisfy $f'(R) > 0$ and $f''(R) > 0$.
- matter described by a massless scalar field with Jordan coupling

Initial data set.

- Geometry of the initial hypersurface $(\overline{M}, \overline{g}, \overline{k})$
- Matter content: initial data ϕ_0, ϕ_1 for the scalar field and its time derivative
- Fourth-order field equations: need two additional data R_0, R_1 related to the spacetime curvature on $\overline{M}$

An **initial data set** $(\overline{M}, \overline{g}, \overline{k}, R_0, R_1, \phi_0, \phi_1)$ consists of:

- a Riemannian 3-manifold $(\overline{M}, \overline{g})$ and a symmetric $(0,2)$-tensor $\overline{k}$
- two scalar fieds ϕ_0, ϕ_1 on $\overline{M}$ (matter field, its time derivative)
- two scalar fields R_0, R_1 on $\overline{M}$ (spacetime curvature, its time derivative)

Hamiltonian constraint of modified gravity $\left[N_{00} = 8\pi T_{00}\right]$

$$
\begin{aligned}
& f'(R_0)\Big(\overline{R} - \overline{k}_{ij}\overline{k}^{ij} + (\overline{k}_j^j)^2\Big) \\
& + 2\,\overline{k}_j^j\, f''(R_0) R_1 - \Big(2\Delta_{\overline{g}} f'(R_0) - f(R_0) + R_0 f'(R_0)\Big) \\
& = 8\pi\left((\phi_1)^2 + \overline{\nabla}_j \phi_0 \overline{\nabla}^j \phi_0\right)
\end{aligned}
$$

Momentum constraint of modified gravity $\left[N_{0j} = 8\pi T_{0j}\right]$

$$
\begin{aligned}
f'(R_0)\Big(\overline{\nabla}_j \overline{k}_i^i - \overline{\nabla}_i \overline{k}_j^i\Big) - \partial_j\big(f''(R_0)R_1\big) - \overline{k}_j^i \partial_i\big(f'(R_0)\big) \\
= 8\pi \phi_1\, \overline{\nabla}_i \phi_0
\end{aligned}
$$

Definition

A **modified gravity Cauchy development**: Lorentzian manifold (M, g) and matter field ϕ defined on M

- Field equations of modified gravity $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$
- An embedding $i : \overline{M} \to M$ such that:
  - pull-back metric $\overline{g} = i^\star g$ and second fundamental form $\overline{k}$
  - R_0 coincides with the restriction of the spacetime curvature R on $\overline{M}$
  - R_1 coincides with the Lie derivative $\mathcal{L}_n R$, n being the normal to $\overline{M}$
  - ϕ_0, ϕ_1 coincide with the restriction of $\phi, \mathcal{L}_n\phi$ on $\overline{M}$.

Theorem. Cauchy developments in the theory of modified gravity

- Given an (asymptotically flat, say) initial data set $(\overline{M} \simeq \mathbb{R}^3, \overline{g}, \overline{k}, R_0, R_1, \phi_0, \phi_1)$, there exists a unique maximal globally hyperbolic development (M, g).
- If an initial data set $(\overline{M}, \overline{g}, \overline{k}, R_0, R_1, \phi_0, \phi_1)$ for modified gravity is "close" (in Sobolev norms) to another initial data set $(\overline{M}, \overline{g}', \overline{k}', \phi_0', \phi_1')$ of Einstein's theory, then its development is also close to the corresponding Einstein development.

Highly singular limit problem.

- Einstein's vacuum corresponds to vanishing $\phi_0 = \phi_1 = R_0 = R_1 \equiv 0$.
- In the limit $f(R) \to R$, we recover Einstein equations.
  - convergence of a fourth-order system (with no well-defined type) to a system of second-order (hyperbolic-elliptic) PDE's
  - recover $R \to 8\pi \, \nabla_\alpha \phi \nabla^\alpha \phi$

Elementary notions of causality

- **Time-like curve:** $\gamma : [0,1] \to M$ whose tangent vector $\dot\gamma(t)$ is time-like: $\dot\gamma(t), \dot\gamma(t)) < 0$
- **Causal curve** (observer or light), whose tangent vector $\dot\gamma(t)$ is time-like or null $\dot\gamma(t), \dot\gamma(t)) \leqslant 0$
- **Global hyperbolicity**.

  Existence of a **Cauchy surface** $H \subset M$, that is,

  - H is spacelike (Riemannian induced metric)
  - every inextendible causal curve intersects H exactly once
- **Maximal globally hyperbolic development**

  Any other such development isometric to a subset of it

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Section 3. THE FORMULATION IN WAVE GAUGE

Field equations of modified gravity $N_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$

- Fourth-order system (with no specific PDE type)

  while Einstein equations $G_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$ are second-order
- *Conformal transformation* leading to a third-order system
- *Wave coordinates* $\Box_g x^\alpha = 0$ associated with the spacetime metric g

  while Einstein equations are hyperbolic with differential constraints
- *Augmented formulation*
  - spacetime scalar curvature taken as an *independent variable*
  - leading to a second-order system of *nonlinear wave-Klein-Gordon equations*
- "Equivalence" with the Einstein-massive field system

THE CONFORMAL FORMULATION

Gravitational curvature tensor of modified gravity

$$N_{\alpha\beta} = f'(R_g)\, R_{\alpha\beta} + \Big(g_{\alpha\beta}\,\Box_g - \nabla_\alpha\nabla_\beta\Big) f'(R_g) - \frac{1}{2} f(R_g) g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$$

Hessian of the scalar curvature $\nabla_\alpha\nabla_\beta f'(R_g)$ (fourth-order term)

CONFORMAL METRIC

$$g^\dagger{}_{\alpha\beta} := e^{2\rho}\, g_{\alpha\beta}, \qquad \rho := \frac{1}{2} \ln f'(R_g)$$

Conformal transformation for the Ricci curvature

$$R_{\alpha\beta} = R^\dagger{}_{\alpha\beta} + \big(2\,\nabla_\alpha\nabla_\beta\rho + g_{\alpha\beta}\,\Box_g\rho\big) + 2\big(-\nabla_\alpha\rho\nabla_\beta\rho + g_{\alpha\beta}\,\nabla^\gamma\rho\nabla_\gamma\rho\big)$$

Notation. Change of variable $R \mapsto \rho =: \frac{1}{2} \ln f'(R)$, assumed to be one-to-one. We also define (function f, its Legendre transform)

- $w_1(\rho) := f(R)$
- $w_2(\rho) := \frac{f(R) - R\, f'(R)}{f'(R)}$
- $w(\rho) := \frac{f(R) - R\, f'(R)}{(f'(R))^2}$

Observing that

$$N_{\alpha\beta} = f'(R_g)R_{\alpha\beta} - \frac{1}{2}f(R_g)g_{\alpha\beta} + \left(g_{\alpha\beta}\Box_g - \nabla_\alpha\nabla_\beta\right)f'(R_g)$$

$$\mathrm{Tr}(N) = f'(R_g)R_g - 2f(R_g) + 3\Box_g f'(R_g)$$

$$\nabla_\alpha\nabla_\beta e^{2\rho} = 2e^{2\rho}\nabla_\alpha\nabla_\beta\rho + 4e^{2\rho}\nabla_\alpha\rho\nabla_\beta\rho$$

$$\Box_g e^{2\rho} = 2e^{2\rho}\Box_g\rho + 4e^{2\rho}\nabla^\gamma\rho\nabla_\gamma\rho$$

we can express the gravity tensor in terms of ρ

$$N_{\alpha\beta} = e^{2\rho}R_{\alpha\beta} - \frac{w_1(\rho)}{2}g_{\alpha\beta} + 2e^{2\rho}\left(g_{\alpha\beta}\Box_g - \nabla_\alpha\nabla_\beta\right)\rho + 4e^{2\rho}\left(g_{\alpha\beta}\nabla^\gamma\rho\nabla_\gamma\rho - \nabla_\alpha\rho\nabla_\beta\rho\right)$$

With the conformal identity for the Ricci curvature, we deduce that

$$N_{\alpha\beta} = e^{2\rho}\left(R^\dagger{}_{\alpha\beta} - \frac{w_1(\rho)}{2e^{2\rho}}g_{\alpha\beta} + 3\,g_{\alpha\beta}\Box_g\rho - 6\,\nabla_\alpha\rho\nabla_\beta\rho + 6\,g_{\alpha\beta}\nabla^\gamma\rho\nabla_\gamma\rho\right)$$

Finally, with the following expression of the trace

$$\left(g^{\alpha'\beta'}N_{\alpha'\beta'}\right) = 6e^{2\rho}\Box_g\rho - w_1(\rho) + 12e^{2\rho}\,\nabla^\gamma\rho\nabla_\gamma\rho - e^{2\rho}w_2(\rho)$$

we arrive at $e^{2\rho}\left(R^\dagger{}_{\alpha\beta} - 6\,\nabla_\alpha\rho\nabla_\beta\rho + \frac{w_2(\rho)}{2}g_{\alpha\beta}\right) = N_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(g^{\alpha'\beta'}N_{\alpha'\beta'}\right)$

Field equations of modified gravity in the Einstein frame

$$R^\dagger{}_{\alpha\beta} - 6\,\nabla^\dagger_\alpha\rho\nabla^\dagger_\beta\rho + \frac{w(\rho)}{2}g^\dagger{}_{\alpha\beta} = 8\pi\, e^{-2\rho}\left(T_{\alpha\beta} - \frac{1}{2}g^\dagger{}_{\alpha\beta}\left(g^{\dagger\,\alpha'\beta'}\,T_{\alpha'\beta'}\right)\right)$$

EULER EQUATIONS. Conformal transformation for Christoffel symbols

$$\Gamma^{\dagger\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + g^{\gamma}_{\alpha}\nabla_{\beta}\rho + g^{\gamma}_{\beta}\nabla_{\alpha}\rho - g_{\alpha\beta}\nabla^{\gamma}\rho$$

$$\nabla^{\dagger\alpha}N_{\alpha\beta} = e^{-2\rho}g^{\alpha\gamma}\nabla^{\dagger}{}_{\gamma}N_{\alpha\beta} = e^{-2\rho}g^{\alpha\gamma}\left(\partial_{\gamma}N_{\alpha\beta} - \Gamma^{\dagger\delta}_{\gamma\alpha}N_{\beta\delta} - \Gamma^{\dagger\delta}_{\gamma\beta}N_{\alpha\delta}\right)$$

$$= e^{-2\rho}g^{\gamma\alpha}\Big(\Big(\partial_{\gamma}N_{\alpha\beta} - \Gamma^{\delta}_{\gamma\alpha}N_{\beta\delta} - \Gamma^{\delta}_{\gamma\beta}N_{\alpha\delta}\Big)$$

$$- \left(g^{\delta}_{\gamma}\nabla_{\alpha}\rho + g^{\delta}_{\alpha}\nabla_{\gamma}\rho - g_{\gamma\alpha}\nabla^{\delta}\rho\right)N_{\beta\delta} - \left(g^{\delta}_{\gamma}\nabla_{\beta}\rho + g^{\delta}_{\beta}\nabla_{\gamma}\rho - g_{\gamma\beta}\nabla^{\delta}\rho\right)N_{\alpha\delta}\Big)$$

Thus we have

$$\nabla^{\dagger\alpha}N_{\alpha\beta} = e^{-2\rho}\Big(\nabla^{\alpha}N_{\alpha\beta} - \left(\nabla^{\delta}\rho + \nabla^{\delta}\rho - 4\nabla^{\delta}\rho\right)N_{\beta\delta}$$

$$- \left(\nabla_{\beta}\rho(g^{\alpha'\beta'}N_{\alpha'\beta'}) + \nabla^{\alpha}\rho N_{\alpha\beta} - \nabla^{\alpha}\rho N_{\alpha\beta}\right)\Big)$$

in which we already have proven that $\nabla^{\alpha}N_{\alpha\beta} = 0$.

Evolution of the matter field in the Einstein frame

$$\nabla^{\dagger\alpha}N_{\alpha\beta} = 2g^{\dagger\gamma\delta}N_{\delta\beta}\nabla^{\dagger}{}_{\gamma}\rho - \left(g^{\dagger\alpha'\beta'}N_{\alpha'\beta'}\right)\nabla^{\dagger}{}_{\beta}\rho$$

Together with $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$, we obtain

$$\nabla^{\dagger\alpha}T_{\alpha\beta} = 2g^{\dagger\gamma\delta}T_{\delta\beta}\nabla^{\dagger}{}_{\gamma}\rho - \left(g^{\dagger\alpha'\beta'}T_{\alpha'\beta'}\right)\nabla^{\dagger}{}_{\beta}\rho$$

RICCI CURVATURE IN GENERAL COORDINATES

Introduce the Christoffel coefficients

$$\Gamma^{\dagger\lambda} := g^{\dagger\alpha\beta}\Gamma^{\dagger\lambda}_{\alpha\beta} \text{ and } \Gamma^{\dagger}_{\lambda} := g^{\dagger}_{\lambda\beta}\Gamma^{\dagger\beta}$$

Motivation. Reduced wave operator $\widetilde{\Box}_{g^\dagger} u := g^{\dagger\alpha\beta}\partial_\alpha\partial_\beta u$, therefore

$$\Box_{g^\dagger} u = g^{\dagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'} u + \Gamma^{\dagger\delta}\partial_\delta u = \widetilde{\Box}_{g^\dagger} u + \Gamma^{\dagger\delta}\partial_\delta u$$

Calculation. To express the Ricci curvature, we proceed as follows. Recalling

$$R^\dagger_{\alpha\beta} = \partial_\lambda\Gamma^{\dagger\lambda}_{\alpha\beta} - \partial_\alpha\Gamma^{\dagger\lambda}_{\beta\lambda} + \Gamma^{\dagger\lambda}_{\alpha\beta}\Gamma^{\dagger\delta}_{\lambda\delta} - \Gamma^{\dagger\lambda}_{\alpha\delta}\Gamma^{\dagger\delta}_{\beta\lambda}$$

$$\Gamma^{\dagger\lambda}_{\alpha\beta} = \frac{1}{2}g^{\dagger\lambda\lambda'}\left(\partial_\alpha g^\dagger_{\beta\lambda'} + \partial_\beta g^\dagger_{\alpha\lambda'} - \partial_{\lambda'} g^\dagger_{\alpha\beta}\right)$$

we obtain

$$\begin{aligned}
&\partial_\lambda\Gamma^{\dagger\lambda}_{\alpha\beta} - \partial_\alpha\Gamma^{\dagger\lambda}_{\beta\lambda}\\
&= \frac{1}{2}\partial_\lambda\Big(g^{\dagger\lambda\delta}(\partial_\alpha g^\dagger_{\beta\delta} + \partial_\beta g^\dagger_{\alpha\delta} - \partial_\delta g^\dagger_{\alpha\beta})\Big) - \frac{1}{2}\partial_\alpha\Big(g^{\dagger\lambda\delta}(\partial_\beta g^\dagger_{\lambda\delta} + \partial_\lambda g^\dagger_{\beta\delta} - \partial_\delta g^\dagger_{\beta\lambda})\\
&= -\frac{1}{2}\partial_\lambda\Big(g^{\dagger\lambda\delta}\partial_\delta g^\dagger_{\alpha\beta}\Big) + \frac{1}{2}\partial_\lambda\Big(g^{\dagger\lambda\delta}(\partial_\alpha g^\dagger_{\beta\delta} + \partial_\beta g^\dagger_{\alpha\delta})\Big) - \frac{1}{2}\partial_\alpha\Big(g^{\dagger\lambda\delta}\partial_\beta g^\dagger_{\lambda\delta}\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\partial_\lambda\Gamma^{\dagger\lambda}_{\alpha\beta} - \partial_\alpha\Gamma^{\dagger\lambda}_{\beta\lambda} = &-\frac{1}{2}g^{\dagger\lambda\delta}\partial_\lambda\partial_\delta g^\dagger_{\alpha\beta}\\
&+\frac{1}{2}g^{\dagger\lambda\delta}\partial_\alpha\partial_\lambda g^\dagger_{\delta\beta} + \frac{1}{2}g^{\dagger\lambda\delta}\partial_\beta\partial_\lambda g^\dagger_{\delta\alpha} - \frac{1}{2}g^{\dagger\lambda\delta}\partial_\alpha\partial_\beta g^\dagger_{\lambda\delta} + \text{l.o.t.}
\end{aligned}$$

On the other hand, we compute the term $\partial_\alpha {\Gamma^\dagger}_\beta + \partial_\beta {\Gamma^\dagger}_\alpha$ as follows:

$$\begin{aligned} {\Gamma^\dagger}^\gamma = {\Gamma^\dagger}^\gamma_{\alpha\beta} {g^\dagger}^{\alpha\beta} &= \frac{1}{2} {g^\dagger}^{\alpha\beta} {g^\dagger}^{\gamma\delta} \left(\partial_\alpha {g^\dagger}_{\beta\delta} + \partial_\beta {g^\dagger}_{\alpha\delta} - \partial_\delta {g^\dagger}_{\alpha\beta} \right) \\ &= {g^\dagger}^{\gamma\delta} {g^\dagger}^{\alpha\beta} \partial_\alpha {g^\dagger}_{\beta\delta} - \frac{1}{2} {g^\dagger}^{\alpha\beta} {g^\dagger}^{\gamma\delta} \partial_\delta {g^\dagger}_{\alpha\beta} \end{aligned}$$

and ${\Gamma^\dagger}_\lambda = {g^\dagger}_{\lambda\gamma} {\Gamma^\dagger}^\gamma = {g^\dagger}^{\alpha\beta} \partial_\alpha {g^\dagger}_{\beta\lambda} - \frac{1}{2} {g^\dagger}^{\alpha\beta} \partial_\lambda {g^\dagger}_{\alpha\beta}$. So, we have

$$\partial_\alpha {\Gamma^\dagger}_\beta = \partial_\alpha \left({g^\dagger}^{\lambda\delta} \partial_\delta {g^\dagger}_{\lambda\beta} \right) - \frac{1}{2} \partial_\alpha \left({g^\dagger}^{\lambda\delta} \partial_\beta {g^\dagger}_{\lambda\delta} \right)$$

and, therefore,

$$\partial_\alpha {\Gamma^\dagger}_\beta + \partial_\beta {\Gamma^\dagger}_\alpha = {g^\dagger}^{\gamma\delta} \partial_\alpha \partial_\lambda {g^\dagger}_{\delta\beta} + {g^\dagger}^{\lambda\delta} \partial_\beta \partial_\lambda {g^\dagger}_{\delta\alpha} - {g^\dagger}^{\lambda\delta} \partial_{\alpha\beta} {g^\dagger}_{\lambda\delta} + \text{l.o.t.}$$

Ricci curvature in general coordinates

$$R^\dagger_{\alpha\beta} = -\frac{1}{2} {g^\dagger}^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} {g^\dagger}_{\alpha\beta} + \frac{1}{2} \left(\partial_\alpha {\Gamma^\dagger}_\beta + \partial_\beta {\Gamma^\dagger}_\alpha \right) + \frac{1}{2} F_{\alpha\beta}(g^\dagger; \partial g^\dagger),$$

where $F_{\alpha\beta}(g^\dagger; \partial g^\dagger)$ are quadratic in $\partial g^\dagger$.

FORMULATION IN WAVE GAUGE

First, we solve a different problem

- Substract $\frac{1}{2}\big(\partial_\alpha {\Gamma^\dagger}_\beta + \partial_\beta {\Gamma^\dagger}_\alpha\big)$ to the principal part (De Turck's trick)
  $N_{\alpha\beta} - \frac{1}{2}{g^\dagger}_{\alpha\beta}\mathrm{Tr}^\dagger(N) - \frac{1}{2}e^{2\rho}\big(\partial_\alpha {\Gamma^\dagger}_\beta + \partial_\beta {\Gamma^\dagger}_\alpha\big) = 8\pi\big(T_{\alpha\beta} - \frac{1}{2}\mathrm{Tr}^\dagger(T){g^\dagger}_{\alpha\beta}\big)$
- The principal part is the wave operator $\tilde{\Box}^\dagger {g^\dagger}_{\alpha\beta} = {g^\dagger}^{\alpha\beta}\partial_{\alpha'}\partial_{\beta'}{g^\dagger}_{\alpha\beta}$
- Standard local existence theorem
- Then, a key observation is the **PROPAGATION PROPERTY**:

  if ${\Gamma^\dagger}^\lambda = 0$ *on a Cauchy hypersurface*
  and the Hamiltonian and momentum constraints hold, then ${\Gamma^\dagger}^\lambda = 0$ *hold in all the spacetime.*

Finally, we impose the WAVE GAUGE ${\Gamma^\dagger}^\lambda = {g^\dagger}^{\alpha\beta}{\Gamma^\dagger}^\gamma_{\alpha\beta} = 0$

Conformal formulation of the field equations in the wave gauge

$$\tilde{\Box}^\dagger {g^\dagger}_{\alpha\beta} = F_{\alpha\beta}(g^\dagger; \partial g^\dagger) - 12\,\partial_\alpha\rho\partial_\beta\rho + w(\rho){g^\dagger}_{\alpha\beta} + 16\pi\, e^{-2\rho}\Big(- T_{\alpha\beta} + \frac{1}{2}{g^\dagger}_{\alpha\beta}\big({g^\dagger}^{\alpha'\beta'}T_{\alpha'\beta'}\big)\Big)$$

$$\nabla^{\dagger\alpha} T_{\alpha\beta} = 2T_{\gamma\beta}\nabla^{\dagger\gamma}\rho - \big({g^\dagger}^{\alpha'\beta'}T_{\alpha'\beta'}\big)\nabla^\dagger_\beta\rho$$

supplemented with ${g^\dagger}^{\alpha\beta}{\Gamma^\dagger}^\gamma_{\alpha\beta} = 0$ and $\rho = \frac{1}{2}f'(R_g)$.

Remark. Trace equation $\tilde{\Box}^\dagger\rho = \frac{1}{6}\big(w(\rho) + e^{-4\rho}w_1(\rho)\big) + \frac{4\pi}{3}\, e^{-2\rho}{g^\dagger}^{\alpha'\beta'}T_{\alpha'\beta'}$

THE AUGMENTED CONFORMAL FORMULATION

Still third-order and not of a specific PDE type !

THE AUGMENTED FORMULATION

- relation $e^{2\rho} = f'(R_g)$ *no longer imposed*
- ρ replaced by a **new independent variable** ϱ
- algebraic constraint $e^{2\rho} = f'(R_g)$ replaced by the trace equation
- new notation for the metric $g^{\ddagger}_{\alpha\beta} = e^{2\varrho} g_{\alpha\beta}$

Main unknowns: the function ϱ and the metric $g^{\ddagger}_{\alpha\beta}$

Introduce the tensor field $N^{\ddagger}$ defined by the relation

$$N^{\ddagger}{}_{\alpha\beta} - \frac{1}{2} g^{\ddagger}_{\alpha\beta} \mathrm{Tr}^{\ddagger}(N^{\ddagger}) := e^{2\varrho}\left(R^{\ddagger}{}_{\alpha\beta} - 6e^{2\varrho}\partial_\alpha\varrho\partial_\beta\varrho + \frac{1}{2} g^{\ddagger}{}_{\alpha\beta} w(\varrho)\right)$$

Definition and Proposition. Conformal augmented formulation of modified gravity (second-order system in $g^\ddagger, \rho$)

$$N^{\ddagger}{}_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\Box_{g^\ddagger}\varrho = \frac{1}{6}\left(w(\varrho) + e^{-4\varrho}w_1(\varrho)\right) + \frac{4\pi}{3}e^{2\varrho}\left({g^\ddagger}^{\delta\gamma}T_{\delta\gamma}\right)$$

which, put together, imply the evolution equation for the matter field

$$\nabla^{\ddagger\alpha}T_{\alpha\beta} = 2\,T_{\gamma\beta}\nabla^{\ddagger\gamma}\varrho - \left({g^\ddagger}^{\delta\gamma}T_{\delta\gamma}\right)\nabla^\ddagger_\beta\varrho$$

(proof given below)

PROPAGATION PROPERTY

- By considering the equation satisfied by $\Box_{g^\ddagger}\left(\varrho - f'(R_g)\right)$, one can check that:

  If $e^{2\varrho} = f'(R_g)$ with $g^{\dagger}{}_{\alpha\beta} = e^{2\varrho}g_{\alpha\beta}$ is satisfied on a Cauchy hypersurface and the wave gauge together with the Hamiltonian and momentum constraints hold, then the condition $e^{2\varrho} = f'(R_g)$ is satisfied everywhere in the spacetime.

Hence, we have truly *extended* the system of modified gravity.

Proof. We need to check that

$$\nabla^{\ddagger\alpha} N^{\ddagger}{}_{\alpha\beta} = e^{-2\varrho}\big(2g^{\alpha\alpha'}\partial_{\alpha'}\varrho N^{\ddagger}{}_{\alpha\beta} - \mathrm{Tr}(N^{\ddagger})\partial_{\beta}\varrho\big)$$

From our definition of the "extended" gravity tensor, we have

$$N^{\ddagger}{}_{\alpha\beta} = e^{2\varrho}G^{\ddagger}_{\alpha\beta} - 6e^{2\varrho}\big(\partial_{\alpha}\varrho\partial_{\beta}\varrho - \frac{1}{2}g^{\ddagger}{}_{\alpha\beta}|\nabla^{\ddagger}\varrho|^2_{g^{\ddagger}}\big) - \frac{1}{2}g^{\ddagger}{}_{\alpha\beta}w_2(\varrho),$$

where $G^{\ddagger}{}_{\alpha\beta} := R^{\ddagger}{}_{\alpha\beta} - \frac{1}{2}g^{\ddagger}{}_{\alpha\beta}R^{\ddagger}$ is the Einstein curvature of $g^{\ddagger}$.

We observe that $\nabla^{\ddagger\alpha}G^{\ddagger}{}_{\alpha\beta} = 0$, as well as the obvious identities

$$\nabla^{\ddagger\alpha}\Big(\partial_{\alpha}\varrho\partial_{\beta}\varrho - \frac{1}{2}g^{\ddagger}{}_{\alpha\beta}|\nabla^{\ddagger}\varrho|^2_{g^{\ddagger}}\Big) = \partial_{\beta}\varrho\,\Box_{g^{\ddagger}}\varrho$$

$$\nabla^{\ddagger\alpha\beta}\big(g^{\ddagger}{}_{\alpha\beta}w_2(\varrho)\big) = \partial_{\beta}\big(w_2(\varrho)\big) = -2e^{-2\varrho}w_1(\varrho)\partial_{\beta}\varrho$$

This allows us to compute the divergence

$$\begin{aligned}\nabla^{\ddagger\alpha} N^{\ddagger}{}_{\alpha\beta} &= 2e^{2\varrho} G^{\varrho}_{\alpha\beta} \nabla^{\ddagger\alpha}\varrho - 12e^{2\varrho}\left(\partial_\alpha\varrho\partial_\beta\varrho - \frac{1}{2}g^{\ddagger}{}_{\alpha\beta}|\nabla^{\ddagger}\varrho|^2_{g^\ddagger}\right)\nabla^{\ddagger\alpha}\varrho \\ &\quad - 6e^{2\varrho}\partial_\beta\varrho\Box_{g^\ddagger}\varrho + e^{2\varrho}\partial_\beta w_1(\varrho) \\ &= 2N^{\ddagger}{}_{\alpha\beta}\nabla^{\ddagger\alpha}\varrho - \partial_\beta\varrho\left(6e^{2\varrho}\Box_{g^\ddagger}\varrho - \frac{w_1(\varrho)}{e^{2\varrho}} - w_2(\varrho)\right),\end{aligned}$$

in which we use

$$\Box_{g^\ddagger}\varrho = \frac{1}{6}\left(w(\varrho) + e^{-4\varrho}w_1(\varrho)\right) + \frac{4\pi}{3}e^{2\varrho}\left(g^{\ddagger\delta\gamma}T_{\delta\gamma}\right)$$

We thus have derived the desired evolution equation for the matter field

$$\nabla^{\ddagger\alpha}T_{\alpha\beta} = 2\,T_{\gamma\beta}\nabla^{\ddagger\gamma}\varrho - \left(g^{\ddagger\delta\gamma}T_{\delta\gamma}\right)\nabla^{\ddagger}_{\beta}\varrho$$

□

Formulation for general matter models

$$
\begin{aligned}
g^{\ddagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}g^{\ddagger}{}_{\alpha\beta} =& F_{\alpha\beta}(g^{\ddagger};\partial g^{\ddagger}) - 12\partial_\alpha\varrho\partial_\beta\varrho + w(\varrho)g^{\ddagger}{}_{\alpha\beta} \\
& - 16\pi\Big(T_{\alpha\beta} - \frac{1}{2}g^{\dagger}{}_{\alpha\beta}g^{\ddagger\alpha'\beta'}T_{\alpha'\beta'}\Big) \\
g^{\ddagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}\varrho =& \frac{1}{6}\Big(w(\varrho) + w_1(\varrho)e^{-4\varrho}\Big) + \frac{4\pi}{3}e^{-2\varrho}g^{\ddagger\alpha'\beta'}T_{\alpha'\beta'} \\
\nabla^{\ddagger\alpha}T_{\alpha\beta} =& 2\,\partial_\gamma\varrho\, g^{\ddagger\delta\gamma}T_{\gamma\beta} - \partial_\beta\varrho\, g^{\ddagger\alpha'\beta'}T_{\alpha'\beta'}
\end{aligned}
$$

in which $F_{\alpha\beta}(g^{\ddagger};\partial g^{\dagger})$ are quadratic in $\partial g^{\ddagger}$.

(Jordan coupling and wave coordinates associated with the Einstein metric)

Remark. $\nabla^{\ddagger\alpha}\Big(\rho^{-2}T_{\alpha\beta}\Big) = \mathrm{Tr}^{\ddagger}(T)\partial_\beta\big(\varrho^{-1}\big) \neq 0.$

A stress-energy tensor which is conserved in the Jordan frame is not conserved in the Einstein frame (and vice versa).

Except if the matter $T_{\alpha\beta}$ is trace-free (conformally invariant)

Finally, we assume that the matter is a (massless) scalar field.

- $\widetilde{\Box}_{g^\ddagger} := {g^\ddagger}^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'}$
- $F_{\alpha\beta}(g^\ddagger; \partial g^\ddagger)$ quadratic in $\partial g^\ddagger$
- $V = V(\varrho)$ and $W = W(\varrho)$ of quadratic order as $\varrho \to 0$

The augmented conformal formulation of modified gravity in wave gauge

$$\widetilde{\Box}_{g^\ddagger} {g^\ddagger}_{\alpha\beta} = F_{\alpha\beta}(g^\ddagger; \partial g^\ddagger) - 12\, \partial_\alpha \varrho \partial_\beta \varrho - 16\pi\, \partial_\alpha \phi \partial_\beta \phi + V(\varrho) g^\ddagger_{\alpha\beta}$$

$$\widetilde{\Box}_{g^\ddagger} \phi = -2\, {g^\ddagger}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \varrho$$

$$\widetilde{\Box}_{g^\ddagger} \varrho - \frac{\varrho}{3\kappa} = -\frac{4\pi}{3e^{2\varrho}} {g^\ddagger}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + W(\varrho)$$

"massive scalaron"

additional gravitational degree of freedom

- supplemented with constraints (propagating from a Cauchy hypersurface)
  - ${g^\ddagger}^{\alpha\beta} {\Gamma^\dagger}^\lambda_{\alpha\beta} = 0$
  - $e^{2\varrho} = f'(R_{e^{-2\varrho} g^\ddagger})$
  - Hamiltonian and momentum constraints of modified gravity

CONCLUSIONS for this chapter

Nonlinear wave-Klein-Gordon system

- Einstein-massive scalar field system
- The theory of modified gravity (additional constraints)
- Brans-Dicke theory, scalar-tensor theories

Nonlinear stability of Minkowski spacetime

- The Klein-Gordon potential drastically modifies the global dynamics.
- Must exclude dynamically unstable, self-gravitating massive modes. (trapped surfaces, black hole)
- Initial data set
  - a small perturbation of an asymptotically flat, spacelike hypersurface in Minkowski space
  - massive scalar field with sufficiently small mass
- This perturbation disperses in timelike directions, and the spacetime is timelike geodesically complete.
- **Main challenge:** time decay, the Hyperboloidal Foliation Method

SELECTED REFERENCES

Modified gravity

- P.G. LeFloch and Y. Ma, Mathematical validity of the f(R) theory of modified gravity, ArXiv:1412.8151

Global nonlinear stability of Minkowski spacetime

- Vacuum or massless scalar fields
  - D. Christodoulou & S. Klainerman (1993)
  - H. Lindblad & I. Rodnianski (2010)
- Massive scalar fields
  - P. LeFloch & Y. Ma (2015)

Numerical work

  - H. Okawa, V. Cardoso, and P. Pani, Collapse of self-interacting fields in asymptotically flat spacetimes: do self-interactions render Minkowski spacetime unstable?, Phys. Rev., 2014.

CHAPTER II. Weakly Regular Rici-flat Spacetimes with T2 Symmetry. The weak formulation

RICCI FLAT LORENTZIAN MANIFOLDS WITH T^2–SYMMETRY

Initial data

- Invariant under a T^2-group action
- Defined on a manifold diffeomorphic to the 3-torus T^3
- Suitable setup for studying the propagation of gravitational waves

Cauchy developments

- Define a suitable class of weakly regular Ricci-flat spacetimes
- Local geometry of Cauchy developments
- Global causal structure (geodesics, late-time asymptotics)

OUTLINE

BACKGROUND MATERIAL

STANDARD FORMULATION for regular data

Initial data set

- a Riemannian 3-manifold (Σ, h), a symmetric 2-tensor field K
- Einstein's constraints ($\nabla^{(3)}$ and $R^{(3)}$ determined by h)

Hamiltonian $R^{(3)} - |K|^2 + (\mathrm{Tr}\, K)^2 = 0$ **Momentum** $\nabla^{(3)}_j K^j_i - \nabla^{(3)}_i K^j_j = 0$

Globally hyperbolic Cauchy developments

- A $(3+1)$-Lorentzian manifold $(\mathcal{M}, g)$
- Ricci-flat condition $R_{\mu\nu} = 0$
- An embedding $\phi : \Sigma \to \mathcal{M}$ such that $\phi(\Sigma)$ is a Cauchy surface in $(\mathcal{M}, g)$.
(intersected exactly once by any inextendible timelike curve)
- The pull-back of the first and second fundamental forms of $\Sigma \subset M$ coincides with h, K.

Gauss equation for a hypersurface $\Sigma^{(n-1)} \subset M^{(n)}$ (Riemannian manifold) with (not nec. unit) normal N and second fundamental form χ

$$R^{(n-1)} = R^{(n)} - 2g(N,N)^{-1} R^{(n)}_{ij} N^i N^j - |\chi|^2 + (\mathrm{Tr}\, \chi)^2$$

SOBOLEV SPACES ON MANIFOLDS

M: connected, oriented, differentiable m-manifold

- **Standard notation.**
  - tangent space $T_x M$ at $x \in M$, co-tangent space $T_x^* M$
  - local moving frame (e_j), $j = 1, \ldots, m$
  - local coordinates $x = (x^j)$, $j = 1, \ldots, m$ with $e_j = \frac{\partial}{\partial x^j}$
- **Regularity** of a vector field $X = (X^j)$ expressed in terms of its components (checked in any local coordinate chart)
  - Lebesgue spaces $L^p_{\mathrm{loc}}(M)$
  - Sobolev spaces $H^k_{\mathrm{loc}}(M)$ and $W^{k,p}_{\mathrm{loc}}(M)$
- **Tensor fields** (see below). $H^k_{\mathrm{loc}} T^q_p(M)$, etc.

Remarks. • Change of coordinates are taken to be C^∞

- Notion of local convergence, but there need not exist a canonical norm in these spaces.

- Space of **SCALAR DISTRIBUTIONS**

  $F \in \mathcal{D}'(M)$: dual of the space $\mathcal{D}\Lambda^m(M)$ of all compactly supported, C^∞ m-form fields

  - **Continuity property**
    $\langle F, \omega^{(k)} \rangle \to \langle F, \omega^{(\infty)} \rangle$ if $\|\omega^{(k)} - \omega^{(\infty)}\|_{C^p(M)} \to 0$ for any p and all $\omega^{(k)}, \omega^{(\infty)}$ smooth and (uniformly) compactly supported

  - Example: **canonical embedding** $f \in L^1_{\text{loc}}(M) \mapsto F \in \mathcal{D}'(M)$, via

$$\prec F, \omega \succ_{\mathcal{D}', \mathcal{D}} := \int_M f\omega, \qquad \omega \in \mathcal{D}\Lambda^m(M)$$

- **LIE DERIVATIVE.**

  - Functions $\mathcal{L}_X f = X(f)$. Vector fields $\mathcal{L}_X Y = [X, Y]$
  - 1-form fields $(\mathcal{L}_X \alpha)(Y) := X(\alpha(Y)) - \alpha([X, Y])$
  - 2-covariant tensor fields (e.g. for a metric)

$$(\mathcal{L}_X h)(T, Z) := X(h(T, Z)) - h(\mathcal{L}_X T, Z) - h(T, \mathcal{L}_X Z)$$

- **DISTRIBUTIONAL DERIVATIVE** XF of a scalar distribution $F \in \mathcal{D}'(M)$ by a smooth vector field X

$$\prec XF, \omega \succ_{\mathcal{D}',\mathcal{D}} := - \prec F, \mathcal{L}_X \omega \succ_{\mathcal{D}',\mathcal{D}}, \qquad \omega \in \mathcal{D}\Lambda^m(M)$$

  - Motivated from Cartan identity: for a smooth function f and a smooth, compactly supported m-form on M^m

$$f \mathcal{L}_X \omega = f\, d(i_X \omega) = d(f\, i_X \omega) - df \wedge i_X \omega$$

    and with Stokes formula

$$\int_M (Xf)\, \omega = - \int_M f\, \mathcal{L}_X \omega$$

  - In local coordinates (x^i) $(i = 1, \ldots, m)$ one has

$$\int X^j \partial_j f\, \overline{\omega} dx^1 \wedge \ldots \wedge dx^m = - \int f\, \partial_j (X^j \overline{\omega})\, dx^1 \wedge \ldots \wedge dx^m$$

- Space of **DISTRIBUTION DENSITIES**
  $\mathcal{D}'\Lambda^m(M)$: dual of the space $\mathcal{D}(M)$ of compactly supported functions.
  - $\prec \Omega, f \succ_{\mathcal{D}',\mathcal{D}}$ for $f \in \mathcal{D}(M)$
  - Example: **canonical embedding** $\omega \in L^1_{\text{loc}}(M) \mapsto \omega \in \mathcal{D}'\Lambda^m(M)$, via

$$\prec \omega, f \succ_{\mathcal{D}',\mathcal{D}} := \int_M f\omega, \qquad f \in \mathcal{D}(M)$$

Notation: $\mathfrak{T}^p_q(M) := \mathcal{C}^\infty T^p_q(M)$

- Space of **TENSOR DISTRIBUTIONS** $\mathcal{D}' T^p_q(M)$: $C^\infty(M)$-multi-linear maps

$$A : \underbrace{\mathfrak{T}^1_0(M) \times \ldots \times \mathfrak{T}^1_0(M)}_{q \text{ times}} \times \underbrace{\mathfrak{T}^0_1(M) \times \ldots \times \mathfrak{T}^0_1(M)}_{p \text{ times}} \to \mathcal{D}'(M)$$

  - $A(aX + bY, c\omega + k\theta) = acA(X,\omega) + bcA(Y,\omega) + akA(X,\theta) + bkA(Y,\theta)$
  - Example: canonical embedding $A \in L^1_{\text{loc}} T^p_q(M) \mapsto A \in \mathcal{D}' T^p_q(M)$:

$$\prec A(X_{(1)},\ldots,X_{(q)},\theta^{(1)},\ldots,\theta^{(p)}),\omega \succ_{\mathcal{D}',\mathcal{D}} := \int_M A(X_{(1)},\ldots,X_{(q)},\theta^{(1)},\ldots,\theta^{(p)})\,\omega$$

Section 1. GEOMETRIC FORMULATION

Section 1.1 WEAKLY REGULAR MANIFOLDS

- **Lie derivative in the weak sense.** $\mathcal{L}_X h$ is defined for a measurable and locally integrable 2-tensor h on a smooth manifold, for any C^1 vector fields X, Y, Z, by

$$(\mathcal{L}_X h)(Y, Z) := X(h(Y, Z)) - h(\mathcal{L}_X Y, Z) - h(Y, \mathcal{L}_X Z)$$

- T^2 **Symmetry** on a smooth (connected, orientable) 3-manifold Σ endowed with a metric $h \in L^1_{\text{loc}}(\Sigma)$
  - **Torus group action:** smooth, linearly independent commuting vector fields X, Y with closed orbits defining an action with no fixed point
  - **Killing property**: $\mathcal{L}_X h = \mathcal{L}_Y h = 0$ in the weak sense

Definition. Weakly regular T^2–symmetric Riemannian manifold (Σ, h)

- L^∞ **Riemannian structure.** $\Sigma \simeq T^3$ (compact, C^∞ 3-manifold) endowed with a Riemannian metric $h \in L^\infty(\Sigma)$
- T^2 **Symmetry.** Two Killing fields X, Y as above $\mathcal{L}_X h = \mathcal{L}_Y h = 0$ in the weak sense
- H^1 **and Lipschitz regularity on the T^2-symmetry orbits.**

$$h_{XX} = h(X,X),\ h_{XY} = h(X,Y),\ h_{YY} = h(Y,Y) \in H^1(\Sigma)$$

- **Lipschitz regularity on the area of T^2–orbits.**

$$\left(\overline{R}\right)^2 := h_{XX}\, h_{YY} - \left(h_{XY}\right)^2 \in W^{1,\infty}(\Sigma)$$

- $W^{1,1}$ **Regularity on the orthogonal of the orbits.**
  - Consider a smooth frame of commuting vector fields (X, Y, Θ) (therefore $\mathcal{L}_X\Theta = \mathcal{L}_Y\Theta = 0$)
  - An **adapted frame** (X, Y, Z) where Z is the (non-smooth!) field $Z := \Theta + \tilde{a}\,X + \tilde{b}\,Y \in \{X, Y\}^\perp$
  - $h_{ZZ} \in W^{1,1}(\Sigma)$

Remarks. • Fully geometric definition, independent of the choice of the Killing fields within the generators of the T^2-symmetry.

- **Regularity**
  - Since h is Riemannian and $\overline{R} > 0$ is continuous on a compact set, one has $\min_\Sigma \overline{R} > 0$.
  - From the T^2–symmetry and $h_{ZZ} \in W^{1,1}(\Sigma)$, we will deduce that h_{ZZ} is continuous and $\inf_\Sigma h_{ZZ} > 0$.
  - The inverse metric components h^{XX}, h^{XY} and h^{YY} are also H^1.
  - No regularity on the derivatives of the coefficients $h_{X\theta}, h_{Y\theta} \in L^\infty(\Sigma)$

- **Isomorphisms transforming vectors into co-vectors** (and vice-versa)
  - Multiplicative operators with L^∞ coefficients.
  - In the frame $(e_1, e_2, e_3) = (X, Y, Z)$

$$V_j = h_{ij} V^i \qquad (\text{with } i, j = 1, 2, 3)$$

with coefficients in L^∞ (or more regular).

Polarized spaces. Special class having $h_{XY} = 0$.

Definition. Weakly regular T^2–symmetric triple (Σ, h, K)

A weakly regular T^2–symmetric Riemannian manifold (Σ, h) with adapted frame (X, Y, Z), endowed with a symmetric 2-tensor field K such that:

- **(Square) integrability conditions.**
  - $K_{UV} \in L^2(\Sigma)$ for all $(U, V) \neq (Z, Z) \in \{X, Y, Z\}^2$
  - $K_{ZZ} \in L^1(\Sigma)$
- L^∞ **Trace on the symmetry orbits.**

$$\mathrm{Tr}^{(2)}(K) := h^{ab} K_{ab} = h^{XX} K_{XX} + 2h^{XY} K_{XY} + h^{YY} K_{YY} \in L^\infty(\Sigma)$$

- T^2 **Symmetry.** K is invariant under the action of the T^2 group generated by (X, Y): $\mathcal{L}_X K = \mathcal{L}_Y K = 0$

Remark. For the Einstein equations, the sup- norm bound will be a bound on the time derivative of the area R (defined later within the spacetime):

- $\mathrm{Tr}^{(2)}(K)$ is essentially $\partial_t R$
- Therefore, $\mathrm{Tr}^{(2)}(K) \in L^\infty(\Sigma)$ is natural in view of $\overline{R} \in W^{1,\infty}(\Sigma)$.

Definition

An L^∞ Lorentzian structure: a $(3+1)$–manifold $\mathcal{M} \simeq I \times T^3$ together with a Lorentzian metric $g \in L^\infty_{\text{loc}}$ whose volume form is bounded below. We assume that the T^2 symmetry property hold.

$(\mathcal{M}, g)$ is said to admit a $(3+1)$–**adapted decomposition** if there exist coordinates (t, x, y, θ):

- Adapted to the product decomposition of $\mathcal{M} \simeq I \times T^3$
  - $t \in I$ and (x, y, θ) periodic on T^3
  - $\frac{\partial}{\partial x} =: X$ and $\frac{\partial}{\partial y} =: Y$ generators of the symmetry group
- Metric decomposition $g = -n^2(t)\, dt^2 + h(t)$
  - a scalar $n(t) \in L^\infty(T^3)$ (lapse function, bounded below > 0)
  - a Riemannian metric $h(t) \in L^\infty(T^3)$ (induced metric)

Remarks.

• We assume a vanishing shift. (existence established below)

• Since $(\mathcal{M}, g)$ is invariant by the group action, $\mathcal{L}_X(n) = \mathcal{L}_Y(n) = 0$ and $\mathcal{L}_X h = \mathcal{L}_Y h = 0$ in the weak sense.

Notation from now on: $(\mathcal{M} \simeq T^3, g)$ is an L^∞ Lorentzian structure

- enjoys the T^2 symmetry, admits an adapted $(3+1)$–decomposition
- adapted frame (T, X, Y, Θ) associated with a global chart (t, x, y, θ)
- we write $\Sigma_t \simeq \{t\} \times T^3$ for the level sets of the function t

Definition. Weakly regular T^2–symmetric Lorentzian manifold

- **Timelike regularity.**
  $\mathcal{L}_T h \in L^1(\Sigma_t)$ for almost every t and L^∞_{loc} in time
  (uniform bounds within any compact subset of I)

- **Spacelike regularity.** For almost every t (and L^∞_{loc} in time)
  - Consider the second fundamental form of the slices
    $K(t) := -\frac{1}{2n(t)} (\mathcal{L}_T h)(t) \in L^1(\Sigma_t)$
  - $(\Sigma_t, h(t), K(t))$ is a weakly regular T^2–symmetric triple
    (the group action being the one induced on Σ_t)

- **Conformal regularity.** Introduce $Z := \Theta + \widetilde{a}\, X + \widetilde{b}\, Y \in \{T, X, Y\}^\perp$

$$\rho^2 := \frac{h_{ZZ}}{n^2} = -\frac{g_{ZZ}}{g_{TT}} \in W^{2,1}(\Sigma_t) \qquad \mathcal{L}_T \rho \in W^{1,1}(\Sigma_t)$$

Application to the Einstein equations. (see below)

- Dependence upon the foliation.
  - We will construct first a specific foliation along which the regularity and integrability conditions hold
  - and, next, deduce the same regularity for more general foliations.
- The conformal quotient metric $-dt^2 + \rho^2 d\theta^2$ determines the relevant wave operator.
- Additional regularity in time (in suitable topologies in space)

Observations.

- $n \in W^{1,1}(\Sigma_t)$ since h_{ZZ} has this regularity.
- From the decomposition $Z = \theta + \widetilde{a}\, X + \widetilde{b}\, Y$ and the commutation properties, we immediately obtain:
$$\mathcal{L}_T Z = T(\widetilde{a})X + T(\widetilde{b})Y \qquad \text{with } T(\widetilde{a}), T(\widetilde{b}) \in L^1(\Sigma_t)$$
- **Expression of the second fundamental form**. Using that T, X, Y, Z commute while Z is orthogonal to X, Y, T, we obtain (with $e_a, e_a \in \{X, Y\}$ and $h_{ab} = h(e_a, e_b)$)

$$K_{ab} = -\frac{1}{2n}\, T(h_{ab}) \qquad \in L^2(\Sigma_t)$$
$$K_{aZ} = \frac{1}{2n} h(e_a, \mathcal{L}_Z T) \qquad \in L^2(\Sigma_t)$$
$$K_{ZZ} = -\frac{1}{2n}\, T\big(h_{ZZ}\big) \qquad \in L^1(\Sigma_t)$$

$(\mathcal{M}, g)$ being a weakly regular T^2–symmetric Lorentzian manifold with adapted frame (T, X, Y, Z)

Definition. Weak version of the second fund. form of the symmetry orbits

To the orbits of symmetry and a.e. in $\mathcal{M}$, we associate the 2-tensor

$$\chi_{ab} := -\frac{1}{2\sqrt{h_{ZZ}}} Z\big(h_{ab}\big) \in L^2(\Sigma_t) \qquad a, b = X, Y$$

Lemma. Normal derivative of the area and volume elements

- Mean curvature of the symmetry orbits

$$\mathrm{Tr}^{(2)}(\chi) := h^{ab}\,\chi_{ab} = \frac{1}{\sqrt{h_{ZZ}}} Z\big(\ln R\big) \in L^\infty_{\mathrm{loc}}(\mathcal{M})$$

- Mean curvature of the slices (with $h := \det h_{ij}$)

$$\mathrm{Tr}\,(K) := h^{ij}\,K_{ij} = -\frac{1}{n} T\big(\ln\sqrt{h}\big) \in L^1(\Sigma_t)$$

QUESTIONS in this weak regularity class

- Einstein's constraints ? Einstein's evolution equations ?
- Existence for the Cauchy problem ?

Objective.

- Christoffel symbols involve ill-defined products (even as distributions)
- Need also to revisit the expressions of the curvature

Let (Σ, h) be a weakly regular T^2–symmetric Riemannian manifold .

- Non-smooth adapted frame (X, Y, Z)

  (Z is orthogonal to X, Y)

  (X need not be orthogonal to Y)

- $\Gamma^{\Theta}_{\Theta\Theta}$ ill-defined, since it involves products $h^{i\Theta}\,\Theta\big(h_{\Theta b}\big)$ with $b = X, Y$ and $h \in L^\infty(\Sigma)$
- Definitions below meaningful only in an adapted frame

Remarks.

- $Z \in L^\infty(\Sigma)$ does not apply to general L^1 functions.
- But it applies to T^2–symmetric functions, provided we set

$$Z(f) := \Theta(f) \qquad \text{for } T^2\text{–symmetric } f \in L^1(\Sigma)$$

Preliminary computation. We first consider regular data:

- X, Y, θ commute
- orthogonality condition $Z \in \{X, Y\}^{\perp}$
- T^2-symmetry properties

For $i = X, Y, Z$ and $a, b = X, Y$

$$
\begin{aligned}
\Gamma^i_{ab} &= \frac{1}{2} h^{ij} \left(h_{aj,b} + h_{jb,a} - h_{ab,j} \right) = -\frac{1}{2} h^{iZ} Z(h_{ab}) \\
\Gamma^Z_{aZ} &= \frac{1}{2} h^{Zj} \left(h_{aj,Z} + h_{jZ,a} - h_{aZ,j} \right) = 0 \\
\Gamma^b_{aZ} &= \frac{1}{2} h^{bj} \left(h_{aj,Z} + h_{jZ,a} - h_{aZ,j} \right) = \frac{1}{2} \left(h^{bX} Z(h_{aX}) + h^{bY} Z(h_{aY}) \right) \\
\Gamma^a_{ZZ} &= \frac{1}{2} h^{aj} \left(2h_{jZ,Z} - h_{ZZ,j} \right) = 0 \\
\Gamma^Z_{ZZ} &= \frac{1}{2} h^{Zj} \left(2h_{Zj,Z} - h_{ZZ,j} \right) = \frac{1}{2} h^{ZZ} Z(h_{ZZ}) = \log \sqrt{h^{ZZ}}
\end{aligned}
$$

Definition-Proposition. Weak version of the Christoffel symbols

$\Gamma^c_{ab} := 0$ $\qquad a, b \in \{X, Y\}$

$\Gamma^b_{aZ} := \frac{1}{2}\Big(h^{bX}\, Z(h_{aX}) + h^{bY}\, Z(h_{aY})\Big)$ $\qquad \Gamma^Z_{ab} := -\frac{1}{2}\, h^{ZZ}\, Z(h_{ab})$

$\Gamma^Z_{aZ} = \Gamma^a_{ZZ} := 0$

$\Gamma^Z_{ZZ} := \frac{1}{2}\, h^{ZZ}\, Z(h_{ZZ})$

These expressions do make sense (in the frame (X, Y, Z))

- $\Gamma^i_{jk} \in L^2(\Sigma)$ for all $(i, j, k) \neq (Z, Z, Z)$
- $\Gamma^Z_{ZZ} \in L^1(\Sigma)$
- trace $\Gamma^a_{aZ} \in L^\infty(\Sigma)$

Furthermore, if (Σ, h) is regular, these functions coincide with the standard Christoffel symbols.

Observation. Mean curvature of the T^2-orbits

$$L^\infty(\Sigma) \ni Z((\overline{R})^2) = Z(h_{XX})\, h_{YY} + h_{XX}\, Z(h_{YY}) - 2\, h_{XY} Z(h_{XY})$$
$$= \frac{1}{\overline{R}^2}\Big(h^{XX}\, Z(h_{XX}) + h^{YY}\, Z(h_{YY}) + 2h_{XY} Z(h_{XY})\Big) = \frac{2}{\overline{R}^2}\Gamma^a_{aZ}$$

WEAK VERSION OF THE HAMILTONIAN CONSTRAINT

Let now (Σ, h, K) be a weakly regular T^2–symmetric triple.

- $\Gamma^Z_{ZZ} \in L^1(\Sigma)$ cannot be multiplied by Christoffel coefficients in $L^2(\Sigma)$ or $L^1(\Sigma)$.
- We will rely on the trace property $\Gamma^a_{aZ} \in L^\infty(\Sigma)$.

Strategy.

- First, we define the Ricci component $R^{(3)}_{ZZ} = \mathbf{Ric}_h(Z, Z)$ of the manifold (Σ, h).
- Then, we rely on the Gauss equation for the T^2–orbits in order to define the scalar curvature $R^{(3)}$ of the manifold (Σ, h).
- Next, we suitably decompose the second fundamental form K of the slices.
- Finally, we arrive at a weak version of the Hamiltonian constraint.

Preliminary computation. Compute $R^{(3)}_{ZZ} = \Omega_1 + \Omega_2$ with

$$\Omega_1 := \Gamma^i_{ZZ,i} - \Gamma^i_{iZ,Z} \qquad \Omega_2 := \Gamma^j_{ij}\Gamma^i_{ZZ} - \Gamma^j_{iZ}\Gamma^i_{jZ}$$

- Since $\Gamma^a_{ZZ,a} = 0$:

$$\Omega_1 = \Gamma^a_{ZZ,a} + \Gamma^Z_{ZZ,Z} - \Gamma^Z_{ZZ,Z} - \Gamma^a_{aZ,Z} = -Z(\Gamma^a_{aZ})$$

- Since $\Gamma^Z_{aZ} = \Gamma^a_{ZZ} = 0$:

$$\begin{aligned}\Omega_2 &= \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ} + \Gamma^Z_{aZ}\Gamma^a_{ZZ} + \Gamma^a_{ab}\Gamma^b_{ZZ} \\ &\quad - \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} - \Gamma^Z_{bZ}\Gamma^b_{ZZ} - \Gamma^a_{ZZ}\Gamma^Z_{aZ} - \Gamma^a_{bZ}\Gamma^b_{aZ} \\ &= \Gamma^a_{aZ}\Gamma^Z_{ZZ} - \Gamma^a_{bZ}\Gamma^b_{aZ}\end{aligned}$$

after cancelling out the ill-defined terms $\pm\Gamma^Z_{ZZ}\Gamma^Z_{ZZ} \in L^1(\Sigma_t)L^1(\Sigma_t)$

Definition–Proposition. Weak version of the Ricci component in the direction (Z, Z)

$$\begin{aligned}R^{(3)}_{ZZ} :=& -Z(\Gamma^a_{aZ}) + \Gamma^a_{aZ}\Gamma^Z_{ZZ} - \Gamma^a_{bZ}\,\Gamma^b_{aZ} \\ & W^{-1,\infty}(\Sigma) + L^\infty(\Sigma)L^1(\Sigma) - L^2(\Sigma)L^2(\Sigma)\end{aligned}$$

where the first term is defined only in the weak sense.
Furthermore, when sufficient regularity is assumed, our definition for $R^{(3)}_{ZZ}$ agrees with the standard definition.

From the Gauss equation and since the T^2 orbits are flat:

$$0 = R^{(3)} - |\chi|^2 + \left(\mathrm{Tr}^{(2)}\chi\right)^2 - \frac{2}{h_{ZZ}} R^{(3)}_{ZZ} \qquad \text{for regular metrics}$$

(Z is not a unit vector field)

This formula does not make sense:

- $R^{(3)}_{ZZ}$ defined in the weak sense only
- Need to "remove' the factor $h_{ZZ} \in W^{1,1}(\Sigma)$

Definition. Weak version of the weighted scalar curvature

$$R^{(w)(3)} := 2\, R^{(3)}_{ZZ} + h_{ZZ} \left(|\chi|^2 - (\mathrm{Tr}^{(2)}\chi)^2\right)$$
$$\in W^{-1,\infty}_{\mathrm{loc}}(\Sigma) + L^1(\Sigma)$$

in the weak sense, in which:

- χ is the second fundamental form of the T^2–orbits.
- $R^{(3)}_{ZZ}$ is the weak version of the Ricci component (Z, Z).

Our final observation is the following **algebraic decomposition** of the tensor K for regular metrics:

$$\left(\mathrm{Tr}K\right)^2 - |K|^2 = \left(\mathrm{Tr}^{(2)}K\right)^2 + 2\left(\mathrm{Tr}^{(2)}K\right)K_Z^Z - K_{ab}K^{ab} - 2K_{aZ}K^{aZ}$$

Definition-Proposition. Weak version of the Hamiltonian constraint

$$R^{(w)(3)} + h_{ZZ}\left(\left(\mathrm{Tr}^{(2)}K\right)^2 + 2\left(\mathrm{Tr}^{(2)}K\right)K_Z^Z - K_{ab}K^{ab} - 2K_{aZ}K^{aZ}\right) = 0$$
$$W^{-1,\infty}(\Sigma) + L^\infty(\Sigma)L^\infty(\Sigma) + L^\infty(\Sigma)L^1(\Sigma) + L^2(\Sigma)L^2(\Sigma)$$

understood in the weak sense.
Furthermore, if (Σ, h, K) is sufficiently regular, then (Σ, h, K) satisfies the weak version of the Hamiltonian constraint equation iff it satisfies this equation in the classical sense.

Recall here that

$$R^{(w)(3)} := 2\,R^{(3)}_{ZZ} + h_{ZZ}\left(|\chi|^2 - (\mathrm{Tr}^{(2)}\chi)^2\right)$$
$$R^{(3)}_{ZZ} := -Z(\Gamma^a_{aZ}) + \Gamma^a_{aZ}\Gamma^Z_{ZZ} - \Gamma^a_{bZ}\,\Gamma^b_{aZ}$$

Definition-Proposition. Weak version of the momentum constraints

$$Z(\mathrm{Tr}^{(2)}K) - \Gamma^a_{aZ}\,K^Z_Z - \Gamma^a_{bZ}K^b_a = 0$$
$$\in W^{-1,\infty}(\Sigma) + L^\infty(\Sigma)L^1(\Sigma) + L^2(\Sigma)L^2(\Sigma)$$

$$Z\big(h^{1/2}_{ZZ}K^Z_a\big) - \Gamma^b_{bZ}h^{1/2}_{ZZ}K^Z_a = 0 \qquad (a = X, Y)$$
$$\in W^{-1,2}(\Sigma) + L^\infty(\Sigma)L^2(\Sigma)$$

understood in the weak sense.

Furthermore, if (Σ, h, K) is sufficiently regular, then (Σ, h, K) satisfies the weak version of the momentum constraint equations iff it satisfies the equations in the classical sense.

(proof given below)

Remark. The last two equations are weighted with the scalar $h^{1/2}_{ZZ}$, while the first equation has a different homogeneity.

COMPATIBILITY WITH THE STANDARD DEFINITION.

Assume sufficient regularity.

Derivation of the momentum constraint in the Z-direction.

$$\begin{aligned}
\nabla^{(3)}_j K^j_Z &= Z(K^Z_Z) + K^a_{Z,a} - \Gamma^i_{jZ} K^j_i + \Gamma^j_{ij} K^i_Z \\
&= Z(K^Z_Z) - \Gamma^Z_{ZZ} K^Z_Z - \Gamma^a_{ZZ} K^Z_a - \Gamma^Z_{aZ} K^a_Z - \Gamma^a_{bZ} K^b_a \\
&\quad + \Gamma^Z_{ZZ} K^Z_Z + \Gamma^Z_{aZ} K^a_Z + \Gamma^b_{bZ} K^Z_Z + \Gamma^b_{ba} K^a_Z \\
&= Z(K^Z_Z) - \Gamma^a_{bZ} K^b_a + \Gamma^b_{bZ} K^Z_Z
\end{aligned}$$

- We used $\Gamma^Z_{aZ} = \Gamma^a_{ZZ} = 0$
- and cancelled out ill-defined terms $\pm \Gamma^Z_{ZZ} K^Z_Z \in L^1(\Sigma) L^1(\Sigma)$.
- Recall that $\mathrm{Tr}^{(2)} K = \mathrm{Tr} K - K^Z_Z$.

We conclude that $\nabla^{(3)}_j K^j_Z - \nabla^{(3)}_Z \mathrm{Tr}\, K = 0$ is equivalent to our formulation above.

Derivation of momentum constraints in the symmetry orbits.

$$\begin{aligned}\nabla^{(3)}_j K^j_a &= K^j_{a,j} - \Gamma^i_{aj} K^j_i + \Gamma^j_{ij} K^i_a \\ &= Z(K^Z_a) - \Gamma^Z_{aZ} K^Z_Z - \sum_{(i,j)\neq(Z,Z)} \Gamma^i_{aj} K^j_i + \Gamma^Z_{ZZ} K^Z_a + \sum_{(i,j)\neq(Z,Z)} \Gamma^j_{ij} K^i_a \\ &= Z(K^Z_a) + \Gamma^Z_{ZZ} K^Z_a + \sum_{(i,j)\neq(Z,Z)} \left(\Gamma^j_{ij} K^i_a - \Gamma^i_{aj} K^j_i \right)\end{aligned}$$

while $\nabla^{(3)}_a K^j_j = 0$.

The product involving $\Gamma^Z_{ZZ} \in L^1(\Sigma)$ and $K^Z_a \in L^2(\Sigma)$ is ill-defined for weakly regular spacetimes. We thus proceed as follows:

- (Formally) multiply the above equations by $h^{1/2}_{ZZ}$
- From the expression of the Christoffel symbol of the second term (using $h^{ZZ} = (h_{ZZ})^{-1}$)

$$\begin{aligned}h^{1/2}_{ZZ} \nabla^{(3)}_j K^j_a = h^{1/2}_{ZZ} &\left(Z(K^Z_a) + \frac{1}{2} h^{-1}_{ZZ} Z(h_{ZZ}) K^Z_a \right) \\ &+ h^{1/2}_{ZZ} \sum_{(i,j)\neq(Z,Z)} \left(\Gamma^j_{ij} K^i_a - \Gamma^i_{aj} K^j_i \right)\end{aligned}$$

- Combine the first two terms in the right-hand side as $Z\left(h^{1/2}_{ZZ} K^Z_a\right)$.

Section 1.3 WEAK VERSION OF EINSTEIN'S EVOLUTION EQUATIONS

Let $(\mathcal{M}, g)$ be a weakly regular T^2–symmetric Lorentzian manifold with adapted frame (T, X, Y, Z), spacelike slices Σ_t $(t \in I)$, and second fundamental form K.

Sketch of the preliminary computation when the spacetime is regular:

- Since the frame $(T, X, Y, Z) = (e_0, e_1, e_2, e_3)$ is not (fully) induced by coordinates:
$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \Big(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} + c_{\delta\beta\gamma} + c_{\delta\gamma\beta} + c_{\beta\gamma\delta} \Big)$$
$$c_{\beta\gamma\delta} := [e_\beta, e_\gamma]_\delta = g_{\delta\rho} \, [e_\beta, e_\gamma]^\rho$$
- $[e_0, e_1] = [e_0, e_2] = [e_1, e_2] = 0$
- Moreover, $[T, Z] = T(\widetilde{a}) \, X + T(\widetilde{b}) \, Y$, which, in particular, is orthogonal to both Z and T.

Components Γ^T_{ij} for $i,j=1,2,3$:

$$g_{TT}\Gamma^T_{ij} = g(T,\nabla_i e_j) = g(nN,\nabla_i e_j) = nK_{ij},$$

where N is the timelike unit normal to Σ_t, therefore since $g_{TT}=-n^2$

$$\Gamma^T_{ij} = -\frac{1}{n}K_{ij}$$

Components Γ^T_{TZ}, Γ^T_{TT}, Γ^T_{Ta} **and** Γ^Z_{TT}. For instance

$$\Gamma^T_{TZ} = \frac{1}{2}g^{TT}\Big(g_{TZ,T}+g_{TT,Z}-g_{TZ,T}+c_{TTZ}+c_{TZT}+c_{ZTT}\Big) = \frac{Z(n)}{n}$$

Remaining components. For $i=1,2,3$

$$g(e_1,\nabla_T e_i) = g_{ZZ}\Gamma^Z_{Ti}$$
$$= g(e_1,\nabla_i T) = -g(\nabla_i e_1, T) = -ng(\nabla_i e_1, N) = -nK_{iZ}$$

- Coefficient Γ^a_{Tb}:

$$g(e_c,\nabla_T e_b) = g_{ca}\Gamma^a_{Tb}$$
$$= g(e_c,\nabla_b T) = -g(\nabla_b e_c, T) = -nK_{bc}$$

- Coefficient Γ^a_{TZ}:

$$g(e_c,\nabla_T Z) = g_{ca}\Gamma^a_{TZ} = -g(\nabla_T e_c, Z)$$
$$= -g(\nabla_c T, Z) = g(T,\nabla_c Z) = nK_{cZ}$$

- Coefficient Γ^a_{TZ}:

$$g(e_c,\nabla_Z T) = g_{ca}\Gamma^a_{TZ}$$
$$= -g(\nabla_Z e_c, T) = -nK_{cZ}$$

We define the **weak version of the Christoffel symbols** as

$$\Gamma^T_{ab} := -\frac{1}{n} K_{ab} \qquad \Gamma^T_{ZZ} := -\frac{1}{n} K_{ZZ} \qquad \Gamma^T_{aZ} := -\frac{1}{n} K_{aZ}$$

$$\Gamma^a_{TZ} := nK^a_Z \qquad \Gamma^Z_{TZ} := -g^{ZZ}\, n\, K_{ZZ}$$

$$\Gamma^Z_{Ta} := -ng^{ZZ}\, K_{aZ} \qquad \Gamma^a_{Tb} := -g^{ac}\, n\, K_{bc}$$

$$\Gamma^T_{TZ} := Z(\log n) \qquad \Gamma^T_{Ta} = \Gamma^a_{TT} := 0 \qquad \Gamma^Z_{TT} := \frac{1}{2} g^{ZZ}\, Z(n^2)$$

$$\Gamma^T_{TT} := T(\log n)$$

Moreover, for $i, j, k = X, Y, Z$, we define Γ^i_{jk} as before, with h replaced by the induced metric $h(t)$.

Proposition. Christoffel symbols in an adapted frame

The coefficients above are well-defined.

- Integrability properties

$$\Gamma^T_{TT},\ \Gamma^Z_{TT},\ \Gamma^T_{ZZ},\ \Gamma^T_{TZ},\ \Gamma^Z_{TZ} \in L^1(\Sigma_t) \qquad L^\infty_{\text{loc}} \text{ in time}$$

$$\Gamma^T_{ab}, \Gamma^T_{iZ},\ \Gamma^Z_{iT},\ \Gamma^i_{TZ} \in L^2(\Sigma_t) \qquad L^\infty_{\text{loc}} \text{ in time}$$

- Additional sup-norm bounds (proof given below)

$$\Gamma^a_{Ta} \in L^\infty_{\text{loc}}(\mathcal{M})$$

$$\Gamma^T_{TT} - \Gamma^Z_{TZ} \in L^\infty_{\text{loc}}(\mathcal{M})$$

$$\Gamma^Z_{ZZ} - \Gamma^T_{TZ} \in L^\infty_{\text{loc}}(\mathcal{M})$$

Furthermore, when $(\mathcal{M}, g)$ is regular, our weak version coincides with the standard definition.

PROOF of the additional PROPERTIES.

- **Trace regularity.**

$$\Gamma^a_{Ta} = -n\mathrm{Tr}^{(2)}(K) \in L^\infty_{\mathrm{loc}}(\mathcal{M})$$

- **Conformal regularity.**

We rely on the (space and time) Lipschitz regularity of the conformal metric $\rho^2 = n^{-2} g_{ZZ} \in W^{1,\infty}(\mathcal{M})$

$$\Gamma^T_{TT} - \Gamma^Z_{TZ} = \frac{1}{2n^2}\left(T(n^2) - T(g_{ZZ})n^2 g^{ZZ}\right) = \frac{1}{2n^2} g_{ZZ} T(\rho^{-2}) \in L^\infty_{\mathrm{loc}}(\mathcal{M})$$

$$\begin{aligned}\Gamma^Z_{ZZ} - \Gamma^T_{TZ} &= \frac{1}{2} g^{ZZ} Z(g_{ZZ}) - \frac{1}{n} Z(n^2) \\ &= \frac{n^2}{2} g^{ZZ} Z\left(\frac{g_{ZZ}}{n^2}\right) = \frac{n^2}{2} g^{ZZ} Z(\rho^2) \in L^\infty_{\mathrm{loc}}(\mathcal{M})\end{aligned}$$

EINSTEIN EVOLUTION EQUATIONS

- Let $(\mathcal{M}, g)$ be a weakly regular T^2–symmetric Lorentzian manifold.
- It remains to consider the components R_{ij}.

Definition

Weak version of the component R_{ZZ} of the Ricci tensor

$$
\begin{aligned}
R_{ZZ} := {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) && \in W^{-1,1}_{\text{loc}}(\mathcal{M}) \\
& + \Gamma^T_{ZZ}\left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right) + \Gamma^T_{TZ}\left(\Gamma^Z_{ZZ} - \Gamma^T_{TZ}\right) && \in L^1(\Sigma_t)L^\infty(\Sigma_t) \\
& + \Gamma^a_{Ta}\,\Gamma^T_{ZZ} + \Gamma^a_{aZ}\,\Gamma^Z_{ZZ} && \in L^1(\Sigma_t)L^\infty(\Sigma_t) \\
& - \Gamma^a_{bZ}\,\Gamma^b_{aZ} + 2\Gamma^T_{aZ}\Gamma^a_{TZ} && \in L^2(\Sigma_t)L^2(\Sigma_t)
\end{aligned}
$$

Weak version of the components R_{aZ} of the Ricci tensor

$$
\begin{aligned}
R_{aZ} := {} & T(\Gamma^T_{aZ}) + \left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right)\Gamma^T_{aZ} + \Gamma^b_{Tb}\Gamma^T_{aZ} \\
& \in W^{-1,\infty}_{\text{loc}}(\mathcal{M}) + L^\infty(\Sigma_t)L^2(\Sigma_t) + L^\infty(\Sigma_t)L^2(\Sigma_t)
\end{aligned}
$$

The components R_{cd}, $c, d = X, Y$ need to be suitably weighted, as follows.

Definition. Weak version of the Ricci components $R_{cd}^{(w)}$

$$R_{cd}^{(w)} := T\left(n\, g_{ZZ}^{1/2} \Gamma_{dc}^{T}\right) + Z\left(n\, g_{ZZ}^{1/2} \Gamma_{dc}^{Z}\right)$$
$$+ n\, g_{ZZ}^{1/2} \Big(\Gamma_{Ta}^{a} \Gamma_{dc}^{T} + \Gamma_{aZ}^{a} \Gamma_{dc}^{Z} - \Gamma_{dZ}^{T} \Gamma_{Tc}^{Z} - \Gamma_{Td}^{Z} \Gamma_{cZ}^{T}$$
$$- \Gamma_{da}^{T} \Gamma_{Tc}^{a} - \Gamma_{Td}^{a} \Gamma_{ac}^{T} - \Gamma_{da}^{Z} \Gamma_{cZ}^{a} - \Gamma_{dZ}^{a} \Gamma_{ac}^{Z}\Big)$$
$$\in W_{\mathrm{loc}}^{-1,2}(\mathcal{M}) + L^{1}(\Sigma_t)$$

Definition. Weak version of Einstein's evolution equations

$$R_{ZZ} = 0 \quad \text{in } W_{\mathrm{loc}}^{-1,1}(\mathcal{M})$$
$$R_{Zd} = 0 \quad \text{in } W_{\mathrm{loc}}^{-1,\infty}(\mathcal{M})$$
$$R_{ab}^{(w)} = 0 \quad \text{in } W_{\mathrm{loc}}^{-1,2}(\mathcal{M})$$

in the weak sense above.

See the proof below

Proposition. Equivalence with the classical definition

For any regular T^2–symmetric spacetime $(\mathcal{M}, g)$:

the weak version of the Einstein's evolution equations is satisfied iff the Ricci flat condition $Ric_g(e_i, e_j) = 0$ for $e_i, e_j \in \{X, Y, Z\}$ holds, where Ric denotes the Ricci tensor of g defined in the classical sense.

Theorem. Weak formulation of the Einstein equations

- If (Σ, h, K) is a weakly regular T^2–symmetric triple, then Einstein's constraint equations make sense in a weak form.
- If $(\mathcal{M}, g)$ is a weakly regular T^2–symmetric Lorentzian manifold, Einstein's evolution equations make sense in a weak form.
- All of the weak notions above coincide with the classical ones when the space is sufficient regular.

Terminology

- **weakly regular T^2–symmetric initial data set**
- **weakly regular T^2–symmetric Ricci-flat spacetime**

OUR NEXT OBJECTIVE will be to express these geometric equations in well-chosen coordinates as a system of nonlinear PDE's.

DERIVATION OF THE WEAK FORM OF THE EVOLUTION EQUATIONS.

Curvature component R_{ZZ}. With $c^{\gamma}_{\alpha\beta} = [e_\alpha, e_\beta]^\gamma$, we have

$$R_{ZZ} = R^{\alpha}_{Z\alpha Z} = \Gamma^{\alpha}_{ZZ,\alpha} - \Gamma^{\alpha}_{\alpha Z,Z} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ZZ} - \Gamma^{\alpha}_{Z\beta}\Gamma^{\beta}_{YZ} - c^{\beta}_{\alpha Z}\Gamma^{\alpha}_{Z\beta}$$

and, therefore,

$$\begin{aligned} R_{ZZ} = {} & Z(\Gamma^{Z}_{ZZ}) + T(\Gamma^{T}_{ZZ}) - \Big(Z(\Gamma^{Z}_{ZZ}) + Z(\Gamma^{T}_{TZ}) + Z(\Gamma^{a}_{aZ}) \Big) \\ & + \Gamma^{T}_{TT}\Gamma^{T}_{ZZ} + \Gamma^{Z}_{ZZ}\Gamma^{Z}_{ZZ} + \Gamma^{T}_{TZ}\Gamma^{Z}_{ZZ} + \Gamma^{Z}_{TZ}\Gamma^{T}_{ZZ} \\ & + \Gamma^{T}_{Ta}\Gamma^{a}_{ZZ} + \Gamma^{a}_{at}\Gamma^{T}_{ZZ} + \Gamma^{Z}_{aZ}\Gamma^{a}_{ZZ} + \Gamma^{a}_{aZ}\Gamma^{Z}_{ZZ} \\ & + \Gamma^{a}_{ab}\Gamma^{b}_{ZZ} - \Big(\Gamma^{T}_{TZ}\Gamma^{T}_{TZ} + \Gamma^{Z}_{ZZ}\Gamma^{Z}_{ZZ} + \Gamma^{T}_{ZZ}\Gamma^{Z}_{TZ} + \Gamma^{Z}_{TZ}\Gamma^{T}_{ZZ} \Big) \\ & - \Big(\Gamma^{T}_{aZ}\Gamma^{a}_{TZ} + \Gamma^{a}_{TZ}\Gamma^{T}_{aZ} + \Gamma^{Z}_{aZ}\Gamma^{a}_{ZZ} + \Gamma^{a}_{ZZ}\Gamma^{Z}_{aZ} \Big) - \Gamma^{a}_{bZ}\Gamma^{b}_{aZ} - c^{b}_{TZ}\Gamma^{T}_{bZ} \end{aligned}$$

- Observe that the only non-vanishing commutator is $[T, Z]$ and that this commutator is orthogonal to Z, T.
- Moreover, $[T, Z]^b = \Gamma^{b}_{TZ} - \Gamma^{b}_{TZ}$

- Take into account the cancellations of $Z(\Gamma^Z_{ZZ})$, $\Gamma^Z_{ZZ}\Gamma^Z_{ZZ}$ and $\Gamma^Z_{TZ}\Gamma^T_{ZZ}$ (ill-defined $L^1 L^1$ products),
- as well as the antisymmetry of Γ^a_{TZ}
- and the fact that $\Gamma^T_{Ta} = \Gamma^Z_{aZ} = 0$

$$
\begin{aligned}
R_{ZZ} = {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) + \left(\Gamma^T_{TT}\Gamma^T_{ZZ} + \Gamma^T_{TZ}\Gamma^Z_{ZZ}\right) \\
& + \left(\Gamma^a_{at}\Gamma^T_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ}\right) + \Gamma^a_{ab}\Gamma^b_{ZZ} \\
& - \left(\Gamma^T_{TZ}\Gamma^T_{TZ} + \Gamma^T_{ZZ}\Gamma^Z_{TZ}\right) - \Gamma^a_{bZ}\Gamma^b_{aZ} + 2\Gamma^T_{aZ}\Gamma^a_{TZ}
\end{aligned}
$$

and, after factoring out Γ^Z_{TZ} and Γ^Z_{TT}:

$$
\begin{aligned}
R_{ZZ} := {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) && \in W^{-1,1}_{\text{loc}}(\mathcal{M}) \\
& + \Gamma^T_{ZZ}\left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right) + \Gamma^T_{TZ}\left(\Gamma^Z_{ZZ} - \Gamma^T_{TZ}\right) && \in L^1(\Sigma_t)L^\infty(\Sigma_t) \\
& + \Gamma^a_{Ta}\,\Gamma^T_{ZZ} + \Gamma^a_{aZ}\,\Gamma^Z_{ZZ} && \in L^1(\Sigma_t)L^\infty(\Sigma_t) \\
& - \Gamma^a_{bZ}\,\Gamma^b_{aZ} + 2\Gamma^T_{aZ}\Gamma^a_{TZ} && \in L^2(\Sigma_t)L^2(\Sigma_t)
\end{aligned}
$$

Curvature components R_{Zd}. We proceed similarly:

$$\begin{aligned}R_{Zd} &= \Gamma^{\alpha}_{dZ,\alpha} - \Gamma^{\alpha}_{\alpha Z,d} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{dZ} - \Gamma^{\alpha}_{d\beta}\Gamma^{\beta}_{\alpha Z} - c^{\beta}_{\alpha d}\Gamma^{\alpha}_{Z\beta} \\ &= T(\Gamma^{T}_{dZ}) + Z(\Gamma^{Z}_{dZ}) + \left(\Gamma^{T}_{TT}\Gamma^{T}_{dZ} + \Gamma^{Z}_{ZZ}\Gamma^{Z}_{dZ} + \Gamma^{T}_{TZ}\Gamma^{Z}_{dZ} + \Gamma^{Z}_{TZ}\Gamma^{T}_{dZ}\right) \\ &\quad + \Gamma^{T}_{Ta}\Gamma^{a}_{dZ} + \Gamma^{a}_{at}\Gamma^{T}_{dZ} + \Gamma^{Z}_{aZ}\Gamma^{a}_{dZ} + \Gamma^{a}_{aZ}\Gamma^{Z}_{dZ} \\ &\quad + \Gamma^{a}_{ab}\Gamma^{b}_{dZ} - \left(\Gamma^{T}_{Td}\Gamma^{T}_{TZ} + \Gamma^{Z}_{dZ}\Gamma^{Z}_{ZZ} + \Gamma^{T}_{dZ}\Gamma^{Z}_{TZ} + \Gamma^{Z}_{Td}\Gamma^{T}_{ZZ}\right) \\ &\quad - \left(\Gamma^{T}_{da}\Gamma^{a}_{TZ} + \Gamma^{a}_{Td}\Gamma^{T}_{aZ} + \Gamma^{Z}_{da}\Gamma^{a}_{ZZ} + \Gamma^{a}_{dZ}\Gamma^{Z}_{aZ}\right) - \Gamma^{a}_{db}\Gamma^{b}_{aZ}\end{aligned}$$

Using $\Gamma^{a}_{ZZ} = \Gamma^{Z}_{aZ} = \Gamma^{T}_{Ta} = \Gamma^{a}_{bc} = 0$ and the fact that X, Y commutes with Z, T:

$$\begin{aligned}R_{Zd} &= T(\Gamma^{T}_{dZ}) + \Gamma^{T}_{TT}\Gamma^{T}_{dZ} + \Gamma^{Z}_{TZ}\Gamma^{T}_{dZ} + \Gamma^{a}_{Ta}\Gamma^{T}_{dZ} \\ &\quad - \left(\Gamma^{T}_{dZ}\Gamma^{Z}_{TZ} + \Gamma^{Z}_{Td}\Gamma^{T}_{ZZ}\right) - \left(\Gamma^{T}_{da}\Gamma^{a}_{TZ} + \Gamma^{a}_{Td}\Gamma^{T}_{aZ}\right)\end{aligned}$$

We cancel the terms $\pm\Gamma^{Z}_{TZ}\Gamma^{T}_{dZ}$ and use the identities

$$\begin{aligned}\Gamma^{T}_{TT}\Gamma^{T}_{dZ} - \Gamma^{Z}_{Td}\Gamma^{T}_{ZZ} &= \Gamma^{T}_{dZ}\left(\Gamma^{T}_{TT} - \Gamma^{Z}_{TZ}\right) \\ \Gamma^{T}_{da}\Gamma^{a}_{TZ} + \Gamma^{a}_{Td}\Gamma^{T}_{aZ} &= 0\end{aligned}$$

We have thus reached:

$$R_{dZ} = T(\Gamma^{T}_{dZ}) + \Gamma^{T}_{dZ}\left(\Gamma^{T}_{TT} - \Gamma^{Z}_{TZ}\right) + \Gamma^{a}_{Ta}\Gamma^{T}_{dZ} \in W^{-1,\infty}_{\mathrm{loc}}(\mathcal{M})$$

Curvature components R_{cd}.

$$
\begin{aligned}
R_{cd} &= \Gamma^{\alpha}_{dc,\alpha} - \Gamma^{\alpha}_{\alpha c,d} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{dc} - \Gamma^{\alpha}_{d\beta}\Gamma^{\beta}_{\alpha c} - c^{\beta}_{\alpha d}\Gamma^{\alpha}_{c\beta} \\
&= T(\Gamma^{T}_{dc}) + Z(\Gamma^{z}_{dc}) + \left(\Gamma^{T}_{TT}\Gamma^{T}_{dc} + \Gamma^{Z}_{ZZ}\Gamma^{Z}_{dc}\right) + \left(\Gamma^{T}_{TZ}\Gamma^{Z}_{dc} + \Gamma^{Z}_{TZ}\Gamma^{T}_{dc}\right) \\
&\quad + \left(\Gamma^{T}_{Ta}\Gamma^{a}_{dc} + \Gamma^{Z}_{aZ}\Gamma^{a}_{dc}\right) + \left(\Gamma^{a}_{Ta}\Gamma^{T}_{dc} + \Gamma^{a}_{aZ}\Gamma^{Z}_{dc}\right) + \Gamma^{a}_{ab}\Gamma^{b}_{dc} - \left(\Gamma^{T}_{Td}\Gamma^{T}_{Tc} + \Gamma^{z}_{dz}\Gamma^{Z}_{cZ}\right) \\
&\quad - \left(\Gamma^{T}_{dZ}\Gamma^{z}_{Tc} + \Gamma^{Z}_{Td}\Gamma^{T}_{cZ}\right) - \left(\Gamma^{T}_{da}\Gamma^{a}_{Tc} + \Gamma^{a}_{Td}\Gamma^{T}_{ac}\right) - \left(\Gamma^{Z}_{da}\Gamma^{a}_{cZ} + \Gamma^{a}_{dZ}\Gamma^{Z}_{ac}\right) - \Gamma^{b}_{da}\Gamma^{a}_{bc}
\end{aligned}
$$

and, using $\Gamma^{Z}_{aZ} = \Gamma^{T}_{Ta} = \Gamma^{a}_{bc} = 0$, we find

$$
\begin{aligned}
R_{cd} &= T(\Gamma^{T}_{dc}) + Z(\Gamma^{z}_{dc}) + \left(\Gamma^{T}_{TT}\Gamma^{T}_{dc} + \Gamma^{Z}_{ZZ}\Gamma^{Z}_{dc}\right) + \left(\Gamma^{T}_{TZ}\Gamma^{Z}_{dc} + \Gamma^{Z}_{TZ}\Gamma^{T}_{dc}\right) \\
&\quad + \left(\Gamma^{a}_{at}\Gamma^{T}_{dc} + \Gamma^{a}_{aZ}\Gamma^{Z}_{dc}\right) - \left(\Gamma^{T}_{dz}\Gamma^{Z}_{Tc} + \Gamma^{z}_{Td}\Gamma^{T}_{cZ}\right) \\
&\quad - \left(\Gamma^{T}_{da}\Gamma^{a}_{Tc} + \Gamma^{a}_{Td}\Gamma^{T}_{ac}\right) - \left(\Gamma^{Z}_{da}\Gamma^{a}_{cZ} + \Gamma^{a}_{dZ}\Gamma^{Z}_{ac}\right).
\end{aligned}
$$

We write

$$T(\Gamma^T_{dc}) + \Gamma^T_{TT}\Gamma^T_{dc} + \Gamma^T_{dc}\Gamma^Z_{TZ} = T(\Gamma^T_{dc}) + \Gamma^T_{dc}\frac{T(n)}{n} + \Gamma^T_{dc}\, g_{ZZ}^{-1/2}\, T(g_{ZZ}^{1/2})$$
$$= n^{-1}(g_{ZZ})^{-1/2} T\left(n\, g_{ZZ}^{1/2}\Gamma^T_{dc}\right)$$

and, similarly,

$$Z(\Gamma^Z_{dc}) + \Gamma^Z_{ZZ}\Gamma^Z_{dc} + \Gamma^Z_{dc}\Gamma^T_{TZ} = Z(\Gamma^z_{dc}) + \Gamma^Z_{dc}\frac{Z(n)}{n} + \Gamma^Z_{dc}\, g_{ZZ}^{-1/2}\, Z(g_{ZZ}^{1/2})$$
$$= n^{-1}(g_{ZZ})^{-1/2} Z\left(n\, g_{ZZ}^{1/2}\Gamma^Z_{dc}\right)$$

and we introduce the weight $n\, g_{ZZ}^{1/2}$ in the expression of R_{cd}

$$\begin{aligned} R^{(w)}_{cd} &:= n\, g_{ZZ}^{1/2} R_{cd} \\ &= T\left(n\, g_{ZZ}^{1/2}\Gamma^T_{dc}\right) + Z\left(n\, g_{ZZ}^{1/2}\Gamma^Z_{dc}\right) \\ &\quad + n\, g_{ZZ}^{1/2}\Big(\Gamma^a_{Ta}\Gamma^T_{dc} + \Gamma^a_{aZ}\Gamma^Z_{dc} - \left(\Gamma^T_{dZ}\Gamma^Z_{Tc} + \Gamma^Z_{Td}\Gamma^T_{cZ}\right) \\ &\qquad - \left(\Gamma^T_{da}\Gamma^a_{Tc} + \Gamma^a_{Td}\Gamma^T_{ac}\right) - \left(\Gamma^Z_{da}\Gamma^a_{cZ} + \Gamma^a_{dZ}\Gamma^Z_{ac}\right)\Big) \\ &\in W^{-1,2}_{\mathrm{loc}}(\mathcal{M}) + L^1(\Sigma_t) \end{aligned}$$

Section 1.4 TWIST COEFFICIENTS

Regular case $\mathcal{E}_{\alpha\beta\gamma\delta}$ being the volume form of $(\mathcal{M}, g)$

$$C_X := \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta \qquad C_Y := \mathcal{E}_{\alpha\beta\gamma\delta} Y^\alpha Y^\beta \nabla^\gamma X^\delta$$

Under weak regularity and within an adapted frame (T, X, Y, Z):

$$C_X := \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{X\rho}, \qquad C_Y := \mathcal{E}_{\alpha\beta\gamma\delta} Y^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{Y\rho} \in L^\infty_{\text{loc}}(L^1(T^3))$$

involving only products $L^\infty(T^3)L^1(T^3)$ or $L^2(T^3)L^2(T^3)$.

Constant twist property

- The twist coefficients of any weakly regular T^2–symmetric, Ricci-flat spacetime are constants.
- Furthermore, one can always choose the Killing fields X, Y in such a way that, one of them vanishes identically.

Special case of interest. T^2 symmetry with vanishing twists (Gowdy symmetry)

Proof. From $\nabla_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$ and the anti–symmetry of the volume form:

$$C_X = \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta = \mathcal{E}_{XYTZ} g^{TT} \Gamma^Z_{TX} + \mathcal{E}_{XYZT}\, g^{ZZ} \Gamma^T_{ZX}.$$

In view of the relations $\Gamma^Z_{TX} = n^2 g^{ZZ} \Gamma^T_{ZX}$ and $g^{TT} n^2 = 1$:

$$C_X = 2\mathcal{E}_{XYZT} g^{ZZ} \Gamma^T_{ZX}$$

and, thanks to $\mathcal{E}_{XYZT} = -\sqrt{n^2 g_{ZZ} R^2}$ and $\rho^2 = \frac{g_{ZZ}}{n^2}$, we find:

$$C_X = -2\frac{R}{\rho} \Gamma^T_{XZ} \in L^\infty_{\mathrm{loc}}(\mathcal{M})$$

- We claim that C_X **is a constant.** *(See details next page.)*
  - From the evolution equation $R_{XZ} = 0$, we obtain $T(C_X) = 0$.
  - From the constraint equation $R_{TX} = 0$, we obtain $Z(C_X) = 0$.
- Same property for C_Y.

One of the twists can be made to vanish:

- Introduce a linear combination $X' = a\,X + b\,Y$ and $Y' = c\,X + d\,Y$
- $ad - bc = 1$ to preserve the area of the T^2 orbits.
- The conclusion follows from

$$C_{X'} = \mathcal{E}_{\alpha\beta\gamma\delta} X'^\alpha Y'^\beta \nabla^\gamma X'^\delta = (ad - bc)\Big(a\,C_X + b\,C_Y\Big).$$

Variation in time: $T(C_X)$**.** From $R_{XZ} = 0$ in the weak sense and the relations $\Gamma^a_{Ta} = -ng^{ab}K_{ab} = -n\mathrm{Tr}^{(2)}(K)$ and $\Gamma^T_{TT} - \Gamma^Z_{TZ} = \frac{1}{2n^2} g_{ZZ} T(\rho^{-2})$ and $\rho^2 n^2 = g_{ZZ}$, we deduce

$$
\begin{aligned}
0 &= T(\Gamma^T_{XZ}) + \Gamma^T_{XZ}\left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right) + \Gamma^b_{Tb}\Gamma^T_{XZ} \\
&= T(\Gamma^T_{XZ}) + \Gamma^T_{XZ}\left(\frac{\rho^2}{2} T(\rho^{-2}) - n\mathrm{Tr}^{(2)}(K)\right) \\
&= T(\Gamma^T_{XZ}) - \Gamma^T_{XZ}\left(\frac{T(\rho)}{\rho} + n\mathrm{Tr}^{(2)}(K)\right)
\end{aligned}
$$

where, with $\sqrt{h} = Rh^{1/2}_{ZZ}$,

$$
\mathrm{Tr}^{(2)}(K) = -\frac{1}{n} T(\ln\sqrt{h}) - K^Z_Z = -\frac{1}{n} T(\ln R) - \frac{1}{n} T(\ln h^{1/2}_{ZZ}) - K^Z_Z
$$

in which the latter term vanishes.

This yields (recalling that $\rho \in W^{2,1}(S^1)$ and $T(\rho) \in W^{1,1}(S^1)$)

$$
0 = T(\Gamma^T_{XZ}) + \Gamma^T_{XZ}\left(-\frac{T(\rho)}{\rho} + \frac{T(R)}{R}\right)
$$

Consequently, the twist function $C_X = -2\frac{R}{\rho}\Gamma^T_{XZ}$ is **constant in time.**

Variation in space: $Z(C_X)$**.** Similarly, from the momentum constraint $R_{XT} = 0$.

Section 2. FORMULATION IN ADMISSIBLE COORDINATES

Section 2.1 REGULARITY IN COORDINATES

Let $(\Sigma \simeq T^3, h)$ be a weakly regular T^2–symmetric Riemannian manifold.

- Functions on T^3 that are invariant by the T^2 action are identified with functions on the circle $S^1 \simeq [0, 2\pi] \ni \theta$.
- **Adapted frame.** (X, Y, Z) with Z being the orthogonal projection on $\{X, Y\}^\perp$ of some vector field Θ commuting with X, Y
- **Admissible coordinates.** (x, y, θ) such that (X, Y, Θ) is the basis of vector fields induced by coordinates (x, y, θ)

Weakly regular T2-symmetric Riemannian manifold in admissible coordinates

The metric h takes the form (6 coefficients)

$$h = \frac{e^{2\overline{\nu}-2\overline{P}}}{\overline{R}} d\theta^2 + e^{2\overline{P}}\overline{R}\Big(\big(\overline{G}+\overline{A}\,\overline{H}\big)\, d\theta + dx + \overline{A}\, dy\Big)^2 + e^{-2\overline{P}}\,\overline{R}\,\Big(\overline{H}\, d\theta + dy\Big)^2$$

$\overline{R}$, $\overline{P}, \overline{A}, \overline{\nu}, \overline{G}, \overline{H}$ depend on $\theta \in S^1$

Regularity

- $\overline{P}, \overline{A} \in H^1(S^1)$ $\qquad$ $\overline{\nu} \in W^{1,1}(S^1)$
- $\overline{G}, \overline{H} \in L^\infty(S^1)$ $\qquad$ $\overline{R} \in W^{1,\infty}(S^1)$ bounded below > 0

Proof. Any metric can be expressed in this form by solving the system $h_{XX} =: e^{2\overline{P}}\overline{R}$, $h_{XY} =: \overline{R}e^{2\overline{P}}\overline{A}, \ldots$

Since the metric is T^2–symmetric, all coefficients are independent of (x, y).

- **Area function.** We have $\overline{R} \in W^{1,\infty}(T^3)$ and, after identifying $\overline{R}$ with a function on S^1, we get $\overline{R} \in W^{1,\infty}(S^1)$.
- **Component $h_{XX} \in H^1(T^3)$.**
  - Since $e^{2\overline{P}}\overline{R} = h_{XX} \in H^1(T^3)$, we obtain $e^{2\overline{P}} \in H^1(T^3)$, and after identifying $e^{2\overline{P}}$ with a function on S^1, we have $e^{2\overline{P}} \in H^1(S^1)$.
  - Since $e^{2\overline{P}} \in C^0(S^1)$ is positive and defined on the compact set S^1, we have $e^{-2\overline{P}} \in C^0(S^1)$.
  - It follows that $\overline{P}_\theta = \frac{1}{2}e^{-2\overline{P}}\left(e^{2\overline{P}}\right)_\theta$ belongs to $L^2(S^1)$.
  - Therefore $\overline{P} \in H^1(S^1)$.
- **Component $h_{YY} \in H^1(T^3)$.** A similar argument yields $\overline{A} \in H^1(S^1)$.

- **Component** $h_{XZ} = e^{2\overline{P}}\overline{R}(\overline{G} + \overline{A}\,\overline{H}) \in L^\infty(T^3)$, in which $e^{2\overline{P}}$ and $\overline{R}$ are bounded away from zero, therefore

$$(\overline{G} + \overline{A}\,\overline{H}) \in L^\infty(S^1)$$

- **Component** $h_{YZ} = \overline{R}e^{2\overline{P}}\overline{A}(\overline{G} + \overline{A}\,\overline{H}) + e^{-2\overline{P}}\,\overline{R}\,\overline{H} \in L^\infty(T^3)$
  - Therefore $e^{-2\overline{P}}\overline{R}\,\overline{H} \in L^\infty(S^1)$.
  - Using the lower bound on $\overline{R}$, we get $\frac{e^{2\overline{P}}}{\overline{R}} \in L^\infty(S^1)$.
  - Thus $\overline{H} \in L^\infty(S^1)$
  - It follows that $\overline{G} \in L^\infty(S^1)$.

- **Component** $h_{ZZ} \in W^{1,1}(T^3)$. From $Z = \Theta + \widetilde{a}X + \widetilde{b}Y$ and the identity

$$h_{ZZ} = h_{\theta\theta} - (\widetilde{a})^2 h_{XX} - 2\widetilde{a}\widetilde{b}h_{XY} - (\widetilde{b})^2 h_{YY} = \frac{e^{2\overline{\nu}-2\overline{P}}}{\overline{R}}$$

we obtain a control of $e^{2\overline{\nu}}$ and, specifically, $\overline{\nu} \in W^{1,1}(S^1)$.

(Σ, h, K): a weakly regular T^2–symmetric triple in admissible coordinates.

Weakly regular tensor fields in admissible coordinates

There exist 6 functions $\underset{0}{\overline{P}}$, $\underset{0}{\overline{A}}$, $\underset{0}{\overline{G}}$, $\underset{0}{\overline{H}}$, $\underset{0}{\overline{R}}$, $\underset{0}{\overline{\nu}}$ defined on S^1 such that

$$K_{ab} = -\frac{\overline{R}^{1/2}}{2} e^{-\overline{\nu}+\overline{P}} \underset{0}{h_{ab}} \qquad \underset{0}{h_{ab}} = \frac{h_{ab}}{\overline{R}} \underset{0}{\overline{R}} + \overline{R} \underset{0}{F_{ab}}$$

$$K_{XZ} = -\frac{1}{2} e^{-\overline{\nu}+3\overline{P}} \big(\underset{0}{\overline{G}} + \overline{A}\, \underset{0}{\overline{H}}\big)$$

$$K_{YZ} = -\frac{1}{2} e^{-\overline{\nu}+\overline{P}} \big(\overline{R}\big)^2 e^{-2\overline{P}} \underset{0}{\overline{H}} + \overline{A}\, K_{XZ}$$

$$K_{ZZ} = -\frac{e^{\overline{\nu}-\overline{P}}}{\overline{R}^{1/2}} \Big(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - \frac{1}{2\overline{R}} \underset{0}{\overline{R}}\Big)$$

$$\underset{0}{F_{ab}} e^{(a)} e^{(b)} = e^{2P} 2\underset{0}{\overline{P}} \big(dx + \overline{A}\, dy\big)^2 - 2\underset{0}{\overline{P}} e^{-2\overline{P}} dy^2 + e^{2\overline{P}} \big(2\underset{0}{\overline{A}}\, dxdy + 2\overline{A}\, \underset{0}{\overline{A}} dy^2\big)$$

Regularity

$$\underset{0}{\overline{P}}, \underset{0}{\overline{A}} \in L^2(S^1) \qquad \underset{0}{\overline{G}}, \underset{0}{\overline{H}} \in L^2(S^1)$$

$$\underset{0}{\overline{R}} \in L^\infty(S^1) \qquad \underset{0}{\overline{\nu}} \in L^1(S^1)$$

Notation: $R_t = n\overline{R}^{1/2} e^{-\overline{\nu}+\overline{P}} \underset{0}{\overline{R}}$ Observe that $\mathrm{Tr}^{(2)} K = -\frac{e^{-\overline{\nu}+\overline{P}}}{\overline{R}^{1/2}} \underset{0}{\overline{R}}$

Let $(\mathcal{M}, g)$ be a weakly regular T^2–symmetric spacetime and (t, x, y, θ) be admissible coordinates.

Weakly regular Lorentzian manifolds in admissible coordinates

Metric in admissible coordinates

$$g = -\,n^2\,dt^2 + \frac{e^{2\nu - 2P}}{R}\,d\theta^2$$

$$+\,e^{2P} R\Big((G + A\,H)\,d\theta + dx + A\,dy\Big)^2 + e^{-2P} R\big(H\,d\theta + dy\big)^2$$

with coefficients R, P, A, ν, G, H depending on $t \in I$ and $\theta \in S^1$:

- $P, A \in L^\infty_{\rm loc}(I, H^1(S^1))$ $\qquad n, \nu \in L^\infty_{\rm loc}(I, W^{1,1}(S^1))$
- $G, H \in L^\infty_{\rm loc}(I, L^\infty(S^1))$
- $R \in L^\infty_{\rm loc}(I, W^{1,\infty}(S^1))$ bounded below > 0

Timelike regularity in admissible coordinates:

$$P_t, A_t \in L^\infty_{\rm loc}(I, L^2(S^1)) \qquad R_t \in L^\infty_{\rm loc}(I, L^\infty(S^1))$$

$$\nu_t \in L^\infty_{\rm loc}(I, L^1(S^1)) \qquad G_t, H_t \in L^\infty_{\rm loc}(I, L^2(S^1))$$

Proof.

$L^1(S^1) \ni 2n\,K_{ZZ} = -\left(e^{2\nu-2P}R^{-1} + Re^{2P}(G+AH)^2 + e^{-2P}H^2R\right)_t$ while all other components are in $L^2(S^1)$

$$2n\,K_{XZ} = -\left(e^{2P}R(G+AH)\right)_t$$

$$2n\,K_{YZ} = -\left(e^{2P}RA\,(G+AH) + e^{-2P}R\,H\right)_t$$

$$2n\,K_{XX} = -\left(e^{2P}R\right)_t \qquad 2n\,K_{XY} = -\left(e^{2P}RA\right)_t$$

$$2n\,K_{YY} = \left(e^{2P}RA^2 + Re^{-2P}\right)_t$$

$$L^\infty(S^1) \ni \mathrm{Tr}^{(2)}(K)$$

$$\sim \left(e^{2P}A^2R^{-1} + e^{-2P}R^{-1}\right)K_{XX} - 2\,Ae^{2P}R^{-1}K_{XY} + e^{2P}R^{-1}K_{YY}$$

Consequently:

- **Components** $K_{XX}, K_{XY}, K_{YY} \in L^2(T^3)$**:** $\left(e^{2P}R\right)_t$ and $\left(e^{-2P}R\right)_t$ and $A_t \in L^2(S^1)$, therefore $P_t, A_t \in L^2(S^1)$
- **Trace regularity** $\quad \mathrm{Tr}^{(2)}(K) \sim e^{-2P}R_t \in L^\infty(S^1)$
- **Components** K_{XZ} **and** $K_{YZ} \in L^2(T^3)$ $\quad G_t, H_t \in L^2(S^1)$
- **Component** $K_{ZZ} \in L^1(T^3)$ $\quad \nu_t \in L^1(S^1)$

Let (Σ, h, K) be a weakly regular T^2–symmetric triple.

THE WEIGHTED SCALAR CURVATURE

$$\begin{aligned} R^{(w)(3)} &= 2R^{(3)}_{ZZ} + h_{ZZ}\left(|\chi|^2 - tr(\chi)^2\right) \\ &= -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - 2\Gamma^a_{bZ}\Gamma^b_{aZ} + h_{ZZ}\left(|\chi|^2 - tr(\chi)^2\right) \\ &\in W^{-1,\infty}(S^1) \end{aligned}$$

Observe that

$$\begin{aligned} \chi_{ab} &= h^{-1/2}_{ZZ} g\left(Z, \nabla_{e_a} e_b\right) \\ &= h^{1/2}_{ZZ}\Gamma^Z_{ab} = -\frac{1}{2}\left(h^{ZZ}\right)^{1/2} Z(h_{ab}) \in L^2(S^1) \end{aligned}$$

thus

$$\begin{aligned} R^{(w)(3)} = & -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - 2\frac{1}{2}h^{ac}Z(h_{cb})\frac{1}{2}h^{bd}Z(h_{da}) \\ & + h_{ZZ}\left(\frac{1}{2}\left(h^{ZZ}\right)^{1/2} Z(h_{ab})\frac{1}{2}\left(h^{ZZ}\right)^{1/2} Z(h_{cd})h^{ac}h^{bd}\right. \\ & \left. - \left(\frac{1}{2}\left(h^{ZZ}\right)^{1/2} Z(h_{ab})h^{ab}\right)^2\right). \end{aligned}$$

Hence, we obtain

$$R^{(w)(3)} = -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - \frac{1}{4}Z(h_{cb})Z(h_{da})h^{ac}h^{bd} - \frac{1}{4}\left(Z(h_{ab})h^{ab}\right)^2 .$$

Using the identity for the variation of the area $R^2 = det(h_{ab})$

$$\frac{1}{2}h^{ab}Z(h_{ab}) = Z(\ln R) = \Gamma^a_{aZ} = -h^{1/2}_{ZZ}\mathrm{Tr}^{(2)}\chi$$

and $\Gamma^Z_{ZZ} = \frac{1}{2}\, h^{ZZ}\, Z(h_{ZZ})$, we find

$$\begin{aligned} R^{(w)(3)} =& -2Z\left(Z(\ln R)\right) + 2Z(\ln R)\Big(-\frac{R_\theta}{2R} + \nu_\theta - P_\theta\Big) - \left(Z(\ln R)\right)^2 \\ & - \frac{1}{4}Z(h_{cb})Z(h_{ad})h^{ac}h^{bd} \end{aligned}$$

and, since there is no dependency in x, y,

$$\begin{aligned} R^{(w)(3)} =& -2\left(\frac{R_\theta}{R}\right)_\theta + 2\frac{R_\theta}{R}\left(\nu_\theta - P_\theta\right) - 2\left(\frac{R_\theta}{R}\right)^2 - \frac{1}{4}h^{ab}h^{cd}Z(h_{ad})Z(h_{bc}) \\ & \in W^{-1,\infty}(S^1) \end{aligned}$$

VARIATIONS OF THE 2-METRIC. Second fund. form of the T^2-orbits

To evaluate $h^{ab}h^{cd}Z(h_{ad})Z(h_{bc}) \in L^1(S^1)$ we decompose h_{ab} in the form $h_{ab} = RF_{ab}$ with $det(F_{ab}) = 1$:

$$\begin{aligned} Z(h_{cb})Z(h_{da})h^{ac}h^{bd} &= Z(RF)(RF)^{-1} \cdot Z(RF)(RF)^{-1} \\ &= Z(F)F^{-1} \cdot Z(F)F^{-1} + 2\,Z(F)\frac{R_\theta}{R}F^{-1} + \frac{2}{R^2}\left(Z(R)\right)^2 \\ &= Z(F)F^{-1} \cdot Z(F)F^{-1} + \frac{2}{R^2}\left(Z(R)\right)^2 \end{aligned}$$

by using $\mathrm{Tr}^{(2)}\left(Z(F)F^{-1}\right) = 0$ (since F has constant determinant).

- Observe that $F = \begin{pmatrix} e^{2P} & Ae^{2P} \\ Ae^{2P} & A^2e^{2P} + e^{-2P} \end{pmatrix}$
- A straightforward computation gives

$$\tfrac{1}{4}Z(F)F^{-1} \cdot Z(F)F^{-1} = 2P_\theta^2 + \tfrac{1}{2}A_\theta^2 e^{4P}$$

$$R^{(w)(3)} = -2\left(\frac{R_\theta}{R}\right)_\theta + 2\frac{R_\theta}{R}\left(\nu_\theta - P_\theta\right) - \frac{5}{2}\left(\frac{R_\theta}{R}\right)^2 - 2P_\theta^2 - \frac{1}{2}A_\theta^2 e^{4P} \in W^{-1,\infty}_{\mathrm{loc}}(S^1)$$

DERIVATION OF THE HAMILTONIAN EQUATION. Need to compute the contribution of the second fundamental form of the 3-slices

$$\begin{aligned}
h_{ZZ}\left((\mathrm{Tr}\,K)^2 - |K|^2\right) &= h_{ZZ}\Big(\mathrm{Tr}^{(2)}K + K_{ZZ}h^{ZZ}\Big)^2 - h_{ZZ}|K|^2 \\
&= \frac{R_0^2}{R^2} + (K_{ZZ})^2\, h^{ZZ} + 2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{\overline{R}}}\,\underset{0}{\overline{R}} \\
&\quad - (K_{ZZ})^2\, h^{ZZ} - 2h_{ZZ}\, K_{aZ}K^{Za} - h_{ZZ}\, K_{ab}K^{ab} \\
&= \frac{R_0^2}{R^2} + 2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{\overline{R}}}\,\underset{0}{\overline{R}} - 2h_{ZZ}K_{aZ}K^{Za} - h_{ZZ}\, K_{ab}K^{ab}
\end{aligned}$$

By the definition of $\underset{0}{\overline{G}}$ and $\underset{0}{\overline{H}}$, we have

$$\begin{aligned}
K_{aZ}K^{Za} &= K_{aZ}K_{bZ}h^{ab} = K_{ZX}^2 h^{XX} + K_{ZY}h^{YY} + 2K_{ZX}K_{ZY}h^{XY} \\
&= \frac{1}{4}e^{-2\overline{\nu}+4\overline{P}}R\left(\underset{0}{\overline{G}} + \overline{A\underset{0}{H}}\right)^2 + \frac{1}{4}e^{-2\nu}\left(\overline{R}\right)^2\overline{\underset{0}{H}}^2
\end{aligned}$$

One has $2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{R}}\underset{0}{\overline{R}} = 2\frac{1}{\overline{R}}\underset{0}{\overline{R}}\left(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - \frac{1}{2\overline{R}}\underset{0}{\overline{R}}\right)$

- Recall that $\underset{0}{F_{ab}}$ is defined by $\underset{0}{h_{ab}} = \frac{1}{\overline{R}}\underset{0}{\overline{R}}\, h_{ab} + \overline{R}\,\underset{0}{F_{ab}}$, and observe that its trace vanishes: $\underset{0}{F_{ab}} h^{ab} = 0$.

$$h_{ZZ}K_{ab}K^{ab} = \frac{1}{2}\left(\frac{1}{\overline{R}}\underset{0}{\overline{R}}\right)^2 + \frac{\left(\overline{R}\right)^2}{4}\underset{0}{F_{ab}}\underset{0}{F_{cd}}h^{ad}h^{bd}$$
$$= \frac{1}{2}\left(\frac{1}{\overline{R}}\underset{0}{\overline{R}}^2\right)^2 + 2\underset{0}{\overline{P}}^2 + \frac{1}{2}\underset{0}{\overline{A}}^2 e^{4P}$$

By collecting all the terms, we reach the following conclusion:

Weak version of the Hamiltonian constraint in admissible coordinates

$$0 = \left(\frac{\overline{R}_\theta}{\overline{R}}\right)_\theta - \frac{\overline{R}_\theta}{R}\left(\overline{\nu}_\theta - \overline{P}_\theta\right) + \frac{5}{4}\left(\frac{\overline{R}_\theta}{R}\right)^2 + \frac{1}{4\left(\overline{R}\right)^2}\left(\underset{0}{\overline{R}}\right)^2$$
$$- \frac{1}{\overline{R}}\underset{0}{\overline{R}}\left(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}}\right) + \left(\overline{P}_\theta^2 + \underset{0}{\overline{P}}^2\right) + \frac{1}{4}\left(\overline{A}_\theta^2 + \underset{0}{\overline{A}}^2\right)e^{4\overline{P}}$$
$$+ \frac{1}{4}e^{-2\overline{\nu}+4\overline{P}}\overline{R}\left(\underset{0}{\overline{G}} + \underset{0}{\overline{AH}}\right)^2 + \frac{1}{4}e^{-2\overline{\nu}}\left(\overline{R}\right)^2\underset{0}{\overline{H}}^2 \in W^{-1,\infty}_{\mathrm{loc}}(S^1)$$

Additional regularity: $\overline{R}_{\theta\theta} \in L^1(\Sigma)$

DERIVATION OF THE MOMENTUM CONSTRAINTS IN THE DIRECTION OF THE T^2 ORBITS

Our geometric formulation is equivalent to

$$Z\left(h_{ZZ}^{1/2}\left(\overline{R}\right)^{-1}K_a^Z\right) = 0$$

and by using the decomposition of K, we arrive at:

Weak version of the twist equations in admissible coordinates

From the components $R_{TX} = R_{TY} = 0$ we obtain

$$\left(\overline{R}e^{4\overline{U}-2\overline{\nu}}\left(\underset{0}{\overline{G}} + \overline{A}\underset{0}{\overline{H}}\right)\right)_\theta = 0$$

$$\left(\overline{R}^3 e^{-2\overline{\nu}}\underset{0}{\overline{H}} + \overline{A}\,\overline{R}e^{4\overline{U}-2\overline{\nu}}\left(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}}\right)\right)_\theta = 0$$

DERIVATION of the MOMENTUM CONSTRAINT in the Z-DIRECTION

$$Z(\mathrm{Tr}^{(2)}K) - \Gamma^a_{aZ}K^Z_Z - \Gamma^a_{bZ}K^b_a = 0$$
$$\in W^{-1,\infty}(\Sigma) + L^\infty(\Sigma)L^1(\Sigma) + L^2(\Sigma)L^2(\Sigma)$$

For the first term:

$$-Z(\mathrm{Tr}^{(2)}K) = -Z\left(e^{-\overline{\nu}+\overline{U}}\underset{0}{\overline{R}}\,(\overline{R})^{-1}\right)$$
$$= -Z(e^{-\overline{\nu}}\underset{0}{\overline{R}})e^{U}(\overline{R})^{-1} + \underset{0}{\overline{R}}\,(\overline{R})^{-2}\overline{R}_\theta e^{-\overline{\nu}+\overline{U}} - \underset{0}{\overline{R}}\,(\overline{R})^{-1}(-\overline{\nu}_\theta - \overline{U}_\theta)e^{-\overline{\nu}+\overline{U}}$$
$$= -Z\left(e^{-\overline{\nu}}\underset{0}{\overline{R}}(\overline{R})^{-1/2}\right)e^{\overline{P}} - Z(\overline{P})e^{-\overline{\nu}+\overline{P}}\underset{0}{\overline{R}}\,(\overline{R})^{-1/2}$$

For the second term:

$$-h^{1/2}_{ZZ}tr\chi K^Z_Z = -\overline{R}_\theta(\overline{R})^{-1}K^Z_Z$$
$$= \overline{R}_\theta R^{-1/2}e^{-\overline{\nu}+\overline{P}}\left(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - 2\underset{0}{\overline{R}}(2\overline{R})^{-1}\right)$$

For the last term:
$-\Gamma^a_{bZ}K^b_a = -\frac{1}{2}h^{ac}Z(h_{bc})h^{bd}K_{ad} = -\frac{1}{4}e^{-\overline{\nu}+\overline{U}}h^{ac}h^{bd}Z(h_{bc})\underset{0}{h_{bd}}$
Using the fact that the traces of $\underset{0}{F_{ab}}$ and $Z(F)$ vanish:

$$-\Gamma^a_{bZ}K^b_a = -\frac{1}{2}e^{-\nu+P}(\overline{R})^{-3/2}\underset{0}{\overline{R}}\,\overline{R}_\theta - \frac{1}{4}e^{-\nu+P}\,\overline{R}^{1/2}h^{bd}h^{ac}Z(F_{bc})\underset{0}{F_{ad}}$$

Finally, $-\frac{1}{4}e^{-\nu+P}\overline{R}^{1/2}h^{bd}h^{ac}Z(F_{bc})\underset{0}{F_{ad}} = -R^{1/2}e^{-\overline{\nu}+\overline{P}}\left(2\underset{0}{\overline{P}}\,\overline{P}_\theta + \frac{1}{2}\underset{0}{\overline{A}}\,\overline{A}_\theta e^{4\overline{P}}\right)$
We have reached the following conclusion:

Remaining momentum equation in admissible coordinates

$$0 = (\underset{0}{\overline{R}})_\theta - (\overline{\nu}_\theta - \overline{P}_\theta)\underset{0}{\overline{R}} - (\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}})\overline{R}_\theta + \frac{1}{2\overline{R}}\overline{R}_\theta\underset{0}{\overline{R}} + \overline{R}\left(2\underset{0}{\overline{P}}\,\overline{P}_\theta + \frac{1}{2}\underset{0}{\overline{A}}\,\overline{A}_\theta e^{4\overline{P}}\right)$$
$$\in W^{-1,\infty}(S^1)$$

Additional regularity

$$(\underset{0}{\overline{R}})_\theta \in L^1(S^1)$$

Section 2.3 WEAK VERSION of the EVOLUTION EQUATIONS in ADMISSIBLE COORDINATES

Let $(\mathcal{M}, g)$ be a weakly regular T^2–symmetric Ricci-flat spacetime with admissible coordinates (t, x, y, θ).

DERIVATION FOR THE RICCI COMPONENTS $R_{aZ} = 0$.

From our expression of the twist constants, we have:

Ricci components $R_{XZ} = R_{YZ} = 0$ — Ordinary diff. equations

$$0 = \left(\rho R^2 e^{-2\nu} H_t\right)_t \qquad 0 = \left(\rho R^2 e^{-2\nu+4P}(G_t + AH_t)\right)_t$$

DERIVATION FOR THE RICCI COMPONENTS $R_{ab} = 0$.

$$0 = T\left(n\, g_{ZZ}^{1/2}\Gamma^{T}_{dc}\right) + Z\left(n\, g_{ZZ}^{1/2}\Gamma^{Z}_{dc}\right) + n\, g_{ZZ}^{1/2}\left(\Gamma^{a}_{Ta}\Gamma^{T}_{dc} + \Gamma^{a}_{aZ}\Gamma^{Z}_{dc} - \Gamma^{T}_{dZ}\Gamma^{Z}_{Tc}\right)$$
$$+ n\, g_{ZZ}^{1/2}\left(-\Gamma^{Z}_{Td}\Gamma^{T}_{cZ} - \Gamma^{T}_{da}\Gamma^{a}_{Tc} - \Gamma^{a}_{Td}\Gamma^{T}_{ac} - \Gamma^{Z}_{da}\Gamma^{a}_{cZ} - \Gamma^{a}_{dZ}\Gamma^{Z}_{ac}\right)$$

First, we note the following identities:

$$n\, g_{ZZ}^{1/2} = \rho n^2 \qquad \Gamma^{T}_{dc} = \frac{1}{2n^2}T(g_{dc}) \qquad T\left(n\, g_{ZZ}^{1/2}\Gamma^{T}_{dc}\right) = \frac{1}{2}T(\rho T(g_{dc}))$$

$$Z\left(n\, g_{ZZ}^{1/2}\Gamma^{Z}_{dc}\right) = -\frac{1}{2}Z(\rho^{-1}Z(g_{dc})) \qquad \Gamma^{a}_{Ta}\Gamma^{T}_{dc} = \frac{R_t}{R}\frac{1}{2n^2}T(g_{dc})$$

$$\Gamma^{a}_{aZ}\Gamma^{Z}_{dc} = -\frac{R_\theta}{R}\frac{1}{2}g^{Z,Z}Z(g_{dc}) \qquad -2\Gamma^{T}_{dZ}\Gamma^{Z}_{Tc} = -\frac{1}{2}\frac{K_d K_c}{R^2}$$

$$\Gamma^{T}_{da}\Gamma^{a}_{Tc} = \frac{1}{4n^2}T(g_{da})T(g_{bc})g^{ab} \qquad \Gamma^{Z}_{da}\Gamma^{a}_{cZ} = -\frac{1}{4\rho^2 n^2}Z(g_{da})Z(g_{bc})g^{ab}$$

where $K_d = K$ if $d = y$ and 0 otherwise.

Investigate the last two expressions: we set $g_{ab} =: RF_{ab}$

$$
\begin{aligned}
T(g_{da})T(g_{bc})g^{ab} &= 2R_t T(F_{dc}) + \frac{R_t^2}{R}F_{dc} + RT(F_{da})T(F_{bc})F^{ab} \\
&= 2\frac{R_t}{R}T(g_{dc}) - \frac{R_t^2}{R^2}g_{dc} + RT(F_{da})T(F_{bc})F^{ab}
\end{aligned}
$$

- For $d = c = x$:

$$
\begin{aligned}
T(F_{ax})T(F_{bx})F^{ab} &= \left(2P_t e^{2P}\right)^2\left(e^{-2P} + A^2 e^{2P}\right) \\
&\quad + 2(-Ae^{2P})\left(2P_t e^{2P}\right)\left(2P_t A e^{2P} + A_t e^{2P}\right) \\
&\quad + \left(2P_t A e^{2P} + A_t e^{2P}\right)^2 e^{2P} \\
&= e^{2P}\left(4P_t^2 + A_t^2 e^{4P}\right)
\end{aligned}
$$

- For $d = c = Y$:

$$
\begin{aligned}
&T(F_{ay})T(F_{by})F^{ab} \\
&= \left(A_t e^{2P} + 2P_t A e^{2P}\right)^2 \left(e^{-2P} + A^2 e^{2P}\right) \\
&\quad + 2(-Ae^{2P}) \left(-2P_t e^{-2P} + 2AA_t e^{2P} + A^2 2P_t e^{2P}\right) \left(A_t e^{2P} + 2P_t A e^{2P}\right) \\
&\quad + \left(-2P_t e^{-2P} + 2AA_t e^{2P} + A^2 2P_t e^{2P}\right)^2 e^{2P} \\
&= \left(4P_t^2 + A_t^2 e^{4P}\right) \left(e^{-2P} + A^2 e^{2P}\right)
\end{aligned}
$$

- For $d = x$ and $c = y$:

$$
\begin{aligned}
T(F_{ax})T(F_{by})F^{ab} &= \left(2P_t e^{2P}\right) \left(A_t e^{2P} + 2P_t A e^{2P}\right) \left(e^{-2P} + A^2 e^{2P}\right) \\
&\quad + \left(-Ae^{2P}\right) \left(2P - te^{2P}\right) \left(-2P_t e^{-2P} + 2AA_t e^{2P} + 2P_t A^2 e^{2P}\right) \\
&\quad + \left(-Ae^{2P} \left(A_t e^{2P} + 2P_t A^2 e^{2P}\right)^2\right. \\
&\quad + e^{2P} \left(A_t e^{2P} + 2P_t A e^{2P}\right) \left(-2P_t e^{-2P} + 2AA_t e^{2P} + 2P_t A^2 e^{2P}\right) \\
&= 4P_t^2 A e^{2P} + AA_t^2 e^{6P} = Ae^{2P} \left(4P_t^2 + A_t^2 e^{4P}\right)
\end{aligned}
$$

Similar expressions are valid for $Z(F_{ac})Z(F_{bd})F^{ab}$ by replacing the t-derivatives by θ-derivatives.

Putting everything together, for $d = c = X$ we obtain:

$$
\begin{aligned}
0 =& T(\rho T(g_{XX})) - Z(\rho^{-1} Z(g_{XX})) - \rho \frac{R_t}{R} T(g_{XX}) + \frac{1}{\rho} \frac{R_\theta}{R} Z(g_{XX}) \\
&- \rho \left(2 \frac{R_t}{R} T(g_{XX}) - \frac{R_t^2}{R^2} g_{XX} + \left(4P_t^2 + A_t^2 e^{4P}\right) g_{XX} \right) \\
&+ \frac{1}{\rho} \left(2 \frac{R_\theta}{R} g_{XX,\theta} - \frac{R_\theta^2}{R^2} g_{XX} + \left(4P_\theta^2 + A_\theta^2 e^{4P}\right) g_{XX} \right)
\end{aligned}
$$

and, finally, substituting $g_{XX} = \left(P_t + \frac{R_t}{2R}\right)$, we conclude:

Ricci components $R_{XX} = 0$ **Nonlinear wave equation for P**

$$
\begin{aligned}
0 =& \left(\rho \left(P_t + \frac{R_t}{2R} \right) \right)_t - \left(\rho^{-1} \left(P_\theta + \frac{R_\theta}{2R} \right) \right)_\theta - \rho \frac{R_t U_t}{R} \\
&+ \rho^{-1} \frac{R_\theta U_\theta}{R} - \frac{\rho}{2} e^{4P} A_t^2 - \frac{\rho^{-1}}{2} e^{4P} A_\theta^2 \\
&\in W^{-1,2}_{\text{loc}}(\mathcal{M})
\end{aligned}
$$

Similarly, we obtain:

Ricci components $R_{YY} = 0$ — Nonlinear wave equation for A

$$
\begin{aligned}
0 &= (\rho A_t)_t - \left(\rho^{-1} A_t\right)_t - \rho \frac{R_t A_t}{R} - \rho^{-1} \frac{R_\theta A_\theta}{R} \\
&\quad - 4 \left(\rho^{-1} A_\theta \left(P_\theta + \frac{R_\theta}{2R} \right) - \rho A_t \left(P_t + \frac{R_t}{2R} \right) \right) \\
&\quad \in W^{-1,2}_{\mathrm{loc}}(\mathcal{M})
\end{aligned}
$$

For the remaining component, we observe that:

$$
\begin{aligned}
& g^{cd} \left(\frac{1}{2} T\left(\rho T(g_{cd})\right) - \frac{1}{2} Z \left(\rho^{-1} Z(g_{cd}) \right) \right) \\
&= T\left(\rho \frac{R_t}{R} \right) - Z \left(\rho^{-1} \frac{R_\theta}{R} \right) \\
&\quad - \frac{1}{2} \rho g^{ad} g^{cd} T(g_{ab}) T(g_{cd}) + \frac{1}{2\rho} g^{ad} g^{cd} Z(g_{ab}) Z(g_{cd})
\end{aligned}
$$

Ricci components $R_{XY} = 0$ — Nonlinear wave equation for R

$$
\begin{aligned}
0 &= (\rho R_t)_t - \left(\rho^{-1} R_\theta\right)_\theta - \frac{1}{2R^3} \rho^{-1} e^{2\nu} K^2 \\
&\quad \in W^{-1,\infty}_{\mathrm{loc}}(\mathcal{M})
\end{aligned}
$$

DERIVATION FOR THE RICCI COMPONENT $R_{ZZ} = 0$.

$$R_{ZZ} = T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) - \Gamma^a_{bZ}\Gamma^b_{az} + \Gamma^a_{aZ}\Gamma^Z_{ZZ} + \Gamma^a_{at}\Gamma^T_{ZZ} + 2\Gamma^T_{aZ}\Gamma^a_{TZ} + \Gamma^T_{ZZ}\left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right) + \Gamma^T_{TZ}\left(\Gamma^Z_{ZZ} - \Gamma^T_{TZ}\right)$$

We evaluate sucessively each of the terms above:

$$T(\Gamma^T_{ZZ}) = T\Big(-\frac{1}{n}K_{ZZ}\Big) = T\Big(\frac{1}{2n^2}T(g_{ZZ})\Big) \qquad \in W^{-1,1}(\mathcal{M})$$

$$-Z(\Gamma^T_{TZ}) = -Z\left(\frac{1}{n}Z(n)\right) \qquad \in W^{-1,1}(\Sigma_t)$$

$$-Z(\Gamma^a_{aZ}) = -Z\left(Z(\ln R)\right) \qquad \in W^{-1,\infty}(\Sigma_t)$$

$$\Gamma^a_{at}\Gamma^T_{ZZ} = T(\ln R)\frac{1}{2n^2}T(g_{ZZ}) \qquad \in L^{\infty}(\Sigma_t)L^1(\Sigma_t)$$

The expressions of

$$\Gamma^a_{bZ}\,\Gamma^b_{aZ} \qquad \Gamma^a_{aZ}\,\Gamma^Z_{ZZ} \qquad \Gamma^T_{aZ}\,\Gamma^a_{TZ}$$

have already been computed in terms of the metric (for the derivation of the constraint equations), i.e.

$$-\Gamma^a_{bZ}\Gamma^b_{aZ} = -\frac{1}{4}Z(h_{cb})Z(h_{ad})h^{ac}h^{bd} = -2P_\theta^2 - \frac{1}{2}A_\theta^2 e^{4P}$$

$$\Gamma^a_{aZ}\Gamma^Z_{ZZ} = Z(\ln R)\left(-\frac{R_\theta}{2R} + \nu_\theta - P_\theta\right)$$

$$\Gamma^T_{aZ}\Gamma^a_{TZ} = K^a_Z K_{aZ} = \rho^2\left(\frac{1}{4}e^{-2\nu+4P}R(G_t + AH_t)^2 + \frac{1}{4}e^{-2\nu}R^2H_t^2\right)$$

$$= \frac{1}{4}R^2e^{-2\nu}K^2$$

where K denotes here the only (possibly) non-vanishing twist constant.

The last two terms in the definition of R_{ZZ}:

$$\Gamma^T_{ZZ}\left(\Gamma^T_{TT} - \Gamma^Z_{TZ}\right) = \frac{1}{4n^4}T(g_{ZZ})\,g^{ZZ}\,T(\rho^{-2})$$

$$\Gamma^T_{TZ}\left(\Gamma^Z_{ZZ} - \Gamma^T_{TZ}\right) = \frac{n}{2}g^{ZZ}Z(n)Z(\rho^2)$$

Adding all the terms together, we obtain:

$$0 = T\left(\rho^2\left(\nu_t - P_t - \frac{R_t}{2R}\right)\right) - \rho\rho_t\left(\nu_t - P_t - \frac{R_t}{2R}\right) - Z\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right)$$
$$+ \frac{\rho_\theta}{\rho}\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right) - Z\left(\frac{R_\theta}{R}\right) + \frac{R_t}{R}\rho^2\left(\nu_t - P_t - \frac{R_t}{2R}\right)$$
$$+ \frac{R_\theta}{R}\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right) - 2P_\theta^2 - \frac{1}{2}A_\theta^2 e^{4P} + \frac{1}{4}R^2 e^{-2\nu} K^2$$

Finally, we eliminate all second-order derivatives of P, R by using the constraint equations derived earlier.

Ricci components $R_{ZZ} = 0$ — Nonlinear wave equation for ν

$$0 = (\rho\nu_t)_t - (\rho^{-1}\nu_\theta)_\theta - \rho\left(P_t - \frac{R_t}{2R}\right)^2 + \rho^{-1}\left(P_\theta - \frac{R_\theta}{2R}\right)^2$$
$$- \frac{e^{4P}}{4}\left(\rho A_t^2 - \rho^{-1}A_\theta^2\right) + \frac{3}{4R^4}\rho^{-1}e^{2\nu}K^2$$
$$\in W^{-1,1}_{\text{loc}}(\mathcal{M})$$

REFERENCES FOR THIS CHAPTER

P.G. LeFloch and J. Smulevici, J. Eur. Math. Soc. (2015).

P.G. LeFloch, Lecture Notes, in preparation.

CONCLUSIONS

In admissible coordinates adapted to the symmetry, our weak version of the Einstein equations is equivalent to a coupled system:

- 4 nonlinear wave equations understood in the weak sense
- 4 ordinary differential equations

Observation

- Robust framework (spherical symmetry as well)
- Allow for matter content
- Compressible fluids
- Shock waves (Chapter IV)

OUR NEXT OBJECTIVE

Techniques of functional analysis (Chapters III and IV)

- Work with weak solutions in suitable functional spaces
- Choice of gauge ? Conformal coordinates, areal coordinates
- Local-in-time existence ?
- Global geometry of these spacetimes ?

CHAPTER III. Weakly Regular Ricci-flat Spacetimes with T2 Symmetry. The Cauchy problem

$(\mathcal{M}, g)$: a Ricci-flat, weakly regular T^2–symmetric spacetime on T^3
Exploit the gauge freedom to simplify the analysis of the Einstein equations

- Coordinate systems adapted to the problem
- Two different gauges: conformal (null coordinates), areal (geometrically-based)
- Functional analysis: energy-type norm, monotonicity (area of the T^2 orbits of symmetry), compactness property, continuation principle

OUTLINE.

- Section 1. **Conformal and areal gauges**
- Section 2. **Weakly regular Cauchy developments**
- Section 3. **Global geometry in areal coordinates**
- Section 4. **Additional properties**

Section 1. CONFORMAL and AREAL GAUGES

CONFORMAL COORDINATES for weakly regular metrics

Let (t, x, y, θ) be admissible coordinates, with $t \in I = [t_0, t_1)$ and $x, y, \theta \in S^1$.

Proposition. Existence of conformal coordinates

There exist functions $\tau, \xi : \mathcal{M} \to \mathbb{R}$ such that:

- τ, ξ depend on (t, θ) and belong to $W^{1,\infty}_{loc}(I \times S^1)$.
- In the chart (τ, ξ, x, y), the metric reads $\quad P = U + \frac{\ln R}{2}$

$$g = \frac{e^{2\nu - 2P}}{R}\left(-d\tau^2 + d\xi^2\right) + e^{2P} R \left((G + A\, H)\, d\xi + dx + A\, dy \right)^2 + e^{-2P} R \left(H\, d\xi + dy\right)^2$$

  with P, A, R, ν, G, H depending on $\tau \in J$ and $\xi \in S^1$

Furthermore:

- The hypersurface $t = t_0$ coincides with a level set $\tau = \tau_0$.
- τ, ξ, x, y determine a global chart on $\mathcal{M}$, hence define a C^∞ differential structure on $\mathcal{M}$.
- The charts (τ, ξ, x, y) and (t, θ, x, y) are $W^{1,\infty}_{\text{loc}}$ compatible (even slightly more regular), but need not be C^∞ compatible.

Proof. We focus our attention to the **quotient metric** on $\mathcal{M}/T^2$

$$\widehat{g} = \frac{e^{2\nu-2P}}{R}\left(-\rho^2\, dt^2 + d\theta^2\right) \qquad \text{with } \rho \in L^\infty_{\text{loc}}(I, W^{2,1}(S^1))$$

and establish the existence of two functions τ, ξ:

$$\widehat{g} = \frac{e^{2\hat{\nu}-2P}}{R}\left(-d\tau^2 + d\xi^2\right)$$

(the relation between ν and $\hat{\nu}$ specified below)

Objective. Construct **null coordinates** $u, v : \mathcal{M}/T^2 \to \mathbb{R}$:

- u, v depend (t, θ) and belong to $W^{1,\infty}_{\text{loc}}(I \times S^1)$.
- $u_t + \rho\, u_\theta = 0, \quad v_t - \rho\, v_\theta = 0$
- Periodicity property: $u(t, \theta + 2\pi) = u(t, \theta) - 2\pi$ and $v(t, \theta + 2\pi) = v(t, \theta) + 2\pi$
- $(t, \theta) \in I \times S^1 \mapsto (u, v)$ is a $W^{1,\infty}_{\text{loc}}$ diffeomorphism on its image.

From the null coordinates to the conformal coordinates.
$\tau := \frac{v+u}{2}$ and $\xi := \frac{v-u}{2}$ satisfy our requirements:

- ν and $\hat{\nu}$ (in the expressions of the metric) are related by writing $du = u_t dt + u_\theta d\theta$ and $dv = v_t dt + v_\theta d\theta \in L^\infty_{\text{loc}}(\mathcal{M})$
- which yields us $e^{2\nu} = e^{2\hat{\nu}} u_\theta v_\theta$.

EXISTENCE OF THE NULL COORDINATES

- Setting $I = [t_0, t_1)$, we choose the initial data for u, v at the initial time as:

$$u(t_0, \cdot) := t_0 - \theta \qquad\qquad v(t_0, \cdot) := t_0 + \theta$$

- Denote by $\overline{\theta}_\pm = \overline{\theta}_\pm(t, \theta)$ the solutions to the characteristic equations with initial conditions on S^1 imposed at the time t_0:

$$\frac{\partial \overline{\theta}_\pm}{\partial t}(t, \theta) = \pm\rho\big(t, \theta_\pm(t, \theta)\big) \qquad \overline{\theta}_\pm(t_0, \theta) = \theta$$

- Since ρ is Lipschitz continuous (in fact slightly more more regular), a standard ODE argument applies and shows the existence of solutions

$$\overline{\theta}_\pm \in W^{1,\infty}_{\text{loc}}(I \times S^1)$$

- The functions

$$\frac{\partial \overline{\theta}_\pm}{\partial \theta}(t,\theta) = \exp\left(\int_{t_0}^{t} \frac{\rho_\theta}{\rho}(t', \overline{\theta}_\pm(t',\theta))\, dt' \right) \in L^\infty_{\text{loc}}(I, L^\infty(S^1))$$

never vanish and the two maps $(t,\theta) \in I \times \mathbb{R} \mapsto (t, \theta_\pm)$ are $W^{1,\infty}$ diffeomorphisms on their image.

- Since the solutions are unique and the data are periodic:

$$\overline{\theta}_\pm(t, \theta + 2\pi) = \overline{\theta}_\pm(t,\theta) \pm 2\pi$$

- Finally, we define the null coordinate by:

$$u(t, \overline{\theta}_+(t,\theta)) := u_0(\theta) = t_0 - \theta \qquad v(t, \overline{\theta}_-(t,\theta)) := v_0(\theta) = t_0 + \theta$$

PRESERVATION OF THE REGULARITY.

$$g = \frac{e^{2\nu-2P}}{R}\left(-\rho^2\,dt^2 + d\theta^2\right) + e^{2P}R\left(\left(G + A\,H\right)d\xi + dx + A\,dy\right)^2 + e^{-2P}R\left(H\,d\xi + dy\right)^2$$

$$g = \frac{e^{2\hat{\nu}-2\hat{P}}}{\hat{R}}\left(-\,d\tau^2 + d\xi^2\right) + e^{2\hat{P}}\hat{R}\left(\left(\hat{G} + \hat{A}\,\hat{H}\right)d\xi + dx + \hat{A}\,dy\right)^2 + e^{-2\hat{P}}\hat{R}\left(\hat{H}\,d\xi + dy\right)^2$$

- $W^{1,\infty}$ coordinate transformation: $u_\theta, v_\theta, u_t, v_t \in L^\infty(S^1)$
- $\hat{P}(\tau,\xi) = P(t,\theta) \in W^{1,2}(S^1)$, same for A
- Concerning the coefficient ν
  - In fact, $W^{2,1}$ coordinate transformation: $u_\theta, v_\theta, u_t, v_t \in W^{1,1}(S^1)$
  - From $e^{2\nu} = e^{2\hat{\nu}}u_\theta v_\theta$, one can check that $\hat{\nu}$ has the same regularity as $\nu \in W^{2,1}$
  - From $u(t,\overline{\theta}_+(t,\theta)) = t_0 - \theta$, we get $u_{\theta\theta} = \left(\frac{\partial\overline{\theta}_+}{\partial\theta}\right)^{-3}\frac{\partial^2\overline{\theta}_+}{\partial\theta^2}$
  - From $\frac{\partial\overline{\theta}_+}{\partial\theta}(t,\theta) = \exp\left(\int_{t_0}^{t}\frac{\rho_\theta}{\rho}(t',\overline{\theta}_+(t',\theta))\,dt'\right)$ we get

$$\int_{S^1}\left|\frac{\partial^2\overline{\theta}_+}{\partial^2\theta}(t,\theta)\right|d\theta \lesssim (t-t_0)\int_{S^1}|\log\rho)_{\theta\theta}|\,d\theta < +\infty$$

NORMALIZATION OF THE TWIST CONSTANTS.

Under weak regularity and within an adapted frame (T, X, Y, Z):

$$C_X := \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{X\rho}, \qquad C_Y := \mathcal{E}_{\alpha\beta\gamma\delta} Y^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{Y\rho} \in L^\infty_{\text{loc}}(L^1(T^3))$$

- We proved that $C_X = -2\frac{R}{\rho}\Gamma^T_{XZ} \in L^\infty_{\text{loc}}(\mathcal{M})$ and $T(C_X) = Z(C_X) = 0$
- Normalize $C_Y = 0$ by a suitable linear combination of the Killing fields X, Y.
- Define $K := C_X$
- Moreover, we normalize K to be positive (or zero) by changing Y into $-Y$, if necessary.

Shift-type coefficients

G, H satisfy equations that are decoupled from the "essential" Einstein equations.

FIELD EQUATIONS in conformal coordinates. $P = U + \frac{\ln R}{2}$

$$g = e^{2(\nu-U)}\left(-d\tau^2+d\xi^2\right)+e^{2U}\left(dx+A\,dy+\left(G+AH\right)d\xi\right)^2+e^{-2U}R^2\left(dy+H\,d\xi\right)^2$$

Four evolution equations

$$U_{\tau\tau} - U_{\xi\xi} = \frac{R_\xi U_\xi}{R} - \frac{R_\tau U_\tau}{R} + \frac{e^{4U}}{2R^2}(A_\tau^2 - A_\xi^2) \qquad W^{-1,2}_{\text{loc}}(\mathcal{M}) \qquad \mathbf{Ric}(X,X)$$

$$A_{\tau\tau} - A_{\xi\xi} = \frac{R_\tau A_\tau}{R} - \frac{R_\xi A_\xi}{R} + 4(A_\xi U_\xi - A_\tau U_\tau) \qquad W^{-1,2}_{\text{loc}}(\mathcal{M}) \qquad \mathbf{Ric}(Y,Y)$$

$$R_{\tau\tau} - R_{\xi\xi} = \frac{e^{2\nu}}{2R^3}K^2 \qquad W^{-1,+\infty}_{\text{loc}}(\mathcal{M}) \qquad \mathbf{Ric}(X,Y)$$

$$\nu_{\tau\tau} - \nu_{\xi\xi} = U_\xi^2 - U_\tau^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 - A_\xi^2) - \frac{3e^{2\nu}}{4R^4}K^2 \qquad W^{-1,1}_{\text{loc}}(\mathcal{M}) \qquad \mathbf{Ric}(Z,Z)$$

Two (non-trivial) constraint equations

$$0 = U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R} - \frac{\nu_\tau R_\tau}{R} - \frac{\nu_\xi R_\xi}{R} + \frac{e^{2\nu}}{4R^4}K^2 \qquad L^1(S^1) \qquad \mathbf{Ric}(T,T)$$

$$0 = 2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R} - \frac{\nu_\xi R_\tau}{R} - \frac{\nu_\tau R_\xi}{R} \qquad L^1(S^1) \qquad \mathbf{Ric}(T,Z)$$

Four twist conditions

$$G_\tau + AH_\tau = 0 \qquad H_\tau = \frac{e^{2\nu}}{R^3}K \qquad \mathbf{Ric}(T,X)\ \&\ \mathbf{Ric}(X,Z)$$

$$K_\xi = 0, \qquad K_\tau = 0 \qquad \mathbf{Ric}(T,Y)\ \&\ \mathbf{Ric}(Y,Z)$$

AREAL COORDINATES for weakly regular metrics

Essential issue: choice of the time function. Here, the area of the orbits of symmetry provides a comon time for a family of observers.

- **Existence.** ∇R is a timelike vector.
- **Regularity.** In these coordinates, the area function is obviously C^∞ but the conformal metric (denoted by ρ or a below) has weak regularity.

Proposition. Properties of the gradient of the area function

$(\mathcal{M}, g)$ being a Ricci-flat T^2–symmetric Lorentzian manifold and (t, θ, x, y) being admisssible coordinates.

- **Additional regularity of the area** $R = R(t, \theta)$:

$$R \in L^\infty_{\text{loc}}(I, W^{2,1}(S^1)) \quad R_t \in L^\infty_{\text{loc}}(I, W^{1,1}(S^1)) \quad R_{tt} \in L^\infty_{\text{loc}}(I, L^1(S^1))$$

- **If this manifold is not flat**, then ∇R is timelike:

$$g(\nabla R, \nabla R) < 0 \qquad \text{in } \mathcal{M}$$

Remarks.

- From the constraint $\mathbf{Ric}(T, T)$ and $\mathbf{Ric}(T, Z)$: $R_{\theta\theta}, R_{t\theta} \in L^1(S^1)$
- From the evolution component $\mathbf{Ric}(X, Y)$: R_{tt} is also $L^1(S^1)$
- This implies the local existence of areal coordinates, which will be shown to exists *globally.*

GRADIENT OF THE AREA OF THE ORBITS.

- ∇R measurable and bounded, defined only almost everywhere.
- From the proposition above, ∇R is in fact continuous in (t, θ).

Decomposition of the Einstein constraints. $\lambda^{\pm} := \pm\rho R_t + R_\theta$

$H := \nu_\theta - P_\theta + \rho(\nu_t - P_t)$

Taking the sum and the difference of two constraint equations:

$$\lambda^{\pm}_{\theta} = \lambda^{\pm} H + N, \qquad N \in L^{\infty}_{loc}(L^1(S^1)), \qquad N \geqslant 0$$

In view of the continuity and the periodicity of $\lambda^{\pm}$

$$\lambda^{\pm}(\theta) = \int_{\theta_0}^{\theta} e^{-\int_{\theta}^{\theta'} H(\theta'')\, d\theta''} N(\theta') d\theta', \qquad \theta_0 \text{ fixed}$$

- Either $\lambda^{+} = 0$ or λ^{+} never vanishes. Same conclusion for λ^{-}.
- The periodicity of R excludes the possibility that $\lambda^{+} > 0$ and $\lambda^{-} > 0$, as well as the possibility that $\lambda^{+} < 0$ and $\lambda^{-} < 0$.
- Thus, either $\lambda^{+}\lambda^{-} < 0$ and the gradient of R is timelike, or else $\lambda_{\pm} \equiv 0$ with $N \equiv 0$.
- In the latter case: $N \equiv 0$ one concludes that the spacetime is flat:
  - U and A are constant functions, while $K = 0$ and G, H are constants
  - ν satisfies the flat wave equation
  - introduce null coordinates: $g = e^{2\nu} dudv$ while $\nu = f(u) + f(v)$

FIELD EQUATIONS in areal coordinates

Let $(\mathcal{M}, g)$ be a Ricci-flat, *non-flat* weakly regular T^2-symmetric spacetime and let (R, x, y, θ) be areal admissible coordinates with $R \in I \subset (0, +\infty)$.

Observations.

- Notation η instead of ν
- Areal coordinates are admissible and the weak regularity holds.

Metric in areal coordinates

$$g = e^{2(\eta - U)}\big(-dR^2 + a^{-2}\,d\theta^2\big) + e^{2U}\big(\big(G + AH\big)\,d\theta + dx + A\,dy\big)^2 + e^{-2U}R^2\big(H\,d\theta + dy\big)^2$$

U, A, η, a, G, H being functions of $R \in I$ and $\theta \in S^1$

Regularity properties:

$$U_R, A_R, U_\theta, A_\theta \in L^\infty_{\mathrm{loc}}(I, L^2(S^1)) \qquad \eta_R, \eta_\theta \in L^\infty_{\mathrm{loc}}(I, L^1(S^1))$$

$$G, H \in L^\infty_{\mathrm{loc}}(I, L^\infty(S^1)) \qquad a \in L^\infty_{\mathrm{loc}}(I, W^{2,1}(S^1))$$

$$G_R, H_R \in L^\infty_{\mathrm{loc}}(I, L^\infty(S^1))$$

Four evolution equations for the metric coefficients U, A, η, a

$$(R\,a^{-1}U_R)_R - (R\,a\,U_\theta)_\theta = 2R\left((2R)^{-2}e^{4U}\left(a^{-1}A_R^2 - a\,A_\theta^2\right)\right) \qquad W^{-1,2}_{\mathrm{loc}}(\mathcal{M})$$

$$(R^{-1}a^{-1}A_R)_R - (R^{-1}a\,A_\theta)_\theta = e^{-2U}\left(4R^{-1}e^{2U}\left(-a^{-1}U_RA_R + a\,U_\theta A_\theta\right)\right) \qquad W^{-1,2}_{\mathrm{loc}}(\mathcal{M})$$

$$(a^{-1}\eta_R)_R - (a\,\eta_\theta)_\theta = -R^{-3/2}\left(R^{3/2}\left(a^{-1}\right)_R\right)_R + \left(-a^{-1}U_R^2 + a\,U_\theta^2\right) + (2R)^{-2}e^{4U}(a^{-1}A_R^2 - a\,A_\theta^2) \qquad W^{-1,1}_{\mathrm{loc}}(\mathcal{M})$$

$$(2\ln a)_R = -R^{-3}K^2\,e^{2\eta} \qquad W^{1,1}_{\mathrm{loc}}(\mathcal{M})$$

Two constraints

$$\eta_\theta = R\left(2U_RU_\theta + 2R^{-2}e^{2U}A_RA_\theta\right) \qquad L^1(S^1)$$

$$\eta_R + \frac{1}{4}R^{-3}e^{2\eta}K^2 = a\,R\left(\left(a^{-1}U_R^2 + a\,U_\theta^2\right) + (2R)^{-2}e^{4U}\left(a^{-1}A_R^2 + a\,A_\theta^2\right)\right) \qquad L^1(S^1)$$

Four twist conditions

$$K_R = 0 \qquad K_\theta = 0$$

$$G_R = -AKe^{2\eta}\,a^{-1}R^{-3} \qquad H_R = Ke^{2\eta}\,a^{-1}R^{-3}$$

Section 2
WEAKLY REGULAR CAUCHY DEVELOPMENTS

OUTLINE OF THE METHOD

Step 0. *Local-in-time smooth solutions from smooth initial data*

standard fixed point arguments

Step 1. *Local existence in conformal coordinates*

- compactness property
- given any weakly regular initial data set, existence of a local-in-time solution in conformal time $[\tau_0, \tau_1)$
- $\tau_1 - \tau_0$ only depends on a natural norm of the initial data

Step 2. *Local existence in areal coordinates*

- arrange that the hypersurface $\{\tau = \tau_1\}$ coincides with $\{R = R_1\}$
- R is strictly increasing with τ
- a weak solution to the conformal Einstein system can be transformed to a weak solution to the areal Einstein system
- a local solution to the equations in areal time $[R_0, R_1)$

Step 3. *Global and causal geometry in areal coordinates* (Chapter V)

- global-in-time control of the natural norms
- all values of the area $R \in (0, +\infty)$ but for some "exceptional" (flat Kasner) spacetimes
- asymptotic behavior when $R \to 0$ or $R \to +\infty$ (some open problems)

Theorem. Local theory in conformal coordinates

A weakly regular T^2–symmetric initial data set $(\Sigma \simeq T^3, h, K)$ and admissible coordinates (ξ, x, y)

- Assume the data are non-flat

$$M_0 := \inf_{S^1} |\underset{0}{\overline{R}} - \overline{R}_{\xi'}| \inf_{S^1} |\underset{0}{\overline{R}} + \overline{R}_{\xi'}| \neq 0 \text{ (time-like gradient)}$$

- Normalize the data to be future expanding: $\mathrm{Tr}^{(2)} K < 0$

Recall that $\mathrm{Tr}^{(2)} K = -e^{-\overline{\nu}+\overline{P}} \, \overline{R}^{-1/2} \underset{0}{\overline{R}}$

Then, there exists a Ricci-flat, weakly regular T^2–symmetric Lorentzian manifold $(\mathcal{M}, g)$ endowed with a global chart of admissible conformal coordinates (τ, ξ, x, y):

- $\mathcal{M} = I \times \Sigma$ with $I := [\tau_0, \tau_1)$ for some $\tau_0 < \tau_1$
- g takes the conformal form where R strictly increasing in τ
- $\tau_1 - \tau_0 > 0$ depends only the natural norm defined from the initial data set

$$N_0 := \|\overline{U}, \overline{A}\|_{H^1(S^1)} + \|\underset{0}{\overline{U}}, \underset{0}{\overline{A}}\|_{L^2(S^1)} + \|\overline{\nu}\|_{W^{1,1}(S^1)} + \|\underset{0}{\overline{\nu}}\|_{L^1(S^1)}$$
$$+ \|\overline{R}\|_{W^{2,1}(S^1)} + \|\underset{0}{\overline{R}}\|_{W^{1,1}(S^1)} + \|1/\overline{R}\|_{L^\infty(S^1)} + \frac{1}{M_0}.$$

Regularity of the metric coefficients

$U, A \in C^0_{\mathrm{loc}}(I, H^1(S^1)) \cap C^1_{\mathrm{loc}}(I, L^2(S^1))$

$\nu \in C^0_{\mathrm{loc}}(I, W^{1,1}(S^1)) \cap C^1_{\mathrm{loc}}(I, L^1(S^1))$

$R \in C^0_{\mathrm{loc}}(I, W^{2,1}(S^1)) \cap C^1_{\mathrm{loc}}(I, W^{1,1}(S^1))$

Initial data set. For the embedding $\psi : \Sigma \to \mathcal{M}$, $(\xi, x, y) \mapsto (\tau_0, \xi, x, y)$:

$$\Big(U, U_\tau, A, A_\tau, \nu, \nu_\tau, R, R_\tau, G, G_\tau, H, H_\tau\Big)(\tau_0)$$

$$= \Big(\overline{U}, \underset{0}{\overline{U}}, \overline{A}, \underset{0}{\overline{A}}, \overline{\nu}, \underset{0}{\overline{\nu}}, \overline{R}, \underset{0}{\overline{R}}, \overline{G}, \underset{0}{\overline{G}}, \overline{H}, \underset{0}{\overline{H}}\Big) \circ \psi$$

Remark. The uniqueness is proven in Section 4 below.

EXISTENCE PROOF

- Regularization of the initial data set
- A priori estimates for sufficiently regular solutions
- Compactness of the set of solutions

To proceed

- a solution (U, A, R, ν) to the Einstein equations in conformal coordinates
- defined on some interval $[\tau_0, \tau_1)$
- non-flat and with the time orientation $R_\tau > 0$

The derivation of a priori estimates relies on a bootstrap argument:

- Energy estimates for the wave equations satisfied by U, A
- Suitable norm for R
- Suitable norm for ν
- Close the argument on $[\tau_0, \tau_1)$ with $\tau_1 - \tau_0 << 1$

REGULARIZATION OF INITIAL DATA SETS

For simplicity in the presentation, we do the analysis in areal coordinates and for a slice with constant area.

Proposition. Regularization of initial data sets

Let $\overline{X} := \left(\overline{U}_0, \overline{A}_0, \overline{U}_1, \overline{A}_1, \overline{a}, \overline{\eta}_0, \overline{\eta}_1\right)$ be a weakly regular initial data set for the (essential) Einstein equations in areal coordinates. Then, there exists a sequence of smooth initial data sets

$$\overline{X}^n = (\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n, \overline{a}^n, \overline{\eta}_0^n, \overline{\eta}_1^n), \qquad n = 1, 2, \ldots$$

satisfying Einstein constraint equations and converging almost everywhere

$$\begin{aligned}
&(\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n) \to (\overline{U}_0, \overline{A}_0, \overline{U}_1, \overline{A}_1) && \text{in } L^2(S^1)\\
&\overline{a}^n \to \overline{a} && \text{in } W^{2,1}(S^1)\\
&(\overline{\eta}_0^n, \overline{\eta}_1^n) \to (\overline{\eta}_0, \overline{\eta}_1) && \text{in } L^1(S^1)
\end{aligned}$$

Remarks.

- The functions G, H do not appear in the constraint equations.
- With the same arguments: existence of weakly regular T^2–symmetric initial data sets with constant R *having precisely the weak regularity*.

Proof. Standard convolution of the data $\overline{X}$.
Pick up an arbitrary regularization $\overline{U}_0^n$, $\overline{A}_0^n$, $\overline{U}_1^n$, $\overline{A}_1^n$, $\overline{a}^n$ defined on S^1, such that:

- $\overline{U}_0^n$, $\overline{A}_0^n$, $\overline{U}_1^n$, $\overline{A}_1^n$ converge in $L^2(S^1)$ toward $\overline{U}_0$, $\overline{A}_0$, $\overline{U}_1$, $\overline{A}_1$
- $\overline{a}^n$ converges to $\overline{a}$ in $W^{2,1}(S^1)$

- It remains to **regularize** $\overline{\eta}_0$ **and** $\overline{\eta}_1$ so that the constraint equations hold. To any regularized data set $\overline{Y}^n := (\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n, \overline{a}^n)$, we associate the average

$$\Lambda[\overline{Y}^n] := \int_{S^1} \lambda[\overline{Y}^n]\, d\theta, \qquad \lambda[\overline{Y}^n] := 2R\left(\overline{U}_0^n \overline{U}_1^n + R^{-2} e^{2\overline{U}^n} \overline{A}_0^n \overline{A}_1^n\right)$$

  - $\lambda[\overline{Y}^n]$ converges in $L^1(S^1)$ toward $\overline{\eta}_1$.
  - The sequence $\Lambda[\overline{Y}^n]$ converges to 0
  ($\overline{\eta}_1$ is the spatial derivative of a periodic function on S^1.

PROVIDED the average $\Lambda[\overline{Y}^n] = 0$ vanishes:

Define $\overline{\eta}^n$. For an arbitrary $\theta_* \in S^1$

$$\overline{\eta}^n(\theta) := \eta(\theta_*) + \int_{\theta_*}^{\theta} \lambda[\overline{Y}^n]\, d\theta'$$

- $\overline{\eta}^n$ converge in $W^{1,1}(S^1)$ toward the initial data $\overline{\eta}(\theta) = \int_0^\theta \overline{\eta}_1\, d\theta$.

Define $\overline{\eta}_0^n$.

$$\overline{\eta}_0^n + \frac{e^{2\overline{\eta}^n} K^2}{4R^3} = R\overline{a}^n E[\overline{Y}^n]$$

$$E[\overline{Y}^n] := (\overline{a}^n)^{-1} (\overline{U}_0^n)^2 + \overline{a}^n (\overline{U}_1^n)^2 + (2R)^{-2} e^{4\overline{U}^n} \Big((\overline{a}^n)^{-1} (\overline{A}_0^n)^2 + \overline{a}^n (\overline{A}_1^n)^2 \Big)$$

Here, K is the twist constant of the given initial data set.
We claim that $\overline{\eta}_0^n$ converges in $L^1(S^1)$ to $\overline{\eta}_0$:

- $(\overline{a}^n)^{-1} e^{2\overline{\eta}^n} K^2 R^{-3}$ converges to $(\overline{a})^{-1} e^{2\overline{\eta}} K^2 R^{-3}$ in $L^1(S^1)$.
- Indeed, $\overline{a}^n$ converges to $\overline{a}$ in $W^{2,1}(S^1)$ and thus in $L^\infty(S^1)$,
- and, moreover, $e^{2\overline{\eta}^n}$ converges to $e^{2\overline{\eta}}$ in $L^1(S^1)$,
- as follows from the convergence of $\overline{\eta}^n$ in $W^{1,1}$ and, thus, in $L^\infty(S^1)$.

In conclusion: $\overline{\eta}^n$ and $\overline{\eta}_0^n$ satisfy Einstein's constraints.

Existence of a regularization such that $\Lambda[\overline{Y}^n] = 0$.

Start from a regularized set $\overline{Y}^n$ that may not satisfy the constraints.

- First of all, when $\int_{S^1}(\overline{U}_1)^2$ and $\int_{S^1}(\overline{A}_1)^2$ both vanish:
  - Then, $\overline{U}$ and $\overline{A}$ must be constant, say $\overline{U} = U_*$, $\overline{A} = A_*$.
  - Keeping the regularization under consideration $\overline{Y}^n_\star = (U_*, \overline{U}^n_0, A_*, \overline{A}^n_0, \overline{a}^n)$, we do obtain $\Lambda[\overline{Y}^n] = 0$.

Otherwise, assume, for instance, that $\int_{S^1}(\overline{U}_1)^2 =: c > 0$. Setting $\delta^n := -\frac{\Lambda[\overline{Y}^n]}{2R\int_{S^1}(\overline{U}^n_1)^2}$, we claim that the rescaled expression

$$\overline{Y}^n_\star := \left(\overline{U}^n, \overline{U}^n_0 + \delta^n\, \overline{U}^n_1, \overline{A}^n, \overline{A}^n_0, \overline{a}^n\right)$$

satisfies Einstein's constraints.

- It is well-defined: for all sufficiently large n we have $\int_{S^1}(\overline{U}^n_1)^2 \geqslant c/2$.
- By construction, $\Lambda[\overline{Y}^n_\star] = 0$.
- **Convergence** to the given initial data set
  - $\Lambda[\overline{Y}^n]$ converges to 0.
  - $|\delta^n| \lesssim |\Lambda[\overline{Y}^n]|$ converges to 0 when $n \to +\infty$.

UNIFORM ESTIMATES

Lemma. Lower bound for the area function

Since ∇R is time like and future-oriented, R is a strictly increasing function of τ. Setting $R_0 := \min_{S^1} R(\tau_0, \cdot)$, one has

$$R(\tau, \xi) \geqslant R_0 \text{ for all } \tau \in [\tau_0, \tau_1) \text{ and } \xi \in S^1$$

Lemma. Gradient of the area function

Since $R_{uv} = \frac{e^{2\nu}}{2R^3} K^2 \geqslant 0$, the functions $R_\tau \pm R_\xi$ are strictly increasing along the lines $\tau \pm \xi = const$, as functions of $\tau \mp \xi$, respectively.
Setting $M(\nabla R)(\tau) := \inf_{S^1} R_u(\tau, \cdot) \inf_{S^1} R_v(\tau, \cdot)$, one has

$$\inf_{S^1} \left(R_\tau^2 - R_\xi^2\right)(\tau) \geqslant M(\nabla R)(\tau) \geqslant M(\nabla R)(\tau_0)$$

$$\|R(\tau, \cdot)\|_{C^1(S^1)} \lesssim \|R(\tau_0)\|_{C^1(S^1)} + (\tau - \tau_0)\|e^{2\nu}\|_{L^\infty([\tau_0,\tau]\times S^1)}$$

where the implied constant only depends on R_0 and the twist constant K.

Proof. Notation: $\partial_u := \partial_\tau - \partial_\xi, \quad \partial_v := \partial_\tau + \partial_\xi$

- For the first claim, we use the fact that $R_\tau^2 - R_\xi^2 = R_u R_v$ and that R_u and R_v are monotonically increasing in v and u, respectively.
- The second claim is easy from the wave equation satisfied by R.

Lemma. Conformal energy estimate

$$\mathcal{E}_{\text{conf}}(\tau) := \int_{S^1} \left(R\left(U_\tau^2 + U_\xi^2\right) + \frac{e^{4U}}{4R}\left(A_\tau^2 + A_\xi^2\right) + \frac{e^{2\nu}K^2}{4R^3} \right)$$

$$\leqslant \mathcal{E}_{\text{conf}}(\tau_0)\, e^{C(R_0)\left(1+\|R\|_{C^1([\tau_0,\tau]\times S^1)}\right)(\tau-\tau_0)}$$

Proof. From the constraint equations:

$$\mathcal{E}_{\text{conf}}(\tau) = \int_{S^1} -R_{\xi\xi} - \nu_\tau R_\tau - \nu_\xi R_\xi = \int_{S^1} -\nu_\tau R_\tau - \nu_\xi R_\xi$$

and, after integrations by parts,

$$\frac{d}{d\tau}\mathcal{E}_{\text{conf}}(\tau) = \int_{S^1} -\nu_\tau(R_{\tau\tau} - R_{\xi\xi}) - R_\tau(\nu_{\tau\tau} - \nu_{\xi\xi}).$$

Using the wave equations for ν and R:

$$\frac{d}{d\tau}\mathcal{E}_{\text{conf}}(\tau) \leqslant C(R_0)\|R\|_{C^1([\tau_0,\tau]\times S^1)}\,\mathcal{E}_{\text{conf}}(\tau) - \int_{S^1} \nu_\tau e^{2\nu}\frac{K^2}{2R^3}.$$

Integrate by parts the term ν_τ and apply Gronwall's lemma:

$$-\int_{\tau_0}^{\tau}\int_{S^1} \nu_\tau e^{2\nu}\frac{K^2}{2R^3} = -\int_{S^1} e^{2\nu(\tau)}\frac{K^2}{4R(\tau)^3} + \int_{S^1} e^{2\nu(\tau_0)}\frac{K^2}{4R(\tau_0)^3} - \int_{\tau_0}^{\tau}\int_{S^1} e^{2\nu}\frac{3K^2}{4R^4}R_\tau$$

Consequence of the (non-trivial) constraint equations

$$\frac{\nu_\tau R_\tau}{R} + \frac{\nu_\xi R_\xi}{R} = U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R} + \frac{e^{2\nu}}{4R^4}K^2 \qquad L^1(S^1)$$

$$\frac{\nu_\xi R_\tau}{R} + \frac{\nu_\tau R_\xi}{R} = 2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R} \qquad L^1(S^1)$$

- Solve for ν_τ, ν_ξ in term of the energy densities.
- Factor $|R_\tau^2 - R_\xi^2|$, which needs to be bounded below.

Lemma. First-order estimate on ν

$$\|\nu_\xi(\tau,\cdot)\|_{L^1(S^1)} + \|\nu_\tau(\tau,\cdot)\|_{L^1(S^1)}$$
$$\lesssim \|R\|_{C^1([\tau_0,\tau]\times S^1)} \Big(\mathcal{E}_{\mathrm{conf}}(\tau) + \|R_{\xi\xi}(\tau,\cdot)\|_{L^1} + \|R_{\xi\tau}(\tau,\cdot)\|_{L^1}\Big),$$

where the implied constant depends on R_0 and $M(\nabla R)(\tau_0)$.

Lemma. Higher-order estimates on the area function

$$\|R_{\xi\xi}(\tau,\cdot)\|_{L^1(S^1)} \leqslant \|R_{\xi\xi}(\tau_0,\cdot)\|_{L^1(S^1)} + C(R_0)\int_{\tau_0}^{\tau}\left(\|\nu_\xi(\tau',\cdot)\|_{L^1} + \|R\|_{C^1([\tau_0,\tau']\times S^1)}\right)\|e^{2\nu(\tau',\cdot)}\|_{L^\infty(S^1)}\,d\tau'$$

$$\|R_{\xi\tau}\|_{L^1(S^1)}(\tau) \leqslant \|R_{\xi\tau}(\tau_0,\cdot)\|_{L^1(S^1)} + C(R_0)\int_{\tau_0}^{\tau}\left(\|\nu_\tau(\tau',\cdot)\|_{L^1(S^1)} + \|R\|_{C^1([\tau_0,\tau']\times S^1)}\right)\|e^{2\nu(\tau',\cdot)}\|_{L^\infty(S^1)}\,d\tau'$$

Proof. We use integration along characteristics for the function R which satisfies a wave equation

$$R_{uv} = \Omega = \frac{e^{2\nu}}{2R^3}K^2$$

- hence $R_{\xi u}(\xi, v) = R_{\xi u}(\xi, v_0) + \int_{v_0}^{v} \partial_\xi \Omega \, dv$
- similarly for the derivatives $R_{\xi v}$, $R_{\tau u}$, and $R_{\tau v}$.

Observe also

$$\|\nu\|_{L^\infty(S^1)}(\tau) \leqslant \frac{1}{2\pi}\|\nu(\tau_0)\|_{L^1(S^1)} + \|\nu_\xi(\tau)\|_{L^1(S^1)} + \frac{1}{2\pi}(\tau-\tau_0)\|\nu_\tau(\tau)\|_{L^1(S^1)}$$

$$\|R\|_{C^1([\tau_0,\tau]\times S^1)} \leqslant 2\left(\|\overline{R}\|_{C^1(S^1)} + \|\underset{0}{\overline{R}}\|_{C^0(S^1)}\right)$$

We arrive at the following conclusion for a *sufficiently small* interval of conformal time.

Proposition. A priori estimates in conformal coordinates

There exist $\delta, C_0 > 0$ depending only on the norm $N_0 = N(\tau_0)$ of the initial data such that on the interval $[\tau_0, \tau_0 + \delta]$:

$$\begin{aligned} N(\tau) :&= \|U, A\|_{H^1(S^1)} + \|U_\tau, A_\tau\|_{L^2(S^1)} + \|\nu\|_{W^{1,1}(S^1)} + \|\nu_\tau\|_{L^1(S^1)} \\ &\quad + \|R\|_{W^{2,1}(S^1)} + \|R_\tau\|_{W^{1,1}(S^1)} + \|R^{-1}\|_{L^\infty(S^1)} + M(\nabla R)^{-1} \\ &\leqslant C_0 = C(N_0) \end{aligned}$$

COMPACTNESS PROPERTY

- Let $(U^{\varepsilon_1}, A^{\varepsilon_1}, \nu^{\varepsilon_1}, R^{\varepsilon_1})$ and $(U^{\varepsilon_2}, A^{\varepsilon_2}, \nu^{\varepsilon_2}, R^{\varepsilon_2})$ be two solutions in conformal coordinates.
- Defined on a cylinder $[\tau_0, \tau_1] \times S^1$, where $\tau_1 = \tau_0 + \delta$, with δ small enough so that
$$N^i(\tau) \leqslant C^i = C(N^i(\tau_0))$$

Define $\Delta U := U^{\varepsilon_2} - U^{\varepsilon_1}$, $\Delta A := A^{\varepsilon_2} - A^{\varepsilon_1}$, ...

$$\begin{aligned} N^{\Delta}(\tau) := {} & \|\Delta U, \Delta A\|_{H^1(S^1)} + \|\Delta U_\tau, \Delta A_\tau\|_{L^2(S^1)} \\ & + \|\Delta \nu_\tau\|_{L^1(S^1)} + \|\Delta \nu, \Delta R_\xi, \Delta R_\tau\|_{W^{1,1}(S^1)} \\ & + \|\Delta R, \Delta(R^{-1})\|_{C^1(S^1)} + \|\Delta R_\tau\|_{C^0(S^1)} \end{aligned}$$

Proposition. Continuous dependence on the initial data

Provided that $\delta << 1$, one has for all $\tau \in [\tau_0, \tau_0 + \delta)$

$$N^{\Delta}(\tau_1) \leqslant C\, N^{\Delta}(\tau_0),$$

where $C > 0$ only depends on the constants $C(N^i(\tau_0))$.

Proof.

- Follows similar lines as in the derivation of the uniform bound.
- We provide a lemma, which is used repeatedly in the proof.
- Express the wave equations as first-order equations in $w_{\pm} = U_\tau \pm U_\theta$, etc.
- Extend the functions by periodicity to the real line $\mathbb{R}$.

Lemma. Higher–integrability estimate

Let $w_-, w_+ : [\tau_0, \tau_1) \times \mathbb{R} \to \mathbb{R}$ be weak solutions in $L^\infty_{\text{loc}}([\tau_0, \tau_1), L^2_{\text{loc}}(\mathbb{R}))$ to the equations $\partial_\tau w_\pm \pm \partial_\theta w_\pm = h_\pm$. with $h_\pm \in L^\infty_{\text{loc}}([\tau_0, \tau_1], L^2_{\text{loc}}(\mathbb{R}))$. Then, for each $\tau < L$, one has

$$\frac{d}{d\tau} N^I + 2\, N^{II} \leqslant N^{III}$$

$$N^I(\tau) := \int_{-L+\tau}^{L-\tau} \int_{\theta_+}^{L-\tau} |w_+(\tau, \theta_+)|^2 \, |w_-(\tau, \theta_-)|^2 \, d\theta_+ \, d\theta_-$$

$$N^{II}(\tau) := \int_{-L+\tau}^{L-\tau} |w_+(\tau, \cdot)|^2 \, |w_-(\tau, \cdot)|^2 \, d\theta$$

$$N^{III}(\tau) := \sum_{\pm} \int_{-L+\tau}^{L-\tau} \Big(|h_\pm(\tau, \cdot)|^2 + |w_\pm(\tau, \cdot)|^2 \Big) \, d\theta \int_{-L+\tau}^{L-\tau} |w_\mp(\tau, \cdot)|^2 \, d\theta$$

Corollary

$U_\tau^2 - U_\xi^2$ and $A_\tau^2 - A_\xi^2$ belong to $L^2_{\text{loc}}([\tau_0, \tau_1) \times S^1)$

Proof. By regularization, we have

$$\partial_\tau |w_\pm|^2 \pm \partial_\theta \left(|w_\pm|^2 \right) \leqslant 2\, |w_\pm|\, |h_\pm|$$

Next, from the definitions:

$$\begin{aligned}
\frac{d}{d\tau} N^I(\tau) \leqslant & \int_{-L+\tau}^{L-\tau} \int_{\theta_+}^{L-\tau} \Big(-\partial_\theta (|w_+|^2) + |h_+| \Big)(\tau, \theta_+)\, |w_-(\tau, \theta_-)|^2 \, d\theta_- d\theta_+ \\
& + \int_{-L+\tau}^{L-\tau} \int_{\theta_+}^{L-\tau} |w_+(\tau, \theta_+)|^2 \Big(\partial_\theta (|w_-|^2) + |h_-|\, |w_-| \Big)(\tau, \theta_-)\, d\theta_- d\theta_+ \\
& - \int_{-L+\tau}^{L-\tau} |w_+(\tau, -L+\tau)|^2\, |w_-(\tau, \theta)|^2 \, d\theta \\
& - \int_{-L+\tau}^{L-\tau} |w_+(\tau, \theta)|^2\, |w_-(\tau, L-\tau)|^2 \, d\theta
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{d}{d\tau} N^I(\tau) \leqslant & -2 \int_{-L+\tau}^{L-\tau} a(\tau, \theta)\, |w_+(\tau, \theta)|^2\, |w_-(\tau, \theta)|^2 \, d\theta \\
& + \int_{-L+\tau}^{L-\tau} \int_{\theta_+}^{L-\tau} \Big(|h_+(\tau, \theta_+)|\, |w_+(\tau, \theta_+)|\, |w_-(\tau, \theta_-)|^2 \\
& \qquad + |w_+(\tau, \theta_+)|^2\, |h_-(\tau, \theta_-)|\, |w_-(\tau, \theta_-)| \Big)\, d\theta_- d\theta_+
\end{aligned}$$

EXISTENCE AND CONTINUATION CRITERION

Theorem. The global geometry

Given any, non-flat, weakly regular T^2-symmetric initial data set (Σ, h, K) with topology T^3 and with constant area $R_0 > 0$ and beign future expanding $\mathrm{Tr}^{(2)}(K) < 0$.

- there exists a weakly regular, vacuum spacetime $(\mathcal{M}, g)$ with T^2–symmetry on T^3,
- which is a future Cauchy development of (Σ, h, K),
- is maximal among all weakly regular T^2-symmetric developments,
- and admits a unique global foliation by the level sets of the area $R \in [R_0, +\infty)$.

First observation:

- The conformal time of existence only depends on the initial norm N_0.
- Hence, the areal time of existence of the solution is bounded below by a constant depending only on N_0.

- We will have the the existence of global solutions in areal coordinates, provided we can derive uniform estimates for the natural norm.
- More precisely, from the previous section, we have immediately:

Proposition. Continuation criterion

Let (U, A, η, a) be a solution to the areal equations defined on an interval of time $R \in [R_0, R_1)$ and with the regularity $U, A \in C^0_R(H^1(S^1)) \cap C^1_R(L^2(S^1))$, $\eta \in C^0_R(W^{1,1}(S^1)) \cap C^1_R(L^1(S^1))$, $a, a^{-1} \in C^0_R(W^{2,1}(S^1)) \cap C^1_R(W^{1,1}(S^1))$. Assume that $R_1 < +\infty$ and

$$N(R) := \|U, A\|_{H^1(S^1)} + \|U_R, A_R\|_{L^2(S^1)} + \|\eta, a_R, a_\theta\|_{W^{1,1}(S^1)} + \|\eta_R, a_{RR}, a_{R\theta}, a_{\theta\theta}\|_{L^1(S^1)} + \|a, a^{-1}\|_{L^\infty(S^1)}$$

is uniformly bounded on $[R_0, R_*)$. Then, the solution can be extended beyond R_1 with the same regularity.

UNIFORM ESTIMATES IN AREAL COORDINATES

We now derive bounds that are uniform on the interval $[R_0, R_\star)$:

- C: constants depending on the norm $N(R_0)$ of the initial data
- $C_\star$: constants also depending on the final time $R_\star$

ENERGY FUNCTIONALS

$$\mathcal{E}(R) := \int_{S^1} E(R,\theta)\, d\theta$$

$$E := a^{-1}(U_R)^2 + a\,(U_\theta)^2 + \frac{e^{4U}}{4R^2}\left(a^{-1}(A_R)^2 + a\,(A_\theta)^2\right)$$

and

$$\mathcal{E}_K(R) := \int_{S^1} E_K(R,\theta)\, d\theta \qquad E_K := E + \frac{K^2}{4R^4}\, e^{2\eta}\, a^{-1}$$

are non-increasing:

$$\frac{d}{dR}\mathcal{E}(R) = -\frac{K^2}{2R^3}\int_{S^1} E\, e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1}\left(a^{-1}\,(U_R)^2 + \frac{1}{4R^2}\, e^{4U}\, a\,(A_\theta)^2\right) d\theta$$

$$\frac{d}{dR}\mathcal{E}_K(R) = -\frac{K^2}{R^5}\int_{S^1} a^{-1} e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1}\left(a^{-1}\,(U_R)^2 + \frac{e^{4U}}{4R^2} a\,(A_\theta)^2\right) d\theta$$

Lemma. H^1 Energy estimates in areal gauge

$$\sup_{R\in[R_0,R_\star)} \mathcal{E}(R) \leqslant \mathcal{E}(R_0) \qquad \sup_{R\in[R_0,R_\star)} \mathcal{E}_K(R) \leqslant \mathcal{E}_K(R_0)$$

$$\int_{R_0}^{\infty}\int_{S^1}\Big(c_1\,(U_R)^2\,a^{-1}+c_2\,(U_\theta)^2\,a+c_3\,(A_R)^2\,a^{-1}+c_4\,(A_\theta)^2\,a\Big)\,dRd\theta \leqslant \mathcal{E}(R_0)$$

$$\int_{R_0}^{+\infty}\int_{S^1}\frac{K^2}{R^5}\,e^{2\eta}\,a^{-1}\,dRd\theta \leqslant \mathcal{E}_K(R_0)$$

$$c_1 := \frac{2}{R}+\frac{K^2}{2R^3}\,e^{2\eta} \qquad c_2 := \frac{K^2}{2R^3}\,e^{2\eta}$$

$$c_3 := \frac{K^2}{8R^5}\,e^{4U+2\eta} \qquad c_4 := \frac{1}{2R^3}\,e^{4U}+\frac{K^2}{8R^5}\,e^{4U+2\eta}$$

Initial slice. Since the function a is bounded above and below on the initial slice $R = R_0$, the initial energy $\mathcal{E}(R_0)$ controls the H^1 norm of the data $\overline{U}, \overline{A}$:

$$\mathcal{E}(R_0) \lesssim \|(\overline{U}, \underset{0}{\overline{U}}, \overline{A}, \underset{0}{\overline{A}})\|_{L^2(S^1)} \lesssim \mathcal{E}(R_0)$$

with implied constants $C_1, C_2 > 0$ depending on the sup norm of the data at time $R = R_0$.

To have similar inequalities at arbitrary times requires a sup-norm bound.

We begin with an upper bound.

Lemma. Upper bound for the function a

$$\sup_{[R_0,R_\star)\times S^1} a \leqslant \sup_{S^1} \overline{a}$$

$$\frac{1}{2R} \int_{S^1} |(1/a)_R| \, d\theta \leqslant \mathcal{E}_K(R_0)$$

Proof. • a decreases when R increases.

• The other estimate follows from since

$$0 \leqslant -2\, a_R \, a^{-2} = \frac{K^2}{R^3} \, e^{2\eta} \, a^{-1} = 4\, R \, (E_K - E) \leqslant 4R \, E_K.$$

Lemma. $W^{1,1}$ estimates for the function η

$$\frac{1}{R}\int_{S^1} |\eta_R| \, \frac{d\theta}{a} \leqslant \mathcal{E}_K(R_0) \qquad \frac{1}{R}\int_{S^1} |\eta_\theta| \, d\theta \leqslant \mathcal{E}(R_0)$$

and the pointwise estimate

$$\sup_{S^1} |\eta(R,\cdot)| \leqslant R\,\mathcal{E}(R_0) + \frac{1}{2\pi}\left|\int_{S^1} \overline{\eta}\, d\theta'\right| + \frac{1}{2}\left(\sup_{S^1} \overline{a}\right)\left(R^2 - R_0^2\right)\mathcal{E}_K(R_0).$$

Proof. $|\eta_\theta| \leqslant R\,E$ $\qquad$ $|\eta_R|\, a^{-1} \leqslant R\,E + \dfrac{a^{-1}}{4R^3} e^{2\eta} K^2 = R\,E_K$

For the pointwise bound, we proceed as follows: any $\theta, \theta' \in S^1$:

$$|\eta(R,\theta) - \eta(R,\theta')| \leqslant R\,\mathcal{E}(R)$$

and by integrating in θ'

$$|\eta(R,\theta)| \leqslant R\,\mathcal{E}(R) + \frac{1}{2\pi}\left|\int_{S^1} \eta(R,\theta')\,d\theta'\right|.$$

On the other hand, we have

$$\left|\int_{S^1} \eta(R,\theta')d\theta'\right| \leqslant \left|\int_{S^1} \overline{\eta}\,d\theta'\right| + \left|\int_{S^1}\int_{R_0}^{R} \eta_R(R,\theta')d\theta'\right|$$

and we evaluate the second term above by

$$\left|\int_{S^1}\int_{R_0}^{R} \eta_R(R,\theta')d\theta'\right| \leqslant \sup_{S^1} a(R)\,\frac{1}{2}\left(R^2 - R_0^2\right)\mathcal{E}_K(R).$$

Finally, we use the energy estimate on $\mathcal{E}_K(R)$ and the upper bound on a.

$$\sup_{S^1}|\eta(R,\cdot)| \leqslant R\,\mathcal{E}(R_0) + \frac{1}{2\pi}\left|\int_{S^1}\overline{\eta}\,d\theta'\right| + \frac{1}{2}\left(\sup_{S^1}\overline{a}\right)\left(R^2 - R_0^2\right)\mathcal{E}_K(R_0).$$

BOUNDS DEPENDING ON THE MAXIMAL TIME R_*

No finite time blow-up ?

Lemma. Lower bound for the conformal metric coefficient a

$$a(R,\theta) \geqslant c_\star > 0$$

Proof. $(a^{-2})_R = \frac{K^2}{2aR^3}\, e^{2\eta} \leqslant C\, R^{-3}\, e^{C\, R^2}$ and, by integration,

$$a(R,\theta)^{-2} - \overline{a}(\theta)^{-2} \leqslant \int_{R_0}^{R} C\, \frac{e^{C\, R'^2}}{R'^3}\, dR' \leqslant C(R) \leqslant C_*.$$

Lemma. H^1 estimates of the functions U, A

$$\int_{S^1} \left(U_t^2 + A_t^2 + U_\theta^2 + A_\theta^2\right) d\theta \leqslant C_\star$$

$$\sup_{[R_0,R_\star]\times S^1} \left(|U| + |A|\right) \leqslant C_\star$$

Proof. From the energy estimates and the lower/upper bounds on a:

$$\int_{S^1} \left(U_\theta^2 + e^{4U}\, A_\theta^2\right) d\theta \leqslant C_\star, \qquad \int_{S^1} \left(U_t^2 + e^{4U}\, A_t^2\right) d\theta \leqslant C.$$

Hence, $U \in L^\infty$.

Lemma. Higher-regularity of the function a

$$\big|(\ln a)_{R\theta}\big| \leqslant \frac{K^2}{2\,R^2}\, e^{2\eta} E$$

$$\sup_{R \in [R_0, R_*)} \big\| a_{R\theta}, a_{RR}, a_{\theta\theta} \big\|_{L^1(S^1)} \leqslant C_\star$$

Term $a_{R\theta}$. By differentiating $(\ln a)_R = -e^{2\eta}\, K^2/(2R^3)$, with respect to θ:

$$\big|(\ln a)_{R\theta}\big| = \Big|\frac{K^2}{4\,R^3} e^{2\eta} 2\eta_\theta\Big| \leqslant \frac{K^2}{2\,R^3} e^{2\eta}\, R\, E,$$

since $|\eta_\theta| \leqslant R\,E$. Then, we use the pointwise estimate on η and the energy bound on E.

Term a_{RR}. We use the L^1 estimate on η_R by writing

$$(\ln a)_{RR} = -\Big(e^{2\eta}\, K^2/(2R^3)\Big)_R = \dots$$

Term $a_{\theta\theta}$**.** Argument more involved.

- From $(\ln a)_R = -\frac{K^2}{2R^3} e^{2\eta}$, we find

$$(\ln a)_{R\theta\theta} = -\frac{K^2}{2R^3}\left(e^{2\eta}\right)_{\theta\theta} = -\frac{K^2}{2R^3}\Big(\left(a^{-2}(e^{2\eta})_R\right)_R + F\Big)$$
$$F = 2e^{2\eta}\left(\left(a^{-2}\eta_R\right)_R - \eta_{\theta\theta}\right) + 4e^{2\eta}\left(a^{-2}\eta_R^2 - \eta_\theta^2\right).$$

  If we can show that F is bounded in $L^1_{\text{loc}}([R_0, R*] \times S^1)$, then the desired conclusion follows

  - by integration in R
  - using integration by parts
  - and the L^1 estimate on η_R in order to control the term arising from $\left(a^{-2}(e^{2\eta})_R\right)_R$.

- On the other hand, concerning the term F:
  - The first term is bounded in L^1_{loc}, thanks to the wave equation for η.
  - The second term involves the product $(a^{-1}\eta_R + \eta_\theta)(a^{-1}\eta_R - \eta_\theta)$, which has the null structure.
    - That is, up to uniformly bounded terms, it a linear combination of the null products

$$(a^{-1}U_R + U_\theta)^2(a^{-1}U_R - U_\theta)^2, \qquad (a^{-1}A_R + A_\theta)^2(a^{-1}A_R - A_\theta)^2$$

    - These terms are bounded in L^1 thanks to the higher-integrability: $a^{-2}U_R^2 - U_\theta^2$ and $a^{-2}A_R^2 - A_\theta^2 \in L^2_{\text{loc}}([R_0, R_*) \times S^1)$

We can now reformulate our existence result in areal coordinates.

Theorem. Global existence theory in areal coordinates

For any weakly regular initial data set with area $R = R_0 > 0$, the system of partial differential equations describing T^2–symmetric spacetimes in areal coordinates

- admits a unique, global weak solution U, A, ν, a, G, H defined on the interval $I = [R_0, +\infty)$
- satisfying the regularity conditions $U_R, A_R, U_\theta, A_\theta \in L^\infty_{\text{loc}}(I, L^2(S^1))$ $\eta_R, \eta_\theta, G, H \in L^\infty_{\text{loc}}(I, L^1(S^1))$ and $a \in L^\infty_{\text{loc}}(I, W^{1,\infty}(S^1))$

Additional regularity

$U, A \in C^0_{\text{loc}}(I, H^1(S^1)) \cap C^1_{\text{loc}}(I, L^2(S^1))$

$\eta \in C^0_{\text{loc}}(I, W^{1,1}(S^1)) \cap C^1_{\text{loc}}(I, L^1(S^1))$

$a, a^{-1} \in C^0_{\text{loc}}(I, W^{2,1}(S^1)) \cap C^1_{\text{loc}}(I, W^{1,1}(S^1))$

$G, H \in C^0_{\text{loc}}(I, L^\infty(S^1))$ $\quad G_R, H_R \in C^0_{\text{loc}}(I, W^{1,1}(S^1)) \cap C^1_{\text{loc}}(I, L^1(S^1))$

Remark. The additional regularity was not required to express Einstein equations in the weak sense.

WEAK REGULARITY FROM CONFORMAL TO AREAL coordinates

- Two coordinate systems
  - local-in-time analysis in conformal coordinates
  - long-time analysis in areal coordinates
- Construction of coordinates depends on the metric
  - Restriction on the regularity of these coordinates as functions of the given coordinates

"The convergence-continuity lemma"

Consider a 2-dimensional Lorentzian manifold $(\mathcal{Q}, \widetilde{g})$ with $\mathcal{Q} = [t_0, t_1) \times S^1$ and $\widetilde{g} = -\rho \, dt^2 + \rho^{-1} \, d\theta^2$ with $\rho = \rho(t, \theta) \in W^{2,1}(\mathcal{Q})$.
Then, any weak solution

$$p \in L^\infty_{\text{loc}}([t_0, t_1), H^1(S^1)) \cap W^{1,\infty}_{\text{loc}}([t_0, t_1), L^2(S^1))$$

to the wave equation $\Box_{\widetilde{g}} p = q$ with $q \in L^2_{\text{loc}}([t_0, t_1), L^2(S^1))$ can be realized as the limit of Lipschitz continuous solutions to the same wave equation, belonging to

$$C^0_{\text{loc}}([t_0, t_1), H^1(S^1)) \cap C^1_{\text{loc}}([t_0, t_1), L^2(S^1)).$$

Proof. Fix some compact time interval $[t_0, t_2] \subset [t_0, t_1)$.

- **Regularization.** Let p_0^ε, p_1^ε, and q^ε be smooth functions, approximating $p(t_0, \cdot)$, $p_t(t_0, \cdot)$, and q in $H^1(S^1)$, $L^2(S^1)$, and $L^2([t_0, t_2] \times S^1)$, respectively.
  - Let p^ε be the solution to the wave equation $\Box_{\tilde{g}} p^\varepsilon = q^\varepsilon$ with regularized initial data $(p_0^\varepsilon, p_1^\varepsilon)$.
  - Observe that p^ε is of class C^1, since ρ is C^1.
- **Energy estimate** on the difference $\Delta p := p - p^\varepsilon$ for all times $t \in [t_0, t_2]$

$$
\begin{aligned}
&\|\Delta p_t(t, \cdot)\|^2_{L^2(S^1)} + \|\Delta p_\theta(t, \cdot)\|^2_{L^2(S^1)} \\
&\lesssim \|p_0 - p_0^\varepsilon\|^2_{H^1(S^1)} + \|p_{t,0} - p_1^\varepsilon\|^2_{L^2(S^1)} + \|\Delta p_t \, \Delta q\|_{L^1([t_0,t_2]\times S^1)} \\
&\lesssim \|p_0 - p_0^\varepsilon\|^2_{H^1(S^1)} + \|p_{t,0} - p_1^\varepsilon\|^2_{L^2(S^1)} + \|\Delta q\|_{L^2([t_0,t_2]\times S^1)} + \|\Delta p_t\|_{L^2([t_0,t_2]\times S^1)}
\end{aligned}
$$

with $\Delta q := q - q^\varepsilon$, where the implied constant depending on t_0, t_1 and $\rho \in W^{1,\infty}$.

- This implies the uniform convergence

$$
\lim_{\varepsilon \to 0} \sup_{t \in [t_0, t_2]} \|(p - p^\varepsilon)(t, \cdot)\|^2_{H^1(S^1)} + \|(p_t - p_t^\varepsilon)(t, \cdot)\|^2_{L^2(S^1)}
$$

  as well as the continuity in time.

Application of the Einstein equations.

$(\mathcal{Q}, \tilde{g})$: quotient space $\mathcal{M}/T^2$ with its induced metric and C^∞ differential structure given by either conformal or areal coordinates

- In conformal coordinates: $\rho \equiv 1$
- In areal coordinates: $\rho = a^{-1}$ proven to be $W^{2,1}$ (and thus C^1)
- The source terms of the wave equations for U, A are $L^2_{\text{loc}}(\mathcal{M})$.

Claim

The weak regularity of the metric coefficients does not change under a change of coordinates from/to conformal to/from areal coordinates.

Proposition. From conformal to areal coordinates

Let $(\mathcal{M}, g, \mathcal{C})$ be a weakly regular Ricci-flat (but non-flat) T^2–symmetric spacetime with admissible conformal coordinates $\mathcal{C} = (\tau, \xi, x, y)$.
There exists an areal coordinate system $\mathcal{A} = (R, \theta, x, y)$ (with ∇R timelike) which is $W^{2,1}$-compatible with $\mathcal{C} = (\tau, \xi, x, y)$. Let $(\mathcal{M}, \mathcal{A})$ be the topological manifold $\mathcal{M}$ endowed with the C^∞ differential structure compatible with $\mathcal{A} = (R, \theta, x, y)$.

- (R, θ, x, y) are admissible coordinates for the manifold $(\mathcal{M}, g, \mathcal{A})$.
- Einstein's field equations hold in areal coordinates.

Proposition. From areal to conformal coordinates

Let $(\mathcal{M}, g, \mathcal{A})$ be a weakly regular, Ricci-flat T^2–symmetric spacetime with admissible areal coordinates let $\mathcal{A} = (R, \theta, x, y)$.
There exists a conformal coordinate system $\mathcal{C} = (\tau, \xi, x, y)$ which is $W^{2,1}$-compatible with $\mathcal{A}$. Let $(\mathcal{M}, \mathcal{C})$ be the topological manifold $\mathcal{M}$ endowed with the C^∞ differential structure compatible with (τ, ξ, x, y).

- (τ, ξ, x, y) are admissible coordinates for the manifold $(\mathcal{M}, g, \mathcal{C})$.
- Einstein‘s field equations hold in conformal coordinates.

GEOMETRIC UNIQUENESS AND MAXIMAL DEVELOPMENT

Definition

Given a weakly regular T^2–symmetric initial data set (Σ, h, K), a **weakly regular T^2–symmetric future development** of (Σ, h, K) is

- a weakly regular T^2-symmetric Lorentzian manifold $(\mathcal{M}, g)$,
- together with a smooth embedding ϕ of Σ onto one of the hypersurfaces Σ_{t_0} of the $3+1$ foliation
- such that (h, K) coincide with $(h(t_0), K(t_0))$
- and $\phi(\Sigma)$ coincides with the past boundary of $\mathcal{M}$.

Definition

- A development $(\mathcal{M}, g)$ **is said to be a proper extension of another development** $(\mathcal{M}', g')$ if there exists a C^1 isometric embedding of $(\mathcal{M}', g')$ into a proper subset of $(\mathcal{M}, g)$.
- A maximal development is a development admitting no proper extension.

Partial order relation induced by this notion of extension.

Theorem. Uniqueness theory for weakly regular T^2–symmetric developments

For any weakly regular T^2–symmetric initial data set (Σ, h, K) with constant area of symmetry R_0, there exists a weakly regular T^2–symmetric future development defined for with $R \geqslant R_0$, which

- is unique (up to C^1 diffeomorphisms) and
- is maximal

and, therefore, coincides with the spacetime we already constructed in areal coordinates.

Proof. Let (Σ, h, K) be a weakly regular initial data set (Σ, h, K) with constant area of symmetry R_0.

- Let $(\mathcal{M}, g)$ be a future development of (Σ, h, K) with $R \geqslant R_0$.
- We have proven that the function R enjoys additional regularity and, in particular, is C^1.
- Hence, we may introduce a new coordinate system, with R as the time function, which is C^1 compatible with the differential structure of $(\mathcal{M}, g)$.
- The regularity of all metric functions is preserved.

Hence, areal coordinates may be introduced on $(\mathcal{M}, g)$ and the essential Einstein system are satisfied.

It then follows from the uniqueness of the solution of the essential Einstein system that $(\mathcal{M}, g)$ can be identified with a subset of the solution that we already constructed.

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OUR NEXT OBJECTIVES (Chapters IV and V)

- Self-gravitating compressible perfect fluids with shock waves
- Global geometry of weakly regular spacetimes
- Causality properties (completeness, etc.)

CHAPTER IV
Self-Gravitating Fluids In Spherical Symmetry. Weakly regular Cauchy developments

Problem of the gravitational collapse of matter

- spherically-symmetric spacetimes
- compressible perfect fluid (physically realistic model)
- shock waves, propagating curvature singularities

Weakly regular Cauchy developments

- initial data set on an incoming light cone
- non-vacuum, weakly regular spacetimes
- existence of a broad class of such spacetimes

Global dynamics

- investigate the global and causal geometry
- weak cosmic censorship (singularities not visible by an observer at infinity)
- dispersion in timelike directions (small mass), formation of trapped spheres (large mass)
- Further results: cosmological spacetimes, weakly regular global foliations

OUTLINE

- **Section 1. The formulation in Eddington-Finkelstein coordinates**

  initial data set prescribed on an incoming light cone

- **Section 2. Einstein-Euler spacetimes with bounded variation (BV)**

  shock waves, curvature discontinuities

- **Section 3. BV regular Cauchy developments**

  converging method of approximation to the Einstein-Euler system

- **Section 4. Riemann problem on an Eddington Finkelstein background**

  Monotonicity properties (sup-norm, total variation)

- **Section 5. Spherically symmetric static Einstein-Euler spacetimes**

  Hawking quasi-local mass

Section 1. THE FORMULATION IN EDDINGTON FINKELSTEIN COORDINATES

EINSTEIN-EULER SPACETIMES

- $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$
- **Einstein equations**

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$

understood in the weak sense

- **Euler equations**

$$\nabla_\alpha T^{\alpha\beta} = 0$$

- Matter content governed by the energy-momentum tensor of compressible fluids

$$T^{\alpha\beta} = (\mu + p) u^\alpha u^\beta + p\, g^{\alpha\beta}$$

  - μ: mass-energy density of the fluid
  - u^α: its velocity vector, normalized to be unit: $u^\alpha u_\alpha = -1$
  - pressure $p = p(\mu) = k^2 \mu$
    - $k \in (0,1)$ represents the sound speed
    - light speed normalized to unit

INITIAL DATA SET

- **Spherical symmetry:** $SO(3)$ acts as an isometry group and
  - there exists a central wordline invariant by the group
  - the group orbits through any point (not on the central line) are spacelike 2-spheres
  - From the area A of these 2-spheres, we define the "area radius" by $A = 4\pi r^2$.
- **Initial hypersurface:**
  - incoming light cone with vertex at $r = 0$
- **Initial data:**
  - spatially compact (and weakly regular) perturbation of a (regular) static solution —representing a fluid at equilibrium
  - Later, we will consider "short-pulse data" (Chapter V).

SCHWARZSCHILD SPACETIME. Black hole of mass $m > 0$

Schwarzschild coordinates $r > 2m$

$$g = -\left(1 - 2m/r\right) dt^2 + \left(1 - 2m/r\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

Suffers an (artificial) singularity on the null hypersurface $r = 2m$.

Eddington-Finkelstein coordinates r and advanced time $v = t + r + 2m\ln(r - 2m)$

$$g = -\left(1 - 2m/r\right) dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

with $(v, r) \in [0, +\infty) \times (0, +\infty)$

- Regular except at the center $r = 0$ (curvature blow-up)
- Useful choordinates
  - Allow us to cross the horizon $r = 2m$
  - **cover the trapped region** $r < 2m$
  - in which the (radially outgoing) null geodesics $\frac{dr}{dv} = 1 - \frac{2m}{r}$ move *toward the center.*

EINSTEIN EQUATIONS IN EDDINGTON-FINKELSTEIN coord.

Coordinates that may include a possibly trapped region of the Cauchy development

$$g = -ab^2\, dv^2 + 2b\, dvdr + r^2\, \overline{g}_{S^2}$$

- advanced time $v \in [v_0, v_*]$ and area radius $r \in [0, +\infty)$
- $\overline{g}_{S^2} = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ with (θ, φ) coord. on S^2
- two metric coefficients
  - $b = b(v, r)$ is positive, but $a = a(v, r)$ may change sign
  - regularity at the center $\lim_{r\to 0}(a, b)(v, r) = (1, 1)$ for all v (no mass concentration at the center)

EINSTEIN EQUATIONS $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$

- Two ordinary differential equations

$$b_r = 4\pi\, r b^3\, T^{00}$$
$$r a_r b + ab - b + 2rab_r = 8\pi\, r^2 b^2\, T^{01}$$

- Two partial differential equations

$$a^2 b - ab + raa_r b + 2ra^2 b_r - ra_v = 8\pi\, r^2 b\, T^{11}$$
$$2a_r b^3 + ra_{rr} b^3 + 3ra_r b^2 b_r + 2ab^2 b_r + 2rab^2 b_{rr} - 2rb_v b_r + 2rbb_{vr} = 16\pi\, r^3 b^3\, T^{22}$$

- The remaining Einstein components impose compatibility/restriction conditions on the matter model:

$$T^{02} = T^{03} = 0$$
$$T^{12} = T^{13} = 0$$
$$T^{23} = T^{22} - (\sin\theta)^2\, T^{33} = 0$$

(cf. below)

Geometry determined from the matter content of the spacetime

1. Combining the T^{00} and T^{01} equations

$$\partial_r(r(1-a)) + r(1-a)\,8\pi r b^2\,T^{00} = 8\pi r^2 b\left(bT^{00} - T^{01}\right)$$

and using the regularity at the center:

$$a(v,r) = 1 + \frac{8\pi}{r}\int_0^r \left(bT^{00} - T^{01}\right)\exp\left(-8\pi\int_{r'}^r T^{00}\,r'' b^2\,dr''\right) b(v,r') {r'}^2 dr'$$

provided the integrand is in $L^1_{\text{loc}}([0,+\infty))$ with respect to r

2. With the T^{00} equation and by using the regularity at the center:

$$b(v,r) = \exp\left(4\pi\int_0^r b^2(v,r')\,T^{00}(v,r')\,r' dr'\right)$$

provided $r \mapsto rb^2 T^{00}$ is in $L^1_{\text{loc}}([0,+\infty))$

3. The T^{11} and T^{22} equations will be deduced from the above two.

(See below.)

Implicit expressions, depending also on the (unknown) matter content of the spacetime.

EULER EQUATIONS IN EDDINGTON-FINKELSTEIN coord.

MATTER MODEL

- Two unknowns: mass-energy density $\mu = \mu(v, r) > 0$ and the velocity vector $u^\alpha = u^\alpha(v, r)$
    - normalization $-1 = u^\alpha u_\alpha$, so that that $u^0 \neq 0$.
- Components $\beta = 0, 1$ of the Euler equations $\nabla_\alpha T^{\alpha\beta} = 0$
- The remaining two components $\beta = 2, 3$ will be deduced from the above two. (See below.)
- Remaining Einstein curvature components

$$T^{02} = T^{03} = T^{12} = T^{13} = T^{23} = T^{22} - (\sin\theta)^2 \, T^{33} = 0$$

- Several components of the energy-momentum tensor do vanish.
- The last two components of the velocity vector vanish: $u^2 = u^3 = 0$

The T^{00} Euler equation $\qquad 0 = \nabla_\alpha T^{\alpha 0}$

$$\partial_v T^{00} + \partial_r T^{01} + \left(\frac{2b_v}{b} + \frac{a_r b}{2} + ab_r\right) T^{00} + \left(\frac{b_r}{b} + \frac{2}{r}\right) T^{01} - \frac{2r}{b} T^{22} = 0$$

The T^{01} Euler equation $\qquad 0 = \nabla_\alpha T^{\alpha 1}$

$$\partial_v T^{01} + \partial_r T^{11} + \left(-\frac{a_v b}{2} + \frac{a a_r b^2}{2} + a^2 b b_r\right) T^{00}$$
$$+ \left(\frac{b_v}{b} - a_r b - 2ab_r\right) T^{01} + \left(\frac{2b_r}{b} + \frac{2}{r}\right) T^{11} - 2ra T^{22} = 0$$

Recall the expression of the energy-momentum tensor

$$T^{\alpha\beta} = (\mu + p) u^\alpha u^\beta + p\, g^{\alpha\beta}$$

Essential Euler equations in Eddington-Finkelstein coordinates

$$
\begin{aligned}
0 =& \partial_v \left(\mu(1+k^2)u^0u^0\right) + \partial_r\left(\mu(1+k^2)u^0u^1 + k^2\frac{\mu}{b}\right) \\
&+ \left(\frac{2b_v}{b} + \frac{a_r b}{2} + ab_r\right)\mu(1+k^2)u^0u^0 \\
&+ \left(\frac{b_r}{b} + \frac{2}{r}\right)\left(\mu(1+k^2)u^0u^1 + k^2\frac{\mu}{b}\right) - \frac{2k^2}{rb}\mu
\end{aligned}
$$

$$
\begin{aligned}
0 =& \partial_v \left(\mu(1+k^2)u^0u^1 + k^2\frac{\mu}{b}\right) + \partial_r\left(\mu(1+k^2)u^1u^1 + k^2\mu a\right) \\
&+ \left(-\frac{a_v b}{2} + \frac{aa_r b^2}{2} + a^2bb_r\right)\mu(1+k^2)u^0u^0 \\
&+ \left(\frac{b_v}{b} - a_r b - 2ab_r\right)\left(\mu(1+k^2)u^0u^1 + k^2\frac{\mu}{b}\right) \\
&+ \left(\frac{2b_r}{b} + \frac{2}{r}\right)\left(\mu(1+k^2)u^1u^1 + k^2\mu a\right) - \frac{2k^2 a}{r}\mu
\end{aligned}
$$

FORMULATION AS A FIRST-ORDER HYPERBOLIC SYSTEM

The normalization $-1 = g_{\alpha\beta} u^\alpha u^\beta$ implies $u^1 = -\frac{1}{2b^2(u^0)^2}$

Normalized fluid variables in generalized Eddington-Finkelstein coordinates

$$M := b^2 \mu \, u^0 u^0 \in (0, +\infty), \qquad V := \frac{u^1}{b\, u^0} - \frac{a}{2} \in (-\infty, 0)$$

Notation: $K^2 := \frac{1-k^2}{1+k^2}$

Energy-momentum tensor in the variables (M, V):

$$T^{00} = (1+k^2)\frac{M}{b^2} \qquad T^{01} = (1+k^2)\frac{M}{b}\left(\frac{a}{2} + K^2 V\right)$$

$$T^{11} = (1+k^2) M \left(\frac{a^2}{4} + K^2 a V + V^2\right) \qquad T^{22} = -\frac{2k^2}{r^2} M V$$

Eliminate the derivative a_v.

- From the T^{01} and T^{11} Einstein equations:

$$\frac{a_v b}{2} = \frac{a_v}{rb}\frac{rb^2}{2} = 4\pi rb^2\left(abT^{01} - T^{11}\right)$$

- Hence, a_v *can be eliminated* from the Euler equations.

Eliminate the derivative b_v.

- Multiply the T^{00} Euler equation by b^2 and the T^{01} Euler equation by b:

$$\partial_v(b^2 T^{00}) + b^2\partial_r T^{01} = b^2\left(-\frac{1}{2}\left(a_r b + 2ab_r\right) T^{00} - \left(\frac{b_r}{b} + \frac{2}{r}\right) T^{01} + \frac{2r}{b} T^{22}\right)$$

$$\begin{aligned}\partial_v(bT^{01}) + b\partial_r T^{11} = b\Bigg(& \left(a_r b + 2ab_r\right)\left(T^{01} - \frac{ab}{2}T^{00}\right) \\ & + 4\pi rb^2(abT^{01} - T^{11})T^{00} - \left(\frac{2b_r}{b} + \frac{2}{r}\right) T^{11} + 2raT^{22}\Bigg)\end{aligned}$$

- Hence, b_v *can be eliminated* from the Euler equations.

Eliminate the derivative b_r.

$$\partial_v(b^2T^{00}) + \partial_r(b^2T^{01}) = -\left(a_r b + 2ab_r\right)\frac{b^2}{2}T^{00} + \left(b_r - \frac{2b}{r}\right)bT^{01} + 2rbT^{22}$$

$$\partial_v(bT^{01}) + \partial_r(bT^{11}) = \left(a_r b + 2ab_r\right)b\left(T^{01} - \frac{ab}{2}T^{00}\right) + 4\pi r b^3\left(abT^{01} - T^{11}\right)T^{00} - \left(b_r + \frac{2b}{r}\right)T^{11} + 2rabT^{22}$$

- $a_r b + 2ab_r$ *can be eliminated* thanks to the T^{01} Einstein equation:

$$a_r b + 2ab_r = \frac{1}{r}\left(8\pi r^2 b^2 T^{01} - (a-1)b\right)$$

- b_r *can be eliminated* thanks to the T^{00} Einstein equation: $b_r = 4\pi r b^3 T^{00}$

Consequently, the right-hand sides are free of derivatives:

$$\partial_v(b^2T^{00}) + \partial_r(b^2T^{01}) = \frac{(a-1)b^3}{r}T^{00} - \frac{2b^2}{r}T^{01} + 2rbT^{22}$$

$$\partial_v(bT^{01}) + \partial_r(bT^{11}) = \frac{1}{r}b\left(\frac{1}{2}ab^2(a-1)T^{00} - (a-1)bT^{01} - 2T^{11}\right) + 2rb\left(4\pi b^2(T^{01})^2 - 4\pi b^2T^{00}T^{11} + aT^{22}\right)$$

Proposition. Essential Euler equations for the normalized fluid variables

$$\partial_v U + \partial_r F(U, a, b) = S(U, a, b)$$

$$U := M \begin{pmatrix} 1 \\ \frac{a}{2} + K^2 V \end{pmatrix} \qquad F(U, a, b) := bM \begin{pmatrix} \frac{a}{2} + K^2 V \\ \frac{a^2}{4} + K^2 aV + V^2 \end{pmatrix}$$

$$S(U, a, b) := \begin{pmatrix} S_1(M, V, a, b) \\ S_2(M, V, a, b) \end{pmatrix} \qquad S_1(M, V, a, b) := -\frac{1}{2r} bM (1 + a + 4V)$$

$$S_2(M, V, a, b) := -\frac{1}{2r} bM \Big(a^2 + 2aV(2 + K^2) - 2K^2 V + 4V^2 \Big) - 16\pi(1 - K^2)\, rb\, M^2 V^2$$

- Curved spacetime geometry determined from its matter content
- No dynamical gravitational degree of freedom

$$a(v, r) = 1 - \frac{4\pi(1 + k^2)}{r} \int_0^r M(v, r') \Big(2K^2 \left| V(v, r') \right| + 1 \Big) \frac{b(v, r')}{b(v, r)} r'^2 dr'$$

$$b(v, r) = \exp \left(4\pi(1 + k^2) \int_0^r M(v, r')\, r' dr' \right)$$

Section 2. EINSTEIN-EULER SPACETIMES WITH BOUNDED VARIATION

NOTION OF BV REGULAR CAUCHY DEVELOPMENTS

Challenge.

- Shock waves arise in the fluid, even from smooth (and small) initial data.
- Global solutions may exist only in a *weak sense*.

Functional spaces.

- Class BV_{loc} of functions with locally bounded variation in r (first-order derivative is a locally bounded measure)
- Total variation $TV_0^{r_0}(f) := \left|\frac{df}{dr}\right|(0, r_0)$, existence of left- and right-hand traces
- $L^\infty_{\text{loc}}(BV_{\text{loc}})$: space of functions depending also on v, whose local total variation (in the radius variable r) is locally bounded in v.
- Locally Lipschitz continuous $\text{Lip}_{\text{loc},v}(L^1_r)$ in the advanced time v

Spherically symmetric Einstein-Euler spacetimes with bounded variation

Generalized Eddington-Finkelstein coordinates

$$g = -ab^2\, dv^2 + 2b\, dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

- Metric coefficients a, b
- Fluid variables $M = b^2\mu\, u^0 u^0 \in (0, +\infty)$ and $V = \frac{u^1}{b\, u^0} - \frac{a}{2} \in (-\infty, 0)$
- Defined for $v \in I := [v_0, v_*]$ and $r \in J := [0, r_0)$

- Regularity at the center: $\lim_{r\to 0}(a, b)(v, r) = (1, 1)$ at all $v \in I$
- BV regularity

$$rM,\ V \in L^\infty_{\mathrm{loc}}(I, BV_{\mathrm{loc}}(J)) \cap \mathrm{Lip}_{\mathrm{loc}}(I, L^1_{\mathrm{loc}}(J))$$

$$a_v,\ ra_r,\ b_r \in L^\infty_{\mathrm{loc}}(I, BV_{\mathrm{loc}}(J)) \cap \mathrm{Lip}_{\mathrm{loc}}(I, L^1_{\mathrm{loc}}(J))$$

- The Einstein and Euler equations are satisfied in the weak sense.

Essential EINSTEIN EQUATIONS

$$b_r = 4\pi rbM\,(1+k^2) \qquad BV_{\text{loc}} \text{ functions}$$

$$a_r = 4\pi rM\,(1+k^2)\left(2K^2V - a\right) + \frac{1-a}{r} \qquad BV_{\text{loc}} \text{ functions}$$

Essential EULER EQUATIONS

$$\partial_v U + \partial_r F(U,a,b) = S(U,a,b) \qquad \text{loc. bounded measures}$$

Additional EINSTEIN EQUATIONS

$$\left(\frac{b_r}{b}\right)_v + \frac{1}{2}(a_r b)_r + (ab_r)_r = -\frac{1}{r}(ab)_r - 16\pi bM\,k^2V \qquad \text{loc. bounded measures}$$

$$a_v = 2\pi rbM(1+k^2)\left(a^2 - 4V^2\right) \qquad BV_{\text{loc}} \text{ functions}$$

Observations.

1. No regularity required on the first-order derivative b_v.
2. The integral formulas for a, b make sense under our integrability conditions.
3. Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ satisfied as equalities within locally bounded measures.

INITIAL DATA SET

- **Initial hypersurface:** an incoming light cone $v = v_0$
- **Initial matter content** on this hypersurface
  - Prescribe the (normalized) matter variables (geometric data)

$$M(v_0, r) = M_0(r), \qquad V(v_0, r) = V_0(r), \qquad r \in J$$

  - Here, $M_0 > 0$ and $V_0 < 0$ are functions with locally bounded variation.
- **Initial geometry** a_0, b_0 on the initial hypersurface
  - determined from the data M_0, V_0
  - satisfy the required regularity conditions:

$$r\partial_r a_0,\ \partial_r b_0, \in BV_{\mathrm{loc}}(J)$$
$$\lim_{r \to 0} a_0(r) = \lim_{r \to 0} b_0(r) = 1$$

REDUCTION TO THE ESSENTIAL EINSTEIN-EULER SYSTEM

- Redundancies as a consequence of the spherical symmetry
- We can recover the "non-essential equations".

Terminology

- *Full system*: 4 metric equations + 4 fluid equations
- *Essential system*: 2 metric equations + 2 fluid equations

Proposition. From the essential system to the full system

Consider self-gravitating compressible fluids in spherical symmetry and work in Eddington-Finkelstein coordinates.
Any solution (M, V, a, b) to the essential system is actually a solution to the full system, that is, if the T^{00} and T^{01} metric equations and the T^{00} and T^{01} fluid equations hold true, then

> *under our weak regularity assumptions and without further initial data or regularity assumptions, all of the Einstein-Euler equations are satisfied in the distributional sense.*

Notation. $B := \log b, \qquad X := a_r b + 2ab_r = \frac{1}{b}(ab^2)_r$

Einstein equations $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$ equivalent to the four partial differential equations

$$B_r = 4\pi\, rb^2\, T^{00}$$
$$rX + b(a-1) = 8\pi\, r^2 b^2\, T^{01}$$
$$ab(a-1) + r(aX - a_v) = 8\pi\, r^2 b\, T^{11}$$
$$2(ab)_r + r(X + 2B_v)_r = 16\pi\, r^3 b\, T^{22}$$

supplemented with (regarded as restrictions on the energy-momentum tensor)

$$T^{02} = T^{03} = T^{12} = T^{13} = T^{23} = 0$$
$$T^{22} = (\sin\theta)^2\, T^{33}$$

Definition

An energy momentum tensor $T^{\alpha\beta}$ is *compatible with spherical symmetry* if

$$T^{02} = T^{03} = T^{12} = T^{13} = T^{23} = 0$$
$$T^{22} = (\sin\theta)^2\, T^{33}$$

For instance, the energy momentum tensor of perfect fluids is compatible with spherical symmetry, provided the velocity vector u^α has $u^2 = u^3 = 0$.

Recovering the T^{02} and T^{03} Euler equations

If the matter tensor is compatible with spherical symmetry, then the T^{02} and T^{03} Euler equations hold:

$$\nabla_\alpha T^{\alpha\beta} = 0, \qquad \beta = 2, 3.$$

Proof. Our assumption of spherical symmetry implies that the partial derivatives in θ, φ vanish.

- $\nabla_\alpha T^{\alpha 2} = 0$ follows from the conditions $T^{02} = T^{12} = 0$ and $T^{22} = (\sin\theta)^2\, T^{33}$ since, in the weak sense,

$$\begin{aligned}
\nabla_0 T^0_2 &= T^0_{2,0} + \Gamma^0_{00} T^{02} = 0\\
\nabla_1 T^{12} &= T^{12}_{,1} + \Gamma^1_{01} T^{02} + \Gamma^1_{11} T^{12} + \Gamma^2_{12} T^{12} = 0\\
\nabla_2 T^{22} &= 2\Gamma^2_{12} T^{12} = 0\\
\nabla_3 T^{32} &= \Gamma^3_{13} T^{12} + \Gamma^3_{23} T^{22} + \Gamma^2_{33} T^{33} = \cot\theta\, T^{22} - \sin\theta\,\cos\theta\, T^{33} = 0.
\end{aligned}$$

- Similarly, $\nabla_\alpha T^{\alpha 3} = 0$ follows from the conditions $T^{03} = T^{13} = T^{23}$.

Recovering the T^{22} Einstein equation

If the T^{00} and T^{01} Einstein equations and T^{00} Euler equation hold and the matter tensor is compatible with spherical symmetry, then the T^{22} Einstein equation holds.

Proof. From the T^{00} Euler equation

$$0 = \nabla_\alpha T^{0\alpha} = T^{00}_{,0} + T^{01}_{,1} + \left(2B_v + \frac{X}{2}\right) T^{00} + \left(B_r + \frac{2}{r}\right) T^{01} - \frac{2r}{b} T^{22}$$

in which, thanks to the T^{00} and T^{01} Einstein equations in the weak sense,

$$T^{00}_{,0} = \frac{1}{4\pi r}\left(\frac{B_r}{b^2}\right)_v = \frac{B_{rv} - 2B_r B_v}{4\pi r b^2} = \frac{B_{rv}}{4\pi r b^2} - 2B_v T^{00}$$

$$T^{01}_{,1} = \frac{1}{8\pi}\left(\frac{rX + b(a-1)}{r^2 b^2}\right)_r = -2\left(B_r + \frac{1}{r}\right) T^{01} + \frac{X + rX_r + (ab)_r}{8\pi r^2 b^2} - \frac{B_r}{8\pi r^2 b}$$

Therefore, again with the T^{00} and T^{01} Einstein equations in the weak sense,

$$\begin{aligned}\frac{2r}{b} T^{22} &= \frac{B_{rv}}{4\pi r b^2} - B_r T^{01} + \frac{X + rX_r + (ab)_r}{8\pi r^2 b^2} - \frac{B_r}{8\pi r^2 b} + \frac{X}{2} T^{00} \\ &= \frac{1}{8\pi r^2 b^2}\Big(r(2B_{rv} + X_r) + X - abB_r + (ab)_r\Big) \\ &= \frac{1}{8\pi r^2 b^2}\Big(r(2B_v + X)_r + 2(ab)_r\Big).\end{aligned}$$

Recovering the T^{11} Einstein equation

If the T^{00}, T^{01}, and T^{22} Einstein equations and the T^{01} Euler equation hold and the matter tensor is compatible with spherical symmetry, then the T^{11} Einstein equation holds provided:

- **Regularity of the metric at the center**
$$\lim_{r\to 0}(a,b) = (1,1)$$
- **Regularity of the fluid at the center**:
$$\lim_{r\to 0} r^2\left(G^{11} - 8\pi\, T^{11}\right) = 0$$

Proof. The Euler equation $\nabla_\alpha T^{\alpha 1} = 0$ reads

$$0 = T^{01}_{,0} + T^{11}_{,1} + (aX - a_v)\frac{b}{2}T^{00} + (B_v - X)T^{01} + 2\left(B_r + \frac{1}{r}\right)T^{11} - 2raT^{22}$$

From the T^{00}, T^{01}, and T^{22} Einstein equations and with the notation

$$\Omega(a,b) := -T^{01}_{,0} + (a_v - aX)\frac{b}{2}T^{00} + (X - B_v)T^{01} + 2raT^{22}$$

we find a linear equation for T^{11} in the variable r

$$T^{11}_{,1} + 2\left(B_r + \frac{1}{r}\right)T^{11} = \Omega(a,b).$$

Claim: The only solution is

$$T^{11} = \left(8\pi r^2 b\right)^{-1}\Big(ab(a-1) + r(aX - a_v)\Big) = \frac{G^{11}}{8\pi}$$

- The solutions to the **homogeneous equation**

$$\partial_r F = -\partial_r \left(\log(rb)^2\right) F$$

  are multiples of $\frac{1}{r^2 b^2}$.
- $\frac{G^{11}}{8\pi}$ is a **particular solution,** as follows from the Bianchi identities:
  - $G^{\alpha\beta}$ is divergence-free
  - In particular, $\nabla_\alpha G^{\alpha 1} = 0$, which reads (using the matter compatibility condition $T^{12} = 0$)

$$\partial_r G^{11} + 2\left(B_r + \frac{1}{r}\right) G^{11} = 8\pi\,\Omega(a, b).$$

- **General solution**

$$T^{11} = \frac{1}{8\pi r^2 b^2}\Big(C + ab^2(a-1) + raa_r b^2 + 2ra^2 bb_r - ra_v b\Big), \qquad C \in \mathbb{R}.$$

- Recall the **regularity condition** at the center:

$$0 = \lim_{r\to 0} r^2\left(G^{11} - 8\pi T^{11}\right) = \lim_{r\to 0} -\frac{C}{b^2} = -C.$$

Section 3. BV REGULAR CAUCHY DEVELOPMENTS

THE COMPACT FLUID PERTURBATION PROBLEM

Consider a self-gravitating compressible fluid in spherical symmetry and seek for weak solutions (M, V, a, b) to the essential Einstein-Euler system in Eddington-Finkelstein coordinates.

INITIAL DATA SET.

- **Fluid at equilibrium.** Consider the unique static spacetime $\widetilde{M}, \widetilde{V}, \widetilde{a}, \widetilde{b}$ with mass-energy fluid density prescribed at the center $r = 0$. (See below.)
- **Initial hypersurface:** an incoming light cone $\mathcal{H}_{v_0} := \{v = v_0\}$
- **Prescribed data**
  - BV regular perturbation of this static solution
    $$M_0 \simeq \widetilde{M} \qquad V_0 \simeq \widetilde{V}, \qquad a_0 \simeq \widetilde{a} \qquad b_0 \simeq \widetilde{b}$$
  - spatially compact: localized within two spheres $S_{r_* - \delta}$ and $S_{r_* + \delta}$
    $$r_* - \delta \leqslant r \leqslant r_* + \delta$$

CAUCHY DEVELOPMENT. Evolution of the fluid toward the future

- **Singularity formation.** Shock waves, curvature measures
- **Domain of dependence.** By the property of finite speed of propagation, the spacetime remains static in a neighborhood of the center of symmetry.

Domain of dependence. Finite speed of propagation

- Perturbation initially localized within $[r_* - \delta, r_* + \delta]$
- Define the radius functions

$$R_*^-(v) = r_* - \delta - C_*(v - v_0) \qquad R_*^+(v) = r_* + \delta + C_*(v - v_0)$$

- C_*: upper bound of the wave speeds λ_1, λ_2 of the Euler equations (cf. the explicit expressions above)
- Perturbation on the hypersurface $\mathcal{H}_v$ supported within the region

$$\text{supp } \big((M, V)(v, \cdot) - (\widetilde{M}, \widetilde{V})\big) \subset \big\{R_*^-(v) \leqslant r \leqslant R_*^+(v)\big\}$$

- solutions defined for times $v \in [v_0, v_*]$
- spacetime is static outside

$$\Big\{v_0 \leqslant v \leqslant v_*, \qquad R_*^-(v) \leqslant r \leqslant R_*^+(v)]\Big\}$$

Theorem. Existence of BV regular Cauchy developments

Consider the compact fluid perturbation problem with data prescribed on some light cone $\mathcal{H}_{v_0}$ and formulated in generalized Eddington-Finkelstein coordinates. Then, **there exists a BV regular Cauchy development**

- spherically symmetric Einstein-Euler spacetime with bounded variation
- satisfying the prescribed initial data M_0, V_0, a_0, b_0 on $\mathcal{H}_{v_0}$
- includes the spacetime region $[v_0, v_*] \times (0, +\infty)$ where the maximal time v_* is only restricted by the condition that the spacetime remains static near the center line $r = 0$.
- **BV regularity** in spacelike directions

$$\sup_{v \in [v_0, v_*]} TV\Big((rM, V, ra_r, b_r, a_v)(v, \cdot) - (r\widetilde{M}, \widetilde{V}, r\widetilde{a}_r, \widetilde{b}_r, 0)\Big) \lesssim N_0$$

- **Lipschitz continuity** in timelike directions $v, v' \in [v_0, v_*]$

$$\|(rM, V)(v, \cdot) - (rM, V)(v', \cdot)\|_{L^1(0,+\infty)} \lesssim N_0 \, |v - v'|$$

with $N_0 := TV_0^{+\infty}\big(r(M_0 - \widetilde{M})\big) + TV_0^{+\infty}\big(V_0 - \widetilde{V}\big)$.

Remark. Geometric singularities typically emanate from the center $r = 0$.

OUTLINE OF THE METHOD

Converging sequence of approximate solutions
$M^\sharp, V^\sharp, a^\sharp, b^\sharp$ containing (many !) shock waves

Discretization of the initial data set on $\mathcal{H}_{v_0}$

- Discretization of (M_0, V_0, a_0, b_0) by a piecewise constant initial data set with finitely many jumps
- Do not increase the total variation

Solve the Riemann problem

- Cauchy problem with a single initial discontinuity
- Approximate solution in a neighborhood of each discontinuity

Advance in timelike directions

- Introduce a discrete foliation by incoming null cones $\mathcal{H}_v$ for $v \geqslant v_0$
- Advance forward in time up to the next slice of the foliation

Global construction

- Randomly choose of a state within each Riemann problem (equidistributed sequence)
- Continue this construction inductively for each hypersurface
- Compactness property
  - Bound on the sup-norm and on the total variation of the unknowns
  - Independent of the discretization parameters

The Riemann problem

- Simplest, non-trivial Cauchy problem
- Initial value problem with piecewise constant data prescribed on a light cone $\mathcal{H}_{v'}$
- The Euler system $\partial_v U + \partial_r F(U, a', b') = S(U, a', b')$
- Initial jump at time v' centered at some radius r':

$$(M^\sharp, V^\sharp)(v', \cdot) = \begin{cases} (M^{\sharp,L}, V^{\sharp,L}) & r < r' \\ (M^{\sharp,R}, V^{\sharp,L}), & r > r' \end{cases}$$

- Cannot be solved in a closed form, except "infinitesimally" near the point (v', r'), as we do below in Section 4.
- Approximate solution, which is sufficiently accurate in a small neighborhood of the intersection $\mathcal{H}_{v'} \cap S_{r'}$

Approximate solution to the Riemann problem

- An approximate solution
  $\mathcal{R}(v,r) = \mathcal{R}(v,r;v',r';M_L,V_L,M_R,V_R,a',b')$
- Homogeneous Riemann problem (See Section 4.)
  $\widehat{\mathcal{R}}(v,r) = \widehat{\mathcal{R}}(v,r;v',r';M_L,V_L,M_R,V_R,a',b')$ satisfying
  $\partial_v U + \partial_r F(U,a',b') = 0$
- Evolution of the geometry described by

$$\mathcal{R}(v,r) := \widehat{\mathcal{R}}(v,r) + \int_{v'}^{v} \widetilde{S}(v'',X(v'',r),a',b')\,dv'',$$

  where $X(v'',r)$ denotes the solution associated with the source-terms of the essential Euler equations:

$$\frac{d}{dv}X = \widetilde{S}(X,a',b'),$$
$$X(v',r) = \widehat{\mathcal{R}}(v',r)$$

- $\widetilde{S}$ takes the geometry into account:

$$\begin{aligned}\widetilde{S} &:= S - a_r\partial_a F - b_r \partial_b F\\ &= -\frac{b'M}{2r}\begin{pmatrix}2+4V\\ a'+4a'V+4V^2\end{pmatrix} - \pi(1+k^2)rb'M^2\begin{pmatrix}8K^2V\\ -(a')^2+4K^2a'V+12V^2\end{pmatrix}\end{aligned}$$

APPROXIMATE SOLUTIONS
to the Compact Fluid Perturbation Problem

- **Discrete foliation of incoming light cones.** $v_0 < v_1 < v_2 < \dots$
  $$\mathcal{H}_{v_0}, \mathcal{H}_{v_1}, \mathcal{H}_{v_1}, \dots$$
- **Discrete foliation of spheres.** $0 < r_1 < r_2 < r_3 < \dots$
  $$S_{r_1}, S_{r_2}, S_{r_3}, \dots$$
- **Converging foliations.** We let $\Delta v := \sup(v_{i+1} - v_i)$ and $\Delta r := \sup(r_{j+1} - r_j)$ tend to zero, while the ratio $\Delta v/\Delta r$ satisfies the stability condition
  $$C_* \Delta v/\Delta r < 1$$
- **Equidistributed sequence** (ω_i) in the interval $(0, 1)$ and set
  $$r_{i,j} := r_{j-1} + \omega_i r_{j+1}$$

(relevant only in the domain of dependence of the initial data)

Step 0. Initial data set on $\mathcal{H}_{v_0}$ approximated by piecewise constant data

$$(M^\sharp, V^\sharp)(v_0, r) := (M_0, V_0)(r_{j+1}), \qquad r \in [r_j, r_{j+2}), \quad j \text{ even}$$
$$a^\sharp(v_0, r) := a_0(r_j), \qquad r \in [r_{j-1}, r_{j+1}), \quad j \text{ even}$$
$$b^\sharp(v_0, r) := b_0(r_j), \qquad r \in [r_{j-1}, r_{j+1}), \quad j \text{ even}$$

If $(M^\sharp, V^\sharp)$ is already determined up to the time $v < v_i$:

Step 1. Using the equidistributed sequence, we define $(M^\sharp, V^\sharp)$ on the light cone $\mathcal{H}_{v_i}$ by:

$$(M^\sharp, V^\sharp)(v_i+, r) := (M^\sharp, V^\sharp)(v_i-, r_{i,j+1}), \qquad r \in [r_j, r_{j+2}), \quad i+j \text{ even}$$

Step 2. Using the equidistributed sequence, we define $a^\sharp$ and $b^\sharp$ on the light cone $\mathcal{H}_{v_i}$ by:

$$a^\sharp(v_i+, r) := a^\sharp(v_i-, r_{i,j}), \qquad r \in [r_{j-1}, r_{j+1}), \quad i+j \text{ even}$$
$$b^\sharp(v_i+, r) := b^\sharp(v_i-, r_{i,j}), \qquad r \in [r_{j-1}, r_{j+1}), \quad i+j \text{ even}$$

Step 3. The approximation within the spacetime region limited by the light cones $\mathcal{H}_{v_i}$ and $\mathcal{H}_{v_{i+1}}$ and the spheres $S_{r_{j-1}}$ and $S_{r_{j+1}}$

$$\{v_i < v < v_{i+1}, \quad r_{j-1} \leqslant r < r_{j+1}, \quad i + j \text{ even}\}$$

from the Riemann problem:

$$(M^\sharp, V^\sharp)(v,r) := \mathcal{R}\Big(v, r; v_i, r_j; (M^\sharp, V^\sharp)(v_i+, r_{j\pm 1}), (a^\sharp, b^\sharp)(v_i+, r_j))\Big)$$

Step 4. Define the metric coefficients using the integral formulas:

$$a^\sharp(v,r) = 1 - \frac{4\pi(1+k^2)}{r} \int_0^r M^\sharp(v,s)\left(1 - 2K^2 V^\sharp(v,s)\right) \frac{b^\sharp(v,s)}{b^\sharp(v,r)}\, s^2 ds$$

$$b^\sharp(v,r) = \exp\left(4\pi(1+k^2)\int_0^r M^\sharp(v,r')\, r' dr'\right), \quad v \in (v_i, v_{i+1})$$

It remains to repeat Steps 1 to 4 within the next region limited by two light cones $\mathcal{H}_{v_{i+1}}$ and $\mathcal{H}_{v_{i+2}}$, and continue inductively for all hypersurfaces $\mathcal{H}_{v_1}, \mathcal{H}_{v_2}, \ldots$

UNIFORM BOUNDS

independent of the parameters of the discrete foliation

Invariant region principles Sup-norm bound

- Positivity $M > 0$ and condition $V < 0$ preserved by the Riemann problem
- Invariant region principle for the Riemann invariants:

Total variation principle BV bound

- $\log M$ enjoys a monotonicity property

IN THE REST OF THIS CHAPTER

- These two principles are now presented in the simplified situation of the *homogeneous Riemann problem* where the coupling to the geometry is "neglected".
- The full proof is quite technical (uniform estimates for the sup-norm and total variation, convergence of the approximation scheme) and will not be presented here (cf. the references below).

Section 4. THE RIEMANN PROBLEM IN EDDINGTON-FINKELSTEIN COORDINATES

LOCAL PROPERTIES OF THE ESSENTIAL EULER SYSTEM

- **Geometry/matter coupling:** we suppress the geometric terms.
- **Local behavior:** metric coefficients a, b treated as prescribed functions and, even, constants

$$\partial_v U + \partial_r F(U, a, b) = 0$$

$$U = M \begin{pmatrix} 1 \\ \frac{a}{2} + K^2 V \end{pmatrix}, \qquad F(U, a, b) = bM \begin{pmatrix} \frac{a}{2} + K^2 V \\ \frac{a^2}{4} + K^2 aV + V^2 \end{pmatrix}$$

- unknowns: mass-energy density $M > 0$ and velocity $V < 0$
- fixed metric coefficients $a \in \mathbb{R}$ and $b > 0$

$K^2 = \frac{1-k^2}{1+k^2} \in (0, 1)$

Strict hyperbolicity property. Fluid variables M and V in terms of U:

$$M = U_1, \qquad V = \frac{1}{K^2}\left(\frac{U_2}{U_1} - \frac{a}{2}\right)$$

Flux vector $F(U,a,b) = b\begin{pmatrix} U_2 \\ \frac{k^2}{(1-k^2)^2}\left(a^2U_1 - 4aU_2\right) + \frac{1}{K^4}\frac{U_2^2}{U_1}\end{pmatrix}$

Eigenvalues (wave speeds of the system)

$$\lambda_1 := b\left(\frac{1+k^2}{(1-k)^2}\frac{U_2}{U_1} - \frac{k}{(1-k)^2}a\right) \qquad \lambda_2 := b\left(\frac{1+k^2}{(1+k)^2}\frac{U_2}{U_1} + \frac{k}{(1+k)^2}a\right)$$

Since $k \in (0,1)$ and $b > 0$ and $V < 0$, one has $\lambda_1 < \lambda_2$.

Genuine nonlinearity property. Right-eigenvectors

$$r_1 := -\begin{pmatrix} 1 \\ \frac{\lambda_1}{b}\end{pmatrix} = -\begin{pmatrix} 1 \\ \frac{1+k}{1-k}V + \frac{a}{2}\end{pmatrix}, \qquad r_2 := \begin{pmatrix} 1 \\ \frac{\lambda_2}{b}\end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-k}{1+k}V + \frac{a}{2}\end{pmatrix}$$

$$\nabla\lambda_1 \cdot r_1 = -\frac{2k(1+k)}{(1-k)^3}\frac{bV}{M} \qquad \nabla\lambda_2 \cdot r_2 = -\frac{2k(1-k)}{(1+k)^3}\frac{bV}{M}$$

$$\nabla\lambda_1 \cdot r_1 > \nabla\lambda_2 \cdot r_2 > 0$$

Basic properties of the fluid model

The homogeneous part of the Euler system on a uniform Eddington-Finkelstein background is a strictly hyperbolic system of conservation laws, with real and distinct wave speeds

$$\lambda_1 = b\left(\frac{1+k}{1-k}V + \frac{a}{2}\right) \qquad \lambda_2 = b\left(\frac{1-k}{1+k}V + \frac{a}{2}\right)$$

satisfying the genuine nonlinearity property: $\nabla\lambda_1 \cdot r_1 > \nabla\lambda_2 \cdot r_2 > 0$.

$$\lambda_1 < 0 < \lambda_2 \quad \text{if and only if} \quad -\frac{1+k}{1-k}\frac{a}{2} < V < \min\left(0, -\frac{1-k}{1+k}\frac{a}{2}\right)$$

$$0 < \lambda_1 < \lambda_2 \quad \text{if and only if} \quad -\frac{1-k}{1+k}\frac{a}{2} < V < 0$$

$$\lambda_1 < \lambda_2 < 0 \quad \text{if and only if} \quad V < \min\left(0, -\frac{1+k}{1-k}\frac{a}{2}\right)$$

In a region where $a < 0$, both eigenvalues λ_1, λ_2 are negative and the fluid flows toward the center $r = 0$.

FORMATION OF SHOCK WAVES

- In view of the genuine nonlinearity property, the gradient of solutions blow-up in finite time.
- A Riccati-type equation along characteristic curves of the hyperbolic system. Analogue to Burgers equation $\partial_v u + u \partial_r u = 0$

THE HOMOGENEOUS RIEMANN PROBLEM

- Initial value problem with data prescribed on $\mathcal{H}_{v'}$
- A single jump located at some $r' \in (0, +\infty)$

$$U(v', r) = \begin{cases} U_L, & r < r' \\ U_R, & r > r' \end{cases}$$

- U_L (determined by M_L, V_L) and U_R (determined by M_R, V_R) are given constants: $M_L, M_R > 0$ and $V_L, V_R < 0$

Observations.

- Invariance of the Riemann problem by self-similar scaling
- Solution depending upon the variable $(r - r')/(v - v')$

Theorem. The Riemann problem on an Eddington-Finkelstein background

The Riemann problem associated with the homogeneous Euler system, on a uniform Eddington-Finkelstein background

- admits a unique self-similar weak solution depending upon the self-similarity variable $(r - r')/(v - v')$
- made of two waves, each being
  - a rarefaction wave
  - or a shock wave (see below)

Moreover, for each constant $\rho > 0$, the set $\Omega_\rho := \{(w, z) \,|\, -\rho \leqslant w, z \leqslant \rho\}$ is an invariant domain for the Riemann problem:

- If the data U_L, U_R belong to Ω_ρ for some $\rho > 0$, then so does the solution for times $v \geqslant v'$.

(Riemann invariants w, z defined below)

Riemann invariants w, z associated with the Euler system

- By definition, w, z are constant along the integral curves of the eigenvectors, that is, $D_U w(U) \cdot r_1(U) = 0$ and $D_U z(U) \cdot r_2(U) = 0$
- In the coordinates (M, V):

$$w(M, V) := \log |V| - \frac{2k}{(1-k)^2} \log M$$
$$z(M, V) := \log |V| + \frac{2k}{(1+k)^2} \log M$$

RAREFACTION WAVES: Continuous solutions determined from integral curves of r_1, r_2.

- Solutions $U = U((r - r')/(v - v'))$ satisfying the equation in the self-similar variable $\xi := (r - r')/(v - v')$, that is,
$$(D_U F(U) - \xi \, Id) \, \partial_\xi U = 0$$
- Characterization of the rarefaction curves passing through (M_0, V_0) in the phase space:
  - The 1-rarefaction curve $\subset \{w(U) = w(U_0)\}$
  - The 2-rarefaction curve $\subset \{z(U) = z(U_0)\}$

In Riemann invariant coordinates:

$$\mathbf{R}_1^{\rightarrow}(U_L) = \Big\{(w,z)\,\Big|\,w(M,V) = w(M_L,V_L) \text{ and } z(M,V) \leqslant z(M_L,V_L)\Big\}$$

$$\mathbf{R}_2^{\leftarrow}(U_R) = \Big\{(w,z)\,\Big|\,w(M,V) \geqslant w(M_R,V_R) \text{ and } z(M,V) = z(M_R,V_R)\Big\}$$

Rarefaction curves associated with the Euler system

The 1-rarefaction curve $\mathbf{R}_1^{\rightarrow}(U_L)$ from $U_L = (M_L, V_L)$

$$\mathbf{R}_1^{\rightarrow}(U_L) = \left\{ M = M_L \left(\frac{V}{V_L}\right)^{\frac{(1-k)^2}{2k}} \ ; \quad \frac{V}{V_L} \in (0,1] \right\}$$

The 2-rarefaction curve $\mathbf{R}_2^{\leftarrow}(U_R)$ from $U_R = (M_R, V_R)$

$$\mathbf{R}_2^{\leftarrow}(U_R) = \left\{ M = M_R \left(\frac{V}{V_R}\right)^{-\frac{(1+k)^2}{2k}} \ ; \quad \frac{V}{V_R} \in [1,+\infty) \right\}$$

Monotonicity properties

- Along $\mathbf{R}_1^{\rightarrow}(U_L)$, the wave speed $\lambda_1(V)$ is increasing when V increases from V_L.
- Along $\mathbf{R}_2^{\leftarrow}(U_R)$, the wave speed $\lambda_2(V)$ is decreasing when V decreases from V_R.

M is decreasing in both cases and

$$\lim_{V\to 0} M|_{\mathbf{R}_1^{\rightarrow}(U_L)} = \lim_{V\to -\infty} M|_{\mathbf{R}_2^{\leftarrow}(U_R)} = 0$$

Proof of the monotonicity property.

- The associated speeds $\lambda_1(V)$ and $\lambda_2(V)$ increase when V increases.
- The monotonicity and limiting behavior for M follows.
- Along the curve $\mathbf{R}_1^{\rightarrow}(U_0)$:

$$\begin{aligned} z(M,V) &= \log|V| + \frac{2k}{(1+k)^2}\log M \\ &= \log|V| + \frac{2k}{(1+k)^2}\log M_0 + \frac{(1+k)^2}{(1-k)^2}\log\frac{V}{V_0} \\ &\leqslant \log|V_0| + \frac{2k}{(1+k)^2}\log M_0 = z(M_0,V_0). \end{aligned}$$

- Similarly, along the curve $\mathbf{R}_2^{\leftarrow}(U_0)$, we obtain $w(M,V) \geqslant w(M_0,V_0)$.

SHOCK WAVES

- **Rankine-Hugoniot conditions**: Two constant states U_L, U_R separated by a single discontinuity, propagating at the speed $s = s(U_L, U_R)$ determined by

$$s\,[U] = [F(U)]$$
$$[U] := U_R - U_L, \qquad [F(U)] := F(U_R) - F(U_L)$$

- **Shock admissibility inequalities**

$$\lambda_i(V_L) > s_i > \lambda_i(V_R) \qquad i\text{-shock}$$

together with $s_1 < \lambda_2(V_R)$ and $\lambda_1(V_L) < s_2$ are imposed in order to guarantee uniqueness (and stability) of the Riemann solution.

Notation.

$$\Phi_\pm(\beta) := \frac{1}{2(1-K^4)\beta^2}\left(1 - 2K^4\beta + \beta^2 \pm (1-\beta)\sqrt{(1+\beta)^2 - 4K^4\beta}\right)$$

$$\Sigma_\pm(V_0, \beta) := \frac{ab}{2} + \frac{bV_0}{2K^2}\left(1 + \beta \pm \sqrt{(1+\beta)^2 - 4K^4\beta}\right)$$

Shock curves associated with the Euler system

The 1-shock curve issuing from a state U_L and the corresponding speed:

$$\mathbf{S}_1^{\rightarrow}(U_L) = \left\{ M = M_L \, \Phi_- \left(\frac{V}{V_L} \right) \quad \text{and} \quad \frac{V}{V_L} \in [1, \infty) \right\}$$

$$s_1(U_L, U) = \Sigma_+ \left(V_L, \frac{V}{V_L} \right)$$

The 2-shock curve issuing from the state U_R and the shock speed:

$$\mathbf{S}_2^{\leftarrow}(U_R) = \left\{ M = M_R \, \Phi_+ \left(\frac{V}{V_R} \right) \quad \text{and} \quad \frac{V}{V_R} \in (0, 1] \right\}$$

$$s_2(U, U_R) = \Sigma_- \left(V_R, \frac{V}{V_R} \right)$$

Monotonicity properties

- The 1-shock speed s_1 is increasing for V decreasing.
- The 2-shock speed s_2 is decreasing for V increasing.
- The shock admissibility inequalities hold.

– Along $\mathbf{S}_1^{\rightarrow}(U_L)$, the mass density M increases and reaches $\frac{M_L}{1-K^4}$ as $V \to -\infty$.
– Along $\mathbf{S}_2^{\leftarrow}(U_R)$, M increases and blows up as $V \to 0$.

Shock curves in Riemann invariant coordinates (w, z)

For $\beta = \frac{V}{V_L} \in [1, +\infty)$:

$$\mathbf{S}_1^{\rightarrow}(U_L): \begin{cases} w - w_L = \log\beta - \frac{2k}{(1-k)^2}\log(\Phi_-(\beta)) \\ z - z_L = \log\beta + \frac{2k}{(1+k)^2}\log(\Phi_-(\beta)) \end{cases}$$

For $\beta = \frac{V}{V_R} \in (0, 1]$:

$$\mathbf{S}_2^{\leftarrow}(U_R): \begin{cases} w - w_R = \log\beta - \frac{2k}{(1-k)^2}\log(\Phi_+(\beta)) \\ z - z_R = \log\beta + \frac{2k}{(1+k)^2}\log(\Phi_+(\beta)) \end{cases}$$

Existence of the Riemann solution. Constructed by pasting together constant states, shock waves, and rarefaction waves.

Construction in the phase space.

- 1-**wave curve** issuing from the data U_L,

$$\mathbf{W}_1^{\rightarrow}(U_L) := \mathbf{R}_1^{\rightarrow}(U_L) \cup \mathbf{S}_1^{\rightarrow}(U_L)$$

  parametrized by $\beta \in (0, 1]$ within the rarefaction part $\mathbf{R}_1^{\rightarrow}(U_L)$, and $\beta \in [1, +\infty)$ within the shock part $\mathbf{S}_1^{\rightarrow}(U_L)$.
- 2-**wave curve** $\mathbf{W}_2^{\rightarrow}(U_R)$ defined similarly

Construction in the physical space. The Riemann problem is solved by observing that

- These two curves intersect at a unique point $U_* \in \mathbf{W}_1^{\rightarrow}(U_L) \cap \mathbf{W}_2^{\leftarrow}(U_R)$.
- The Riemann solution is a 1-wave connecting U_L to U_*, followed by a 2-wave from U_* to U_R.

Validity of this construction. For any initial states U_L, U_R satisfying $M_L, M_R > 0$ and $V_L, V_R < 0$:

- The wave speeds arising in the Riemann solution do increase from left to right.
- V decreases from 0 toward $-\infty$, while M increases from 0 to $\frac{M_L}{1-K^4}$ along the curve $\mathbf{W}_1^{\rightarrow}(U_L)$.
- Along $\mathbf{W}_2^{\rightarrow}(U_R)$, the velocity V decreases from 0 to $-\infty$, while the mass density M decreases from $+\infty$ toward 0.

In view of these global monotonicity properties, the intersection point $U_* \in \mathbf{W}_1^{\rightarrow}(U_L) \cap \mathbf{W}_2^{\leftarrow}(U_R)$ exists and is unique.

Invariant domains. Any Ω_ρ is an invariant region for the Riemann problem.

- We write w_L for $w(U_L)$, etc. and, for definiteness, we suppose $U_* \in \mathbf{R}_1^{\leftarrow}(U_L) \cap \mathbf{R}_2^{\rightarrow}(U_R)$.
- Then, we have $w = w_L$ and $z \leqslant z_L$ for all states between U_L and U_*,
- while $w \geqslant w_R$ and $z = z_R$ for all states between U_* and U_R.

Therefore:

$$w_R \leqslant w = w_L, \qquad z_R = z \leqslant z_L$$

along the solution of the Riemann problem, and, in particular $w, z \in [-\rho, \rho]$ if $w_L, w_R, z_L, z_R \in [-\rho, \rho]$.

Furthermore, both shock curves $\mathbf{S}_1^{\rightarrow}(U_L)$ and $\mathbf{S}_2^{\leftarrow}(U_R)$

- remain within an upper-left triangle in the (w, z)-plane, so that
- if intersected with each other or with $\mathbf{R}_2^{\leftarrow}(U_R)$ and $\mathbf{R}_1^{\rightarrow}(U_L)$, respectively, the corresponding Riemann solution belongs to the region Ω_ρ.

Tangent to the shock curve $\mathbf{S}_1^{\rightarrow}(U_L)$ in the (w,z)-plane:

$$\frac{dw}{dz} = \frac{d(w-w_L)}{d(z-z_L)} = \frac{d(w-w_L)}{d\beta}\left(\frac{d(z-z_L)}{d\beta}\right)^{-1}$$

$$= \frac{(1+k)^2\left(1+k^2-\frac{2k(1+\beta)}{\sqrt{(1+\beta)^2-4K^4\beta}}\right)}{(1-k)^2\left(1+k^2+\frac{2k(1+\beta)}{\sqrt{(1+\beta)^2-4K^4\beta}}\right)} < 1$$

The shock curve $\mathbf{S}_1^{\rightarrow}$ is convex in z:

$$\frac{d}{d\beta}\frac{dw}{dz} = \frac{8k(1+k)^2K^2(-1+\beta)}{\sqrt{(1+\beta)^2-4K^4\beta}\left(\sqrt{(1+\beta)^2-4K^4\beta}+k\left(2+2\beta+k\sqrt{(1+\beta)^2-4K^4\beta}\right)\right)}$$

is non-negative since $\beta \geqslant 1$.

Since $\frac{2k}{\sqrt{1-K^4}} = 1+k^2$, we have

$$\lim_{\beta\to 1+}\frac{dw}{dz} = \frac{(1+k)^2\left(1+k^2-\frac{2k}{\sqrt{1-K^4}}\right)}{(1-k)^2\left(1+k^2+\frac{2k}{\sqrt{1-K^4}}\right)} = 0,$$

and the second–order derivative being positive, we conclude that $\frac{dw}{dz} \in [0,1]$.

Similarly, the curve $\mathbf{S}_2^{\leftarrow}(U_R)$ is increasing and concave in the (w,z)-plane.

TOTAL VARIATION MONOTONICITY PRINCIPLE

BV estimate for a pair of Riemann solutions associated with the Euler system

- Assume that the initial fluid data consists of three equilibria U_L, U_M, U_R.
- Given $0 < r' < r'' < +\infty$, prescribe the following initial data at $v = v'$

$$U(v', r) = \begin{cases} U_L, & r < r' \\ U_M, & r' < r < r'' \\ U_R, & r > r'' \end{cases}$$

Combine the Riemann solutions associated with the data U_L, U_M and U_M, U_R

Total variation $\mathcal{E}(U_L, U_R)$ of a Riemann solution (U_L, U_R)

- Sum the magnitude of shock and rarefaction waves
- Use the variable $\log M$ Recall that $M := b^2 \mu \, u^0 u^0$

We introduce

$$\mathcal{E}(U_L, U_R) := |\log M_R - \log M_*| + |\log M_* - \log M_L|$$

where M_* denotes the mass-energy density for the intermediate state $U_* \in \mathbf{W}_1^{\leftarrow}(U_L) \cap \mathbf{W}_2^{\rightarrow}(U_R)$.

Proposition. Total Variation Monotonicity Principle

Given arbitrary states U_L, U_M, U_R, the total variation of the solutions to the Riemann problems (U_L, U_M), (U_M, U_R), and (U_L, U_R) satisfy

$$\mathcal{E}(U_L, U_R) \leqslant \mathcal{E}(U_L, U_M) + \mathcal{E}(U_M, U_R).$$

Proof. Consider the wave curves in the plane of the Riemann invariants.

- Rarefaction curves are straightlines
- The shock curves have the same shape independently of the base points U_L or U_R. and are described by the functions $\Phi_\pm$

We observe the identity

$$\begin{aligned}\Phi_-(\beta)\Phi_+(\beta) &= \Big(4(1-K^4)^2\beta^4\Big)^{-1}\Big(\big(1-2K^4\beta+\beta^2\big)^2-(1-\beta)^2\big((1+\beta)^2-4K^4\beta\big)\Big)\\ &= \Big(4(1-K^4)^2\beta^4\Big)^{-1}\\ &\quad\cdot\Big(\big(1-\beta)^2+2\beta(1-K^4)\big)^2-(1-\beta)^2\big((1-\beta)^2-4\beta(1-K^4\beta)\big)\Big)\\ &= 1\end{aligned}$$

$$\log(\Phi_-(\beta)) = -\log(\Phi_+(\beta))$$

- The expressions for the shock curves coincide up to a change of the role of w, z.
- The shock curves are symmetric with respect to the $w = z$ axis.
- We observe that the total variation, by definition, is measured along the $w = z$ axis in the Riemann invariant coordinates.
- These symmetry properties imply the desired monotonicity property.

Section 5. SPHERICALLY SYMMETRIC STATIC EINSTEIN-EULER SPACETIMES

FORMULATION of the STATIC FLUID PROBLEM

Static solutions (M, V) satisfying $\partial_v M = \partial_v V = 0$, thus

$$\partial_r F(U, a, b) = S(U, a, b).$$

- Timelike Killing field Y^α, foliation whose leaves are orthogonal to Y^α
- Example: outer domain of communication of Schwarzschild ($r > 2m$)

Rescaled mass-energy density $M = M(r)$ and **Hawking mass** $m = m(r) = \frac{r}{2}\left(1 - |\nabla r|^2\right)$ (defined by $a = 1 - \frac{2m}{r}$):

$$m_r = 2\pi r^2 M(1 + k^2)\left(1 - \frac{2m}{r} + 2K^2 |V|\right)$$

$$m_v = \pi r^2 bM(1 + k^2)\left(4V^2 - \left(1 - \frac{2m}{r}\right)^2\right)$$

- The function $r \mapsto m(v, r)$ is increasing, provided $\frac{a}{2V} < K^2$.
  - holds if, for instance, a is positive
  - but remains true even for negative values of a (trapped region) if the normalized velocity is sufficient large.
- The function $v \mapsto m(v, r)$ is decreasing provided the ratio $\frac{|a|}{2|V|}$ is greater than 1.

Curved geometry

- The condition $\partial_v U = 0$ implies that $M_v = 0$ and, thus, b_v and a_v (resp. m_v) vanish.
- By the T^{11} Einstein equation, we have $a^2 = 4V^2$.

Fluid variables. Returning to the definitions of M, V, we find

- $u^1 = 0$ and $ab^2(u^0)^2 = 1$
- which implies $aM = \left(1 - \frac{2m}{r}\right) M = \mu$.
- Velocity $V = -\frac{a}{2} = \frac{m}{r} - \frac{1}{2}$

Formulation of the static problem

All solution to the static Euler equations having $a > 0$ satisfy a system of first-order ordinary differential equations in m, μ defined for all radius $r \in (0, +\infty)$:

$$m_r = 4\pi r^2 \mu$$

$$\mu_r = -\frac{(1+k^2)\mu}{r-2m}\left(4\pi r^2 \mu + \frac{m}{rk^2}\right)$$

- The functions V, M, a are recovered by algebraic formulas
- The metric coefficient b is given by

$$b(r) = \exp\left(4\pi(1+k^2)\int_0^r \frac{r'^2\mu(r')}{r'-2m(r')}dr'\right)$$

provided $\frac{r^2\mu}{r-2m}$ is integrable at the center.

EXISTENCE OF STATIC EQUILIBRIA

– Prescribe initial conditions at the center $r = 0$, specifically $\mu_0 > 0$ and $m_0 = 0$.

– The condition on the initial value on m is consistent with m being non-negative, however, we need (in the proof of the following theorem) to check whether the second equation makes sense.

Theorem

There exists a unique global solution (m, μ) to the static Einstein-Euler system with prescribed initial conditions $m_0 = 0$ and $\mu_0 > 0$ at the center:

$$\lim_{r\to 0} m(r) = 0, \qquad \lim_{r\to 0} \mu(r) = \mu_0.$$

- The functions m, μ are smooth and positive on $(0, +\infty)$ and $\lim_{r\to +\infty} \mu(r) = 0$.
- These static solutions (M, V, a, b) satisfy our weak regularity conditions.
- In particular, the regularity at the center is satisfied

$$\lim_{r\to 0} (a, b)(r) = (1, 1)$$

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Weakly regular self-gravitating fluids (Gowdy and plane symmetry)

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NEXT OBJECTIVE. Some results on the global geometry and causal properties of weakly regular spacetimes

CHAPTER V. The Geometry of Weakly Regular Spacetimes with Symmetry

Properties of Cauchy developments

- Global issues, late-time asymptotics
- Weak regularity (natural energy, shock waves, gravitational waves)
- Non-trivial effects generated by the matter

OUR MAIN OBJECTIVE in this chapter

- Section 1. **Formation of trapped spheres in self-gravitating fluids** spherical symmetry, compressible fluid, shock waves
- Section 2. **Crushing singularities in self-gravitating fluids in Gowdy symmetry**
- Section 3. **Future geodesic completeness of weakly regular Gowdy spacetimes** define geodesics, rate of energy decay, sharp asymptotics
- Section 4. **Geodesic completeness of polarized, weakly regular T^2 symmetric spacetimes**

Section 1. FORMATION OF TRAPPED SPHERES in self-gravitating fluids

Section 1.1 THE BOUNDED VARIATION (BV) FORMULATION

Gravitational collapse of compressible matter with shocks

Two typical behaviors

- Dispersion of the matter in future timelike directions (sufficiently small mass-energy)
- Collapse of the matter (focusing of light rays)

Set-up

- Spherically symmetric, weakly regular spacetimes with bounded variation
- Converging sequence of solutions to the Einstein-Euler system
- The Riemann problem (evolution of a single discontinuity)

Objective

- Short-pulse ansatz (following Christodoulou)
- Formation of trapped surfaces

MAIN STATEMENT in a rough form. Initial value problem for

- spherically-symmetric initial data sets $(\mathcal{H}, \overline{g}, \overline{\rho}, \overline{j})$
- a large class specified explicitly by a short-pulse ansatz
- weak regularity (bounded variation)

a spherically symmetric, Einstein-Euler future Cauchy development $(\mathcal{M}, g, \mu, u)$ containing shock waves

- the initial hypersurface does not contain trapped spheres and
- the Cauchy development does contain trapped spheres.

Remark. Large data result. Geodesically incomplete spacetime (Penrose-Hawking's singularity theorem)

Eddington-Finkelstein coordinates

- $g = -ab^2\,dv^2 + 2b\,dvdr + r^2\,g_{S^2}$ with $v \in [v_0, v_*)$, $r \in [0, r_0)$
- $b = b(v, r) > 0$ but $a = a(v, r)$ may change sign
- regularity at the center: $\lim_{r\to 0}(a, b)(v, r) = (1, 1)$

ESSENTIAL EINSTEIN-EULER SYSTEM IN REDUCED FORM

Normalized mass-energy $M := b^2 \mu \, u^0 u^0 \in (0, +\infty)$

Normalized fluid velocity $V := \frac{u^1}{b\, u^0} - \frac{a}{2} \in (-\infty, 0)$ with $u^1 = \frac{1}{2}\left(abu^0 - \frac{1}{bu^0}\right)$

System of two nonlinear hyperbolic equations $K^2 := \frac{1-k^2}{1+k^2} \in (0,1)$

$$\partial_v U + \partial_r F(U,a,b) = S(U,a,b)$$

$$U := M \begin{pmatrix} 1 \\ \frac{a}{2} + K^2 V \end{pmatrix} \qquad F(U,a,b) := bM \begin{pmatrix} \frac{a}{2} + K^2 V \\ \frac{a^2}{4} + K^2 aV + V^2 \end{pmatrix}$$

$$S(U,a,b) := \begin{pmatrix} S_1(M,V,a,b) \\ S_2(M,V,a,b) \end{pmatrix} \qquad S_1(M,V,a,b) := -\frac{1}{2r} bM \Big(1 + a + 4V\Big)$$

$$S_2(M,V,a,b) := -\frac{1}{2r} bM \Big(a^2 + 2aV(2+K^2) - 2K^2 V + 4V^2\Big) - 16\pi(1-K^2)\, rb\, M^2 V^2$$

$$a(v,r) = 1 - \frac{4\pi(1+k^2)}{r} \int_0^r \frac{b(v,r')}{b(v,r)} M(v,r') \big(2K^2 |V(v,r')| + 1\big) \, r'^2 dr'$$

$$b(v,r) = \exp\left(4\pi(1+k^2) \int_0^r M(v,r')\, r' dr'\right)$$

BOUNDED VARIATION SOLUTIONS

- strictly hyperbolic, genuinely nonlinear
- formation of shocks in finite time

Definition

A **spherically symmetric, Einstein-Euler spacetime with bounded variation** in Eddington-Finkelstein coordinates

$$g = -ab^2\, dv^2 + 2b\, dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

with $v \in I := [v_0, v_*)$ and $r \in J := [0, r_0)$ is a weak solution to the Einstein-Euler system with the following regularity:

- Normalized mass density and velocity

$$rM,\ V \in L^\infty_{\mathrm{loc}}(I, BV_{\mathrm{loc}}(J)) \cap \mathrm{Lip}_{\mathrm{loc}}(I, L^1_{\mathrm{loc}}(J))$$

- Metric coefficients a, b

$$a_v,\ ra_r,\ b_r \in L^\infty_{\mathrm{loc}}(I, BV_{\mathrm{loc}}(J)) \cap \mathrm{Lip}_{\mathrm{loc}}(I, L^1_{\mathrm{loc}}(J))$$

STATIC EINSTEIN-EULER EQUATIONS

Closed system for the Hawking mass $m = \frac{r}{2}\left(1 - |\nabla r|^2\right) = \frac{r}{2}(1-a)$ and the fluid density μ

$$m_r = 4\pi r^2 \mu$$

$$\mu_r = -\frac{(1+k^2)\mu}{r-2m}\left(4\pi r^2\mu + \frac{m}{rk^2}\right)$$

The remaining unknowns V, M, a, b are recovered by

$$M(r) = \frac{\mu(r)}{1 - \frac{2m(r)}{r}} \qquad V(r) = \frac{m(r)}{r} - \frac{1}{2}$$

$$a(r) = 1 - \frac{2m(r)}{r} \qquad b(r) = \exp\left(4\pi(1+k^2)\int_0^r \frac{r'^2\mu(r')}{r' - 2m(r')}dr'\right)$$

Theorem

Given any mass-energy density $\mu_0 > 0$ at the center $r = 0$, there exists a unique global solution (m, μ) to the static Einstein-Euler system in Eddington-Finkelstein coordinates with prescribed values at the center of spherical symmetry

$$\lim_{r\to 0} m(r) = 0, \qquad \lim_{r\to 0}\mu(r) = \mu_0.$$

m, μ are smooth and positive on $(0, +\infty)$ with $\lim_{r\to+\infty}\mu(r) = 0$.

Section 1.2 THE FORMATION OF TRAPPED SURFACES

- A given static spacetime $(\widetilde{M}, \widetilde{V}, \widetilde{a}, \widetilde{b})$: a fluid at equilibrium
- Initial data set prescribed on the incoming null cone $\mathcal{H}_{v_0}$
- Localized (compactly supported) and large perturbation

$$M_0 = \widetilde{M} + M_0^{(1)} \qquad V_0 = \widetilde{V} + V_0^{(1)}$$
$$a_0 = \widetilde{a} + a_0^{(1)} \qquad b_0 = \widetilde{b} + b_0^{(1)}$$

- Initial data assumed to be *untrapped*

$a_0 > 0$ on the hypersurface $\mathcal{H}_{v_0}$

Challenge

- existence on a sufficiently long interval $[v_0, v_*)$ so that trapped surfaces do form in the development
- need to carefully monitor the evolution (and sign) of the metric coefficient a !
- bounds uniforms along the sequence of approximate solutions to the Einstein-Euler system

THE SHORT PULSE DATA

A perturbation with "strength" $1/h$ which is *initially localized* on the interval $[r_* - \delta, r_* + \delta]$ for some small $\delta > 0$ and some finite radius r_*.

- **Normalized velocity**: a short-pulse, step-like function

$$V_0^{(1)}(r) := \begin{cases} 0, & r < r_* - \delta \\ \frac{\tilde{V}(r)}{h}, & r \in [r_* - \delta, r_* + \delta] \\ 0, & r > r_* + \delta \end{cases}$$

- **Normalized mass-energy density**: no perturbation needed

$$M_0^{(1)} = 0$$

(Our method applies to more general perturbations.)

Heuristics.

- Recall that $a_v = 2\pi rbM(1 + k^2)\left(a^2 - 4V^2\right)$
- Therefore, a_v is initially large (negative) within a small interval $[r_* - \delta, r_* + \delta]$.
- Expect a to become negative as time evolves.
- Challenge: *show that this is the dominant effect by "monitoring" the evolution of the short -pulse !*

Domain of influence / Domain of dependence

The spacetime remains static in a neighborhood of the center of symmetry

- Radius functions $R^{\pm}_{*\mp}(v) = r_* \pm \delta \mp C_*(v - v_0)$
- C_*: Bound on the sound speed, determined by the speed of propagation of the (rarefaction, shock) waves
- The support of the solution expands in time:
  - domain of influence of the initial data
    perturbation on the hypersurface $\mathcal{H}_v$ supported within
    $$\text{supp}\ \big((M, V)(v, \cdot) - (\widetilde{M}, \widetilde{V})\big) \subset \big\{R^{-}_{*-}(v) \leqslant r \leqslant R^{+}_{*+}(v)\big\}$$
  - domain of dependence of the initial data $\big\{R^{-}_{*+}(v) \leqslant r \leqslant R^{+}_{*-}(v)\big\}$
  - uniform a priori bound on the support: we will work in the interval $[r_* - \Delta, r_* + \Delta]$ within the time interval of interest.

Recall that

$$a(v,r) = 1 - \frac{4\pi(1+k^2)}{r}\int_0^r \frac{b(v,r')}{b(v,r)} M(v,r')\left(2K^2\,|V(v,r')| + 1\right) r'^2 dr'$$

$$b(v,r) = \exp\left(4\pi(1+k^2)\int_0^r M(v,r')\, r'\, dr'\right)$$

Observe that $b_0^{(1)} = 0$ and $a_0 = \tilde{a} + a_0^{(1)}$

$$a_0(r) = 1 - \frac{4\pi(1+k^2)}{r}\int_0^r \frac{b^{(0)}(s)}{b^{(0)}(r)} M^{(0)}(s)\left(1 + K^2\left(1 + \frac{1}{h}\chi_{[r_* - \delta, r_* + \delta]}\right)\tilde{a}(s)\right) s^2 ds$$

where $\chi_{[r_* - \delta, r_* + \delta]}$ denotes the characteristic function of the interval.

Lemma

Given $r_* > \Delta > 0$, there exist constants $C_1, C_2, C_3 > 0$ depending on r_* and Δ such that for all $\delta, h > 0$ with $\frac{\delta}{h} \leqslant \frac{1}{C_1}$

$$0 < a_0(r) \leqslant \tilde{a}(r), \qquad r \in [0, r_* + \Delta] \qquad \text{(non-trapped)}$$

$$a_v(v_0, r) \begin{cases} = 0, & r \in [0, r_* - \delta) \qquad \text{(static region)} \\ \leqslant -C_2 \frac{\delta}{h^3} & r \in [r_* - \delta, r_* + \delta] \qquad \text{(short-pulse region)} \\ \leqslant -C_3 \frac{\delta}{h}, & r \in (r_* + \delta, r_* + \Delta] \end{cases}$$

SEQUENCE OF APPROXIMATE SOLUTIONS $M^\sharp, V^\sharp, a^\sharp, b^\sharp$

(defined in Chapter IV) to the Einstein-Euler system based on the RIEMANN PROBLEM

$$\partial_v U + \partial_r F(U, a, b) = S(U, a, b) \qquad (M, V)(v', r) = \begin{cases} (M_L, V_L), & r < r' \\ (M_R, V_R), & r' < r \end{cases}$$

- Blow-up near the initial discontinuity: geometric effects suppressed
- Fundamental scale-invariant wave structures: shocks, rarefactions

Proposition

The homogeneous Riemann problem associated with the Euler system on a uniform Eddington-Finkelstein background admits a unique self-similar solution $U = U((r - r'/(v - v'))$ made of shock waves or rarefaction waves.

Rarefaction curves

$$\mathbf{R}_1^{\rightarrow}(U_L) = \left\{ M = M_L \left(\frac{V}{V_L}\right)^{\frac{(1-k)^2}{2k}}; \quad \frac{V}{V_L} \in (0, 1] \right\}$$

$$\mathbf{R}_2^{\leftarrow}(U_R) = \left\{ M = M_R \left(\frac{V}{V_R}\right)^{-\frac{(1+k)^2}{2k}}; \quad \frac{V}{V_R} \in [1, +\infty) \right\}$$

Shock curves (for some explicit functions $\Phi_\pm$)

$$\mathbf{S}_1^{\rightarrow}(U_L) = \left\{ M = M_L \, \Phi_- \left(\frac{V}{V_L}\right); \quad \frac{V}{V_L} \in [1, +\infty) \right\}$$

$$\mathbf{S}_2^{\leftarrow}(U_R) = \left\{ M = M_R \, \Phi_+ \left(\frac{V}{V_R}\right); \quad \frac{V}{V_R} \in (0, 1] \right\}$$

Definition

An approximate solution $M^\sharp, V^\sharp, a^\sharp, b^\sharp$ (as defined in Chapter IV) to the Euler–Einstein system is said to satisfy the **short-pulse property** if

- there exist constants $C_0, C, C_b, \Lambda > 0$ (depending only on the chosen static solution)
- an exponent $\kappa > 1$ (depending on the sound speed k),
- defined for all $v \in [v_0, v_*)$ with $v_* := v_0 + \tau h^\kappa$

- **In the domain of influence of the initial pulse**

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)^{-\kappa_0} \leqslant M^\sharp(v, r) \leqslant C_0 e^{C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)^{\kappa_0}$$

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \leqslant -V^\sharp(v, r) \leqslant C_0 e^{C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)$$

- **In the domain of dependence of the initial pulse** (bounded matter density + very large negative velocity)

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \leqslant M^\sharp(v, r) \leqslant C_0 e^{C\frac{v-v_0}{h^\kappa}}$$

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right) \leqslant -V^\sharp(v, r) \leqslant C_0 e^{C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)$$

Theorem

Fix $k \in (0,1)$ small enough. Given any $\mu_0 > 0$, let $\widetilde{M}, \widetilde{V}, \widetilde{a}, \widetilde{b}$ be the static solution with density μ_0 at the center.

Initial data set

- Fix any $r_* > \Delta > 0$ together with perturbation parameters $h, \delta > 0$ satisfying $\delta \leqslant \frac{h}{C_1}$ with C_1 as above.
- Let (M_0, V_0, a_0, b_0) be a short-pulse perturbation of this static solution.

There exist $\tau, \kappa > 0$ such that for all light-cones $\mathcal{H}_{v_0}, \mathcal{H}_{v_1}, \mathcal{H}_{v_2}, \dots$ and spheres $S_{r_1}, S_{r_2}, S_{r_3}, \dots$ of the discrete spacetime foliation:

- the approximate solutions $M^\sharp, V^\sharp, a^\sharp, b^\sharp$ to the Einstein-Euler system (defined in Chapter IV) are **well-defined** on the (large) time interval $[v_0, v_*)$ with $v_* = v_0 + \tau h^\kappa$
- satisfy the **short-pulse property**

- **BV regularity** in spacelike directions

$$\sup_{v \in [v_0, v_*)} TV\Big((rM^\sharp, V^\sharp, ra_r^\sharp, b_r^\sharp, a_v^\sharp)(v, \cdot) - (r\widetilde{M}, \widetilde{V}, r\widetilde{a}_r, \widetilde{b}_r, 0)\Big) \lesssim N_0$$

- **Lipschitz continuity** in timelike directions $v, v' \in [v_0, v_*)$

$$\|(rM, V)(v, \cdot) - (rM, V)(v', \cdot)\|_{L^1(0,+\infty)} \lesssim N_0 \, |v - v'|$$

with $N_0 := TV_0^{+\infty}\big(r(M_0 - \widetilde{M})\big) + TV_0^{+\infty}\big(V_0 - \widetilde{V}\big)$.

Moreover, the sequence of approximate solution $(M^\sharp, V^\sharp, a^\sharp, b^\sharp)$ **converges pointwise to a limit** (M, V, a, b) which

- is a bounded variation solution to the Einstein-Euler system in spherical symmetry
- satisfies an initial condition on $\mathcal{H}_{v_0}$ without any trapped surface
- admits trapped surfaces in the future of the initial hypersurface $\mathcal{H}_{v_0}$.

Section 2. CRUSHING SINGULARITIES in self-gravitating fluids in Gowdy symmetry

- **Einstein equations** in the weak sense $R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = T_{\alpha\beta}$
- **Compressible fluids** $T_{\alpha\beta} = (\mu + p(\mu)) u_\alpha u_\beta + p(\mu) g_{\alpha\beta}$

  future unit timelike velocity vector u^α, mass-energy density μ
  isothermal pressure law $p(\mu) = k^2 \mu$
- **Gowdy symmetry on** T^3
  - Action by a group of isometries generated by two commuting, spacelike Killing fields X, Y
  - Vanishing twists $\varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta = \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta = 0$
  - Inhomogeneous cosmology: big bang / big crunch
  - Include gravitational waves (contrary to spherical symmetry)
- **Weak solutions to the Einstein equations**
  - Weak formulation of the Einstein equations
  - Allows for impulsive gravitational waves, shock waves
- **Existence of weakly regular Cauchy developments**
  - foliation by spacelike hypersurfaces, areal time function
  - behavior near the future boundary of the spacetime

We state here a typical result based on the methods in Chapters II to IV.

Metric in areal coordinates $R \in [t_0, t_1)$, $\theta \in S^1$ and $x, y \in S^1$

$$g = e^{2(\nu-U)}\left(-dR^2 + \alpha^{-1} d\theta^2\right) + e^{2U}\Big(dx + A\,dy\Big)^2 + e^{-2U} R^2 dy^2$$

Einstein evolution equations in a weak sense

$$\left(t^{-1}\, a^{-1} A_t\right)_t - \left(t^{-1}\, a A_\theta\right)_\theta = -\frac{4}{at}\left(U_t A_t - a^2 U_\theta A_\theta\right)$$

$$\left(t\, a^{-1}(U_t - 1/(2t))\right)_t - \left(t\, a U_\theta\right)_\theta = \frac{e^{4U}}{2ta}\left(A_t^2 - a^2 A_\theta^2\right)$$

$$\left(ta^{-1}(\nu_t + te^{2(\nu-U)}\mu(1-k^2))\right)_t - (ta\nu_\theta)_\theta = 2atU_\theta^2 + \frac{e^{4U}}{2at}A_t^2 + ta^{-1}\, e^{2(\nu-U)}\mu\, \frac{(1+k^2)}{1-v^2}$$

Einstein constraint equations in a weak sense

$$a_t = -ate^{2(\nu-U)}\mu(1-k^2)$$

$$\nu_t = t(U_t^2 + a^2 U_\theta^2) + \frac{e^{4U}}{4t}(A_t^2 + a^2 A_\theta^2) + te^{2(\nu-U)}\mu\, \frac{k^2+v^2}{1-v^2}$$

$$\nu_\theta = -2tU_t U_\theta - \frac{e^{4U}}{2t}A_t A_\theta - a^{-1} te^{2(\nu-U)}\mu\, \frac{(1+k^2)v}{1-v^2}$$

Theorem. Existence of Cauchy developments and global areal foliations

$(\overline{\mathcal{M}} \simeq T^3, \overline{g}, \overline{k}, \overline{\mu}, \overline{J})$: BV regular Gowdy symmetric, initial data set with constant area $R_0 = t_0$ and which is either expanding or contracting.

- **Existence of BV regular Cauchy developments**
  - BV regular Gowdy symmetric spacetime $(\mathcal{M}, g, \mu, u^\alpha)$
  - A future Cauchy development, globally covered by a single coordinate chart
- **Global foliation** by spacelike hypersurfaces
  - A global geometrically-defined time function t coinciding with $\pm$ the area R
  - R : two-dimensional spacelike orbits of the T^2 isometry group
- **R increasing or decreasing toward the future**
  - Expanding spacetime $t = R \in [t_0, +\infty)$ $\quad t_0 > 0$ (the area grows without bound)
  - Contracting spacetime $t = -R \in [t_0, t_1)$ $\quad t_0 < t_1 \leqslant 0$
- **Crushing singularity property for Gowdy-symmetric matter spacetimes**

  For future contracting Einstein-Euler spacetimes with Gowdy symmetry, and for generic initial data, one has $t_1 = 0$, that is, the area of the T^2 orbits of symmetry approaches 0 in the future.

Remark. Our genericity condition is optimal within the class of spatially homogeneous spacetimes.

Spatially homogeneous spaces - Bianchi type I solutions (three-dim. Abelian group of isometries) $\alpha_* = \frac{3k^2+1}{4}$

$$\frac{d}{dt}\left(\frac{t\,U_t - 1/2}{a}\right) = \frac{e^{4U}}{2at}A_t^2$$

$$\frac{d}{dt}\left(\frac{e^{2U}A_t}{at}\right) = -2\frac{e^{2U}}{a\,t}U_t A_t$$

$$\frac{d}{dt}a = -ate^{2(\nu-U)}\mu(1-k^2)$$

$$\frac{d}{dt}\left(\frac{te^{2(\nu-U)}}{a}\mu\right) = \frac{te^{2(\nu-U)}}{a}\mu(1-k^2)\left(-\frac{1}{(1-k^2)}\frac{\alpha_*}{t} + at\,E\right)$$

$$a\,E = \left(U_t - \frac{1}{2t}\right)^2 + \frac{e^{4U}}{4t^2}A_t^2$$

- Normalized mass-energy density $M := \frac{4t^2}{3}e^{2(\nu-U)}\mu$
- New time function $\tau \in [0,+\infty)$ $\tau = -\log\left(\frac{t}{t_0}\right)^{1-\alpha_*}$

Proposition

(A) **Generic regime** $E > 0$: solution *defined for all* $t \in [t_0, 0)$

- $a(t)$, $tU_t(t)$, $A_t(t)$ remain globally bounded
- $M(t) \to 0$ as $t \to 0$

crushing singularity
"matter does not matter"

(B) **Exceptional regime** $E = 0$

(B1) *Small mass density* $M_0 < 1$ — same as above

(B2) *Critical case* $M_0 = 1$: defined for times, but

- $a(t)$ blows-up at $t = 0$
- The normalized mass density $M(t) = 1$ remains constant

crushing singularity
"matter matters"

(B3) *Large mass density* $M_0 > 1$
Only defined on a sub-interval $[t_0, t_1)$ with $t_1 \in (t_0, 0)$

- $a(t)$ blows–up at t_1
- $M(t)$ blows–up at t_1
- the Kretschmann scalar blows-up
- null singularity: non-unique extensions across a null hypersurface

Cauchy horizon
"matter matters"

Section 3. FUTURE GEODESIC COMPLETENESS OF WEAKLY REGULAR GOWDY SPACETIMES

Section 3.1 MAIN STATEMENT

From our result in Chapter III.

Theorem

Given any non–flat, weakly regular T^2-symmetric initial data set (Σ, h, K) with topology T^3 whose initial area $R_0 > 0$ of the orbits of symmetry is a constant, *there exists a weakly regular, Ricci-flat spacetime* $(\mathcal{M}, g)$ *with* T^2*–symmetry on* T^3

- a development of (Σ, h, K), maximal among all such developments
- a unique global foliation by level sets of the area R

  $R \in [R_0, \infty)$ in the future expanding direction

  $R \in (0, R_0]$ in the past contracting direction, except for flat Kasner (homogeneous) spacetimes

In other words:

- Generic initial data sets lead to a global foliation
- But there exist exceptional solutions (incomplete geodesics, null singularities)

Theorem

Furthermore, future expanding, Gowdy spacetimes are **geodesically complete in future timelike directions.**

In other words, every affinely parametrized, timelike geodesic can be extended indefinitely toward the future.

Generalization. Result also valid for *polarized*, Ricci-flat, T^2 symmetric spacetimes, but the problem remains open in full generality, *even* within the class of regular spacetimes.

ELEMENTS of the proof

- **Modified energy functionals.**
  - inspired by the Gowdy-to-Ernst transformation
- **Weakly regular spacetimes.**
  - estimates at the regularity level imposed by the natural energy
- **Regularity along timelike curves.**
  - define the geodesic curves under our weak regularity conditions
- **Global analysis of the geodesic equation.**
  - geodesic completeness without using pointwise estimates on the Christoffel symbols

COORDINATES general T^2 symmetry

Areal coordinates. R coincides with the time variable

$$g = e^{2(\eta-U)}\left(-dR^2 + a^{-2}\,d\theta^2\right) + e^{2U}\Big(dx + A\,dy + (G + AH)\,d\theta\Big)^2 + e^{-2U}R^2\big(dy + H\,d\theta\big)^2$$

- unknowns U, A, η, a, G, H Two Killing fields X, Y
- functions of the area (of the symmetry orbits) R and $\theta \in S^1$
- R describes an interval $[R_0, R_1)$, and x, y, θ describe S^1

WEAK REGULARITY CONDITIONS

$U, A \in C^0([R_0, R_1), H^1(S^1))$ $\eta \in C^0([R_0, R_1), W^{1,1}_\theta(S^1))$

$a, a^{-1} \in C^0([R_0, R_1), W^{2,1}(S^1))$ $G, H \in C^0([R_0, R_1), L^\infty(S^1))$

WEAK VERSION OF THE EINSTEIN EQUATIONS

1. **Nonlinear wave equations** for the coefficients U, A, η

$$
\begin{aligned}
(R\, a^{-1} U_R)_R - (R\, a\, U_\theta)_\theta &= 2R\, \Omega^U \\
(R^{-1}\, a^{-1} A_R)_R - (R^{-1}\, a\, A_\theta)_\theta &= e^{-2U} \Omega^A \\
(a^{-1} \eta_R)_R - (a\, \eta_\theta)_\theta &= \Omega^\eta - R^{-3/2} \big(R^{3/2} \big(a^{-1}\big)_R\big)_R
\end{aligned}
$$

$$
\begin{aligned}
\Omega^U &:= (2R)^{-2} e^{4U} \left(a^{-1}\, A_R^2 - a\, A_\theta^2\right) \\
\Omega^A &:= 4R^{-1} e^{2U} \left(- a^{-1}\, U_R A_R + a\, U_\theta A_\theta\right) \\
\Omega^\eta &:= (-a^{-1} U_R^2 + a\, U_\theta^2) + (2R)^{-2} e^{4U} (a^{-1} A_R^2 - a\, A_\theta^2)
\end{aligned}
$$

2. **Evolution equation** for the coefficient a

$$
(2 \ln a)_R = -R^{-3} K^2\, e^{2\eta}
$$

K being the twist constant (vanishing in Gowdy symmetry)

3. **Constraint equations** for the coefficient η

$$\eta_R + \frac{1}{4} R^{-3}\, e^{2\eta} K^2 = a\, RE \qquad \eta_\theta = R\, F$$

$$E := \left(a^{-1}\, U_R^2 + a\, U_\theta^2\right) + (2R)^{-2} e^{4U} \left(a^{-1}\, A_R^2 + a\, A_\theta^2\right)$$
$$F := 2 U_R U_\theta + 2 R^{-2} e^{2U} A_R A_\theta$$

4. **Coefficients** G, H

$$G_R = -AKe^{2\eta}\, a^{-1} R^{-3} \qquad H_R = Ke^{2\eta}\, a^{-1} R^{-3}$$

- **Four equations for the (constant) twists**

$$\left(R\, e^{4U-2\eta} a \left(G_R + AH_R\right)\right)_\theta = 0 \qquad \left(R^3\, e^{-2\eta} a\, H_R\right)_\theta = 0$$
$$\left(R\, e^{4U-2\eta}\, a \left(G_R + AH_R\right)\right)_R = 0 \qquad \left(R^3\, e^{-2\eta}\, aH_R\right)_R = 0$$

EINSTEIN EQUATIONS FOR GOWDY SPACETIMES

Nonlinear wave equations

$$(R\,U_R)_R - (R\;U_\theta)_\theta = 2R\,\Omega^U$$
$$(R^{-1}\,A_R)_R - (R^{-1}\;A_\theta)_\theta = e^{-2U}\Omega^A$$
$$\eta_{RR} - \eta_{\theta\theta} = \Omega^\eta$$

in the weak sense

$$\Omega^U := \frac{e^{4U}}{4R^2}\left(A_R^2 - A_\theta^2\right) \qquad \Omega^A := \frac{4e^{2U}}{R}\left(-\;U_R A_R + U_\theta A_\theta\right)$$
$$\Omega^\eta := \left(-\,U_R^2 + U_\theta^2\right) + \frac{e^{4U}}{4R^2}\left(A_R^2 - A_\theta^2\right)$$

Constraint equations $\quad \eta_R = RE$ and $\eta_\theta = R\,F$

$$E := \left(U_R^2 + U_\theta^2\right) + \frac{e^{4U}}{4R^2}\left(A_R^2 + A_\theta^2\right)$$
$$F := 2\,U_R U_\theta + \frac{e^{4U}}{2R^2}\,A_R A_\theta$$

ENERGY FUNCTIONALS. Basic energy

$$\mathcal{E}(R) := \int_{S^1} E(R,\theta)\, d\theta \qquad E := E^U + E^A$$

$$E^U := (U_R)^2 + (U_\theta)^2 \qquad E^A := \frac{e^{4U}}{4R^2}\Big((A_R)^2 + (A_\theta)^2\Big)$$

$$\frac{d}{dR}\mathcal{E}(R) = -\frac{2}{R}\int_{S^1}\Big((U_R)^2 + \frac{e^{4U}}{4R^2}(A_\theta)^2\Big)\, d\theta$$

Proposition. Integral energy decay for U_R and A_θ

$$\mathcal{E}(R) \leqslant \mathcal{E}(R_0) \qquad R \in [R_0, +\infty)$$

$$\frac{1}{R}\|\eta_R(R,\cdot)\|_{L^1(S^1)} \leqslant \mathcal{E}(R_0)$$

$$\frac{1}{R}\|\eta_\theta(R,\cdot)\|_{L^1(S^1)} \leqslant \mathcal{E}(R_0)$$

$$\Big\|\frac{4}{R}(U_R)^2\Big\|_{L^1((R_0,+\infty)\times S^1))} + \Big\|\frac{1}{R^3}e^{4U}(A_\theta)^2\Big\|_{L^1((R_0,+\infty)\times S^1))} \leqslant 2\,\mathcal{E}(R_0)$$

Gowdy-to-Ernst transformation and decay property for A_R

- Only a partial dissipation estimate on U_R and A_θ
- Decay for A_R derived now from another functional

Recall the *Gowdy-to-Ernst transformation:*

$$U' := \ln(R^{1/2}) - U$$

$$A'_R := e^{4U} R^{-1} A_\theta \qquad A'_\theta := e^{4U} R^{-1} A_R$$

U', A' satisfy the same evolution equations:

$$(R\, U'_R)_R - (R\ U'_\theta)_\theta = 2R\,\Omega^{U'}$$

$$(R^{-1}\, A'_R)_R - (R^{-1}\ A'_\theta)_\theta = e^{-2U'} \Omega^{A'}$$

Modified energy an arbitrary parameter $b \in \mathbb{R}$ (interpolation between $b = 0$ and $b = 1/2$)

$$\mathcal{E}_b := \int_{S^1} \Big(U_R - bR^{-1} \Big)^2 + (U_\theta)^2 + \frac{e^{4U}}{4R^2} \Big((A_R)^2 + (A_\theta)^2 \Big)$$

$$= \mathcal{E} - \frac{2b}{R} \int_{S^1} U_R + \frac{b^2}{R^2} \int_{S^1} d\theta$$

$$
\begin{aligned}
\frac{d\mathcal{E}_b}{dR} =& \frac{d\mathcal{E}}{dR} + \frac{4b}{R^2}\int_{S^1} RU_R d\theta - \frac{2b}{R^2}\int_{S^1}\left((RU_\theta)_\theta + 2R\Omega^U\right)d\theta - \frac{4\pi b^2}{R^3} \\
=& -\frac{2}{R}\int_{S^1}\left((U_R)^2 + \frac{1}{4R^2}e^{4U}(A_\theta)^2\right)d\theta \\
&+ \frac{4b}{R^2}\int_{S^1} RU_R d\theta - \frac{b}{R^3}\int_{S^1} e^{4U}\left(A_R^2 - A_\theta^2\right)d\theta - \frac{4b^2\pi}{R^3} \\
=& -\frac{2}{R}\int_{S^1}\left(U_R - bR^{-1}\right)^2 d\theta \\
&- \int_{S^1}\frac{1}{2R^3}e^{4U}\left(2bA_R^2 + (1-2b)(A_\theta)^2\right)d\theta - \frac{4\pi b^2}{R^3}
\end{aligned}
$$

We discover that, when $b \in [0, 1/2]$, all terms have a favorable sign !

Therefore, $\frac{d\mathcal{E}_b}{dR} \leqslant 0$ and $\mathcal{E}_b(R) \geqslant 0$ is bounded uniformly for all $R \geqslant R_0$.

Proposition. Integral energy decay for A_R and A_θ

For all $b \in [0, 1/2]$:

$$
\int_{R_0}^{+\infty}\frac{2}{R}\int_{S^1}\frac{e^{4U}}{4R^2}\left(2bA_R^2 + (1-2b)(A_\theta)^2\right)d\theta dR \leqslant \mathcal{E}_b(R_0) + b^2\mathcal{E}(R_0)
$$

We ned **yet another energy functional** Fix some $b \in [0, 1/2)$

$$\widehat{\mathcal{E}}(R) := \mathcal{E}_b(R) + \mathcal{G}^U(R)$$
$$\mathcal{G}^U := \frac{1}{R}\int_{S^1} \left(U - \langle U \rangle\right) U_R \, d\theta$$

with the average of a function $f = f(\theta)$ defined by $\langle f \rangle := \frac{1}{2\pi}\int_{S^1} f \, d\theta$.

Specifically, we will use this strategy

- with $b = 1/4$ and therefore $\widehat{\mathcal{E}}(R) = \mathcal{E}_{1/4}(R) + \mathcal{G}^U(R)$
- only for sufficiently large R

Theorem. The rate of decay of energy

For all Gowdy spacetimes, one has

$$\mathcal{E}(R) \simeq \widehat{\mathcal{E}}(R) \leqslant \frac{C_0}{R} \qquad R \in [R_0, +\infty)$$

where $C_0 > 0$ depends only on the initial data $U_0 = U(R_0, \cdot)$ and $A_0 = A(R_0, \cdot)$.

Heuristics when $R \to +\infty$

- one might expect that

$$\frac{d}{dR}\widehat{\mathcal{E}}(R) \leqslant -\frac{2}{R}\widehat{\mathcal{E}}(R) \qquad \text{(modulo higher order terms)}$$

- $\widehat{\mathcal{E}}$ should decay like $\frac{1}{R^2}$
- indeed correct for spatially homogeneous (Gowdy or general T^2) spacetimes

However, for non-spatially homogeneous spacetimes

- asymptotic equi-partition of energy
- space derivatives = time derivatives
- we expect

$$\frac{d}{dR}\widehat{\mathcal{E}}(R) \leqslant -\frac{1}{R}\widehat{\mathcal{E}}(R) \qquad \text{(modulo higher order terms)}$$

- $\widehat{\mathcal{E}}$ should decay like $\frac{1}{R}$

Time evolution of the correction term

$$\mathcal{G}^U(R) = \frac{1}{R}\int_{S^1}\left(U - \frac{\ln R}{4} - \left\langle U - \frac{\ln R}{4}\right\rangle\right)\left(U_R - \frac{1}{4R}\right)d\theta$$

by using the equation $(R\,U_R)_R - (R\;U_\theta)_\theta = 2R\,\Omega^U$ and an integration by parts

$$\begin{aligned}\frac{d\mathcal{G}^U(R)}{dR} &= -\frac{2}{R}\mathcal{G}^U(R) + \frac{2}{R}\int_{S^1}(U - \langle U\rangle)\Omega^U\,d\theta \\ &\quad - \frac{1}{R}\int_{S^1} U_\theta^2 d\theta - \frac{2\pi}{R}\left\langle U_R - \frac{1}{4R}\right\rangle^2 \\ &\quad + \int_{S^1}\left(U_R - \frac{1}{4R}\right)^2 d\theta\end{aligned}$$

$$\frac{d\widehat{\mathcal{E}}(R)}{dR} = -\frac{1}{R}\widehat{\mathcal{E}}(R) - \frac{2\pi}{R}\left\langle U_R - \frac{1}{4R}\right\rangle^2 - \frac{1}{R}\mathcal{G}^U(R) + \frac{2}{R}\int_{S^1}(U - \langle U\rangle)\Omega^U\,d\theta$$

- The first two terms have a favorable sign.
- We need to control two error terms.

- With Cauchy-Schwarz and Poincaré inequalities:

$$\frac{|\mathcal{G}^U(R)|}{R} \leqslant \frac{1}{4\pi^2}\frac{\widehat{\mathcal{E}}(R)}{R^2}.$$

- Nonlinear source-term

$$\frac{2}{R}\int_{S^1}(U-\langle U\rangle)\Omega^U\,d\theta \leqslant \|U(R)-\langle U(R)\rangle\|_{L^\infty(S^1)}\frac{\mathcal{E}(R)}{R}$$
$$\leqslant (2\pi)^{1/2}\frac{\mathcal{E}(R)^{3/2}}{R} \leqslant C_0\frac{\widehat{\mathcal{E}}(R)^{3/2}}{R}$$

(for R sufficiently large)

In conclusion, for some $C_0 > 0$

$$\frac{d\widehat{\mathcal{E}}(R)}{dR} \leqslant -\frac{\widehat{\mathcal{E}}(R)}{R} + C_0\frac{\widehat{\mathcal{E}}(R)}{R^2} + C_0\frac{\widehat{\mathcal{E}}(R)^{3/2}}{R}$$

$$\widehat{\mathcal{E}}(R) \leqslant \frac{K}{R}, \qquad R \in [R_0, +\infty)$$

provided $\widehat{\mathcal{E}}(R_1)$ is sufficiently small at some time $R_1 \geqslant R_0$.

The energy decreases to zero.

- Recall that an integral decay property was already established for U_R^2, A_R^2, A_θ^2.
- To control U_θ, we return to our functional $\mathcal{G}^U(R)$

$$\frac{d\mathcal{G}^U(R)}{dR} \leqslant -\frac{2}{R}\mathcal{G}^U(R) + \frac{2}{R}\int_{S^1} \Omega^U (U - \langle U \rangle) d\theta + \int_{S^1} \left(U_R - \frac{1}{4R}\right)^2 d\theta - \frac{1}{R}\int_{S^1} U_\theta^2 d\theta$$

- All the terms therein are controled by U_R^2, A_R^2, A_θ^2, except U_θ^2 which however has a *favorable sign.*
- Use the integrability of $\frac{\mathcal{E}^A}{R}$, etc. in order to deduce that $\frac{1}{R}\int_{S^1} U_\theta^2 d\theta$ is integrable in time.

Therefore, $\frac{1}{R}\widehat{\mathcal{E}}(R)$ is integrable in time and, since $\widehat{\mathcal{E}}(R)$ is decreasing, $\widehat{\mathcal{E}}(R) \to 0$.

Rate of decay of energy for all Gowdy spacetimes: $\mathcal{E}(R) \leqslant \frac{C_0}{R}$

THE GEODESIC EQUATION

Future expanding, weakly regular Gowdy spacetime $(\mathcal{M}, g)$ in areal coordinates

$$g = e^{2(\eta-U)}\left(-dR^2 + d\theta^2\right) + e^{2U}\left(dx + A\,dy\right)^2 + e^{-2U}R^2\,dy^2$$

- $\xi : [s_0, s_1] \to \mathcal{M}$ satisfying the parallel transport equation

$$\ddot{\xi}^\alpha = -\left(\Gamma^\alpha_{\beta\gamma} \circ \xi\right)\dot{\xi}^\beta\dot{\xi}^\gamma$$

- Frame (T, X, Y, Z) adapted to the symmetry (the Killing fields need not be orthogonal)
- Recall the weak regularity $U, A \in C^0((R_0, +\infty), H^1(S^1))$ and $\eta \in C^0((R_0, +\infty), W^{1,1}(S^1))$
- the Christoffel coefficients are L^p functions on spacelike hypersurfaces

Too weak to deal directly with the geodesic equation

Theorem. Existence of weakly regular geodesics

Given $\xi_0 \in \mathcal{M}$, and a future-oriented timelike vector $\xi_1 \in T_{\xi_0}\mathcal{M}$ and some initial affine parameter s_0:

- There exist $s_1 > s_0$ and a curve $\xi \in W^{1,\infty}((s_0, s_1), \mathcal{M})$ satisfying
$$\ddot{\xi}^\alpha = -\left(\Gamma^\alpha_{\beta\gamma} \circ \xi\right) \dot{\xi}^\beta \dot{\xi}^\gamma, \qquad \xi(s_0) = \xi_0, \quad \dot{\xi}(s_0) = \xi_1$$
- $(\Gamma^\alpha_{\beta\gamma} \circ \xi) \in L^1(s_0, s_1)$ along any (uniformly) timelike curve, so that $\dot{\xi} \in W^{1,1}(s_0, s_1)$.
- For every (ξ_0, ξ_1), there exists at least one maximal geodesic defined on some interval $[s_0, s_\star)$.

Elements of proof

- Uniform bounds for a sequence of curves in weakly regular spacetimes
- L^1 regularity of the Christoffel symbols along timelike curves
- Formulation of the geodesic problem as a nonlinear equation for the time component $\dot{\xi}^R$ with algebraic constraints

UNIFORMLY TIMELIKE CURVES $\xi \in W^{1,\infty}((s_0, s_1), \mathcal{M})$

- ξ **uniformly timelike** if there exists $C > 0$ such that

$$g\big(\dot{\xi}(s), \dot{\xi}(s)\big) < -C \qquad \text{for almost all } s \in (s_0, s_1)$$

- If ξ uniformly timelike, then there exists $D > 0$ (depending on the sup-norms of $\dot{\xi}$, $U \circ \xi$, and $\eta \circ \xi$) such that

$$|\dot{\xi}^R| \geqslant |\dot{\xi}^\theta| + D \qquad \text{for almost all } s \in (s_0, s_1)$$

Indeed, write

$$-C > e^{2(\eta - U)}\big(- |\xi^R|^2 + |\xi^\theta|^2\big) + e^{2U}|\theta^X|^2 + 2e^{2U}\xi^X\xi^Y + \Big(e^{2U}A^2 + e^{-2U}R^2\Big)|\xi^Y|^2$$
$$\geqslant e^{2(\eta - U)}\big(- |\xi^R|^2 + |\xi^\theta|^2\big)$$

in which $\eta - U$ is bounded on any compact time interval.

Interpretation.

- Curves projected to the quotient spacetime $\widetilde{\mathcal{M}} = [R_0, +\infty) \times S^1$ endowed with the (conformally equivalent) flat metric $\widetilde{g}_{\text{conf}} := -dR^2 + d\theta^2$
- **uniformly timelike** for the flat quotient metric: $|\dot{\xi}^R| \geqslant |\dot{\xi}^\theta| + D$ on (s_0, s_1) for some $D > 0$

Using our earlier analysis of weakly regular T^2 symmetric spacetimes

- the structure of the Einstein equations
- energy arguments
- integration in a domain bounded by two timelike curves ξ_1, ξ_2 and two spacelike hypersurfaces

Proposition. Higher-integrability of Christoffel symbols along timelike curves

Let $\xi \in W^{1,\infty}((s_0, s_1), \mathcal{M})$ be a uniformly timelike curve.

- The metric coefficients $(U \circ \xi)$ and $(A \circ \xi)$ along this curve belong to $H^1(s_0, s_1)$. Consequently, for $\alpha = R, \theta$ and $a, b = X, Y$, the trace of $\Gamma_\gamma^{\alpha\beta}$ on ξ is well-defined, with $\Gamma_{ab}^\alpha \circ \xi \in L^2(s_0, s_1)$
- $\eta \circ \xi$ belongs to $W^{1,1}(s_0, s_1)$. Consequently, for all $\alpha, \beta, \gamma = R, \theta, x, y$, the trace of $\Gamma_\gamma^{\alpha\beta}$ on ξ is well-defined and $\Gamma_\gamma^{\alpha\beta} \circ \xi \in L^1(s_0, s_1)$.

Stability of Christoffel symbols along timelike curves:

- $\|\Gamma_{\alpha\beta}^\gamma \circ \xi\|_{L^1(s_0,s_1)} \leqslant \delta(s_0 - s_1)$ (equi-integrability)
  - δ is a continuous function satisfying $\delta(0) = 0$
  - depending only on D, $R(\xi(s_0))$, and $R(\xi(s_1))$
  - independent of ξ
- For every sequence ξ_ε approaching ξ in $W^{1,\infty}((s_0, s_1), \mathcal{M})$:

$$\|(\Gamma_{\alpha\beta}^\gamma \circ \xi_\varepsilon)\dot{\xi}_\varepsilon^\alpha - (\Gamma_{\alpha\beta}^\gamma \circ \xi)\dot{\xi}^\alpha\|_{L^1(s_0,s_1)} \to 0$$

EXISTENCE OF GEODESICS

X, Y being Killing fields, the geodesic equation for $\xi \in W^{2,1}((s_0, s_1), \mathcal{M})$ is equivalent to

$$\frac{d}{ds}\left(g(\dot\xi, \dot\xi)\right) = 0, \qquad \frac{d}{ds}\left(g(\dot\xi, X)\right) = 0 \qquad \frac{d}{ds}\left(g(\dot\xi, Y)\right) = 0$$

$$\ddot\xi^\theta + \left(\Gamma^\theta_{\beta\gamma} \circ \xi\right) \dot\xi^\beta \dot\xi^\gamma = 0$$

- Given ξ_0, ξ_1, we introduce the constants
  $N^2 := -g(\xi_1, \xi_1) J^X := g(\xi_1, X), \qquad J^Y := g(\xi_1, Y)$
- Let $\hat g$ be the Riemannian metric induced by g on the torus T^2 and associated with each coordinate (R, θ) in the quotient manifold $\mathcal{M}/T^2$.
- Fix some $p \in \mathcal{M}$ and denote by $(\zeta^X, \zeta^Y) = (\zeta^X_p, \zeta^Y_p)$ the unique vector on T^2 with prescribed components

$$\hat g\Big((\zeta^X, \zeta^Y), X\Big) = J^X \qquad \hat g\Big((\zeta^X, \zeta^Y), Y\Big) = J^Y$$

(cf. the explicit formula, below)

- Given any ζ^θ, we define the component ζ^R (as a function of ζ^θ) and therefore the vector $\zeta = \left(\zeta^R(\zeta^\theta), \zeta^\theta, \zeta^X, \zeta^Y\right)$ by solving

$$g(\zeta, \zeta) = -N^2, \qquad \mathrm{sgn}(\zeta^R) = \mathrm{sgn}(\xi_1^R)$$

- To any map $\zeta \in L^1(s_0, s_1)$ as above, we associate the curve $\xi(s') := \xi_0 + \int_{s_0}^{s'} \zeta(s)\, ds, \qquad s' \in (s_0, s_1)$

Geodesic mapping $\qquad W^{2,1}(s_0, s_1) \ni \xi \mapsto \Psi[\xi] \in W^{2,1}(s_0, s_1)$

$$\Psi[\xi]^\alpha(s) := \xi_0^\alpha + \int_{s_0}^{s} \dot{\Psi}[\xi]^\alpha\, ds$$

$$\dot{\Psi}[\xi]^X(s) = \zeta^X(\xi(s)), \quad \dot{\Psi}[\xi]^Y(s) = \zeta^Y(\xi(s)), \quad \dot{\Psi}[\xi]^R(s) = \zeta^R(\xi(s))$$

$$\dot{\Psi}[\xi]^\theta(s) := \xi_1^\theta - \int_{s_0}^{s} \left(\Gamma_{\alpha\beta}^\theta \circ \xi\right) \dot{\xi}^\alpha \dot{\xi}^\beta\, ds$$

Fixed point argument

- compactness argument: equi-continuity property and Ascoli-Arzela theorem
- stability property of the Christoffel symbols along timelike curves
- uniformly timelike bound

Conclusion: existence of a maximal (possibly non-unique) geodesic $\xi \in W^{2,1}([s_0, s_1), \mathcal{M})$

Section 4.4 THE GEODESIC COMPLETENESS

Asymptotic behavior of the metric coefficients

$$g = e^{2(\eta - U)} \left(- dR^2 + d\theta^2 \right) + e^{2U} \left(dx + A\, dy \right)^2 + e^{-2U} R^2\, dy^2$$

with $R \in [R_0, +\infty)$ and $\theta \in S^1$

LATE-TIME ASYMPTOTICS for the SUP-NORM

Proposition. Sup-norm bounds for U, A, and η

There exist $C_0, c_0 > 0$ depending only on the natural norm of the initial data U_0, A_0 such that for all $R \in [R_0, +\infty)$ and $\theta \in S^1$

$$|U(R, \theta)| \leqslant C_0\, R^{1/2}$$

$$|A(R, \theta)| e^{2U(R,\theta)} \leqslant C_0\, e^{C_0 R^{1/2}}$$

$$\eta(R, \theta) - U(R, \theta) \geqslant c_0 R - C_0$$

for all *but spatially homogeneous* Gowdy spacetimes.

Arguments for U.

$U(\theta, R) = \frac{1}{2\pi} \int_{S^1} \left(\int_{\theta'}^{\theta} U_\theta(\theta'', R) d\theta'' \right) d\theta' + \frac{1}{2\pi} \int_{S^1} U(\theta', R)\, d\theta'$

Using Cauchy-Schwarz inequality, for all $\theta \in S^1$ and $R \geqslant R_0$

$$|U(\theta, R)| \leqslant (2\pi)^{1/2} \left(\int_{S^1} |U_\theta|^2(\theta'', R) d\theta'' \right)^{1/2} + \left| \frac{1}{2\pi} \int_{S^1} U(\theta', R)\, d\theta' \right|$$

- First term in $R^{-1/2}$ thanks to our energy decay property
- Second term

$$\left| \int_{S^1} U(\theta', R)\, d\theta' \right| \leqslant \left| \int_{S^1} \int_{R_0}^{R} U_R(\theta', R') dR' d\theta' \right| + \left| \int_{S^1} U_0\, d\theta' \right|$$

  - first term

$$\left| \int_{S^1} \int_{R_0}^{R} U_R(\theta', R') dR' d\theta' \right| \lesssim \int_{R_0}^{R} \left| \int_{S^1} U_R^2(\theta', R')\, d\theta' \right|^{1/2} dR'$$

$$\lesssim \int_{R_0}^{R} (R')^{-1/2} dR' \lesssim R^{1/2}$$

  - second term controlled by the H^1 norm of U_0

Arguments for $\widetilde{A} := Ae^{2U}$. For all $\theta \in S^1$ and $R \geqslant R_0$

$$\widetilde{A}(R,\theta) = \frac{1}{2\pi}\int_{S^1}\widetilde{A}(R,\cdot)\,d\theta' + \frac{1}{2\pi}\int_{S^1}\int_{\theta'}^{\theta}\left(Ae^{2U}\right)_{\theta}(R,\theta'')\,d\theta''d\theta'$$

- First term

$$\left|\int_{S^1}\widetilde{A}(R,\theta')\,d\theta'\right| = \left|\int\widetilde{A}_0\,d\theta' + \int_{R_0}^{R}\int_{S^1}(A_R e^{2U} + 2AU_R e^{2U})\,d\theta' dR'\right|$$
$$\lesssim C_0 + (2\pi)^{1/2}\int_{R_0}^{R}\left(\int_{S^1}A_R^2 e^{4U}\,d\theta'\right)^{1/2} d\theta' dR'$$
$$+ \int_{R_0}^{R}\left(\int_{S^1}A^2 e^{4U}\,d\theta'\right)^{1/2}\left(\int_{S^1}U_R^2\,d\theta'\right)^{1/2} dR'.$$

By setting $N^\infty(R) := \|A(R,\cdot)e^{2U(R,\cdot)}\|_{L^\infty(S^1)}$

$$\left|\int_{S^1}\widetilde{A}(R,\theta')\,d\theta'\right| \lesssim 1 + R^{3/2} + \int_{R_0}^{R}\frac{N^\infty(R')}{R'^{1/2}}\,dR'$$

- Second term

$$\left|\int_{\theta'}^{\theta}\left(Ae^{2U}\right)_{\theta}(R,\theta'')\,d\theta''\right| \leqslant \int_{S^1}|A_\theta|\,e^{2U}(R,\theta'')\,d\theta'' + \int_{S^1}\left|Ae^{2U}2U_\theta\right|(R,\theta'')\,d\theta''$$
$$\lesssim R^{1/2} + R^{-1/2}N^\infty(R)$$

Hence for the norm $N^\infty(R) := \|A(R,\cdot)e^{2U(R,\cdot)}\|_{L^\infty(S^1)}$

$$N^\infty(R) \lesssim 1 + R^{1/2} + R^{3/2} + \frac{N^\infty(R)}{R^{1/2}} + \int_{R_0}^{R} \frac{N^\infty(R')}{R'^{1/2}}\, dR'$$

- on any bounded interval, N^∞ can be estimated directly using the energy
- we can assume that R is sufficiently large
- absorb $\frac{N^\infty(R)}{R^{1/2}}$ in the left-hand side
- apply Gronwall lemma

Similar arguments for η but more involved.

For all *but spatially homogeneous* Gowdy spacetimes, there exists a constant $c > 0$ such that $\left\|\eta_R - U_R - c\right\|_{L^1(S^1)}(R) \lesssim \frac{1}{R}$

Consequently $e^{cR} \lesssim e^{(\eta-U)(R,\theta)} \qquad R \in [R_0, +\infty),\ \theta \in S^1$

FORMULATION IN TERMS OF THE TIME COMPONENT $\dot\xi^R$

Reduction based on the angular momentum

One says that a geodesic curve is *future complete* if it is global in the future direction, i.e. its interval of definition in the affine parameter is of the form $[s_0, +\infty)$ $\qquad \nabla_{\dot\xi}\dot\xi = 0 \qquad \dot\xi^R > 0$

- Let $\xi \in W^{2,1}(([s_0, s_1), \mathcal{M})$ be a future directed, *maximal* solution to the geodesic equation.
- Denote by $K > 0$ the (conserved) norm of $\dot\xi$ defined by $-K^2 = g(\dot\xi, \dot\xi)$
- Denote by J_X and J_Y the (conserved) angular momenta $g(\dot\xi, X)$ and $g(\dot\xi, Y)$

$$\nabla_{\dot\xi} g(\dot\xi, X) = g(\dot\xi, \nabla_{\dot\xi} X) = \dot\xi^i \dot\xi^j \nabla_i X_j = \tfrac{1}{2}\dot\xi^i \dot\xi^j \left(\nabla_i X_j + \nabla_j X_i\right) = 0$$

Conservation along ξ of the two angular momenta

$$J_X(s_0) = J_X(s) := g(\dot\xi, X) = e^{2U}\left(\dot\xi^X + A\dot\xi^Y\right)$$
$$J_Y(s_0) = J_Y(s) := g(\dot\xi, Y) = e^{2U}A\left(\dot\xi^X + A\dot\xi^Y\right) + e^{-2U}R^2\dot\xi^Y$$

$$\dot\xi^X = \left(R^{-2}A^2e^{2U} + e^{-2U}\right)J_X - Ae^{2U}R^{-2}J_Y$$
$$\dot\xi^Y = -Ae^{2U}R^{-2}J_X + R^{-2}e^{2U}J_Y$$

$$\dot{\xi}^X = \left(R^{-2}A^2e^{2U} + e^{-2U}\right)J_X - Ae^{2U}R^{-2}J_Y$$
$$\dot{\xi}^Y = -Ae^{2U}R^{-2}J_X + R^{-2}e^{2U}J_Y$$

Recall our sup-norm bounds

$$|U(R,\theta)| \leqslant C_0\, R^{1/2} \qquad\qquad |A(R,\theta)|e^{2U(R,\theta)} \leqslant C_0\, e^{C_0 R^{1/2}}$$

Lemma. Asymptotics of the X, Y components of a geodesic

There exists a constant $C_0 > 0$ depending on the initial metric U_0, A_0 and the initial momenta J_X, J_Y of the geodesic ξ such that for all $s \in [s_0, s_1)$

$$|\dot{\xi}^X| + |\dot{\xi}^Y| \leqslant C_0\, e^{C_0\, R(\xi)^{1/2}} \qquad \text{in } W^{1,1}([s_0,s_1))$$

Geodesic equation

$$\begin{aligned}\ddot{\xi}^R &= -\Gamma^R_{\alpha\beta}\dot{\xi}^\alpha\dot{\xi}^\beta \\ &= -\Gamma^R_{RR}\left(\dot{\xi}^R\right)^2 - 2\Gamma^R_{R\theta}\dot{\xi}^R\xi^\theta - \Gamma^R_{\theta\theta}\left(\dot{\xi}^\theta\right)^2 - \Gamma^R_{ab}\dot{\xi}^a\xi^b\end{aligned}$$

$a, b = X, Y$ and we have used $\Gamma^R_{Ra} = \Gamma^R_{\theta a} = 0$ and we can also recall the expressions of the Christoffel symbols

$$\Gamma^R_{RR} = \eta_R - U_R \in L^1(S^1) \qquad \Gamma^R_{R\theta} = \eta_\theta - U_\theta \in L^1(S^1)$$

$$\Gamma^R_{\theta\theta} = \eta_R - U_R \in L^1(S^1) \qquad \Gamma^R_{ab} = -\frac{1}{2}e^{-2(\eta-U)}g_{ab,R} \in L^2(S^1)$$

$$\ddot{\xi}^R = -\left(\eta_R - U_R\right)\left((\dot{\xi}^R)^2 + (\dot{\xi}^\theta)^2\right) - 2\left(\eta_\theta - U_\theta\right)\dot{\xi}^\theta\dot{\xi}^R - \Gamma^R_{ab}\dot{\xi}^a\xi^b$$

Recall now the Einstein equations for η

$$\eta_R - U_R = R\left(U_R - \frac{1}{2R}\right)^2 + R\,U_\theta^2 + \frac{e^{4U}}{4R^2}\left(A_R^2 + A_\theta^2\right) - \frac{1}{4R}$$

$$\eta_\theta - U_\theta = 2R\left(U_R - \frac{1}{2R}\right)U_\theta + \frac{e^{4U}}{2R}\,A_R A_\theta$$

$$|\eta_\theta - U_\theta| \leqslant \left(\eta_R - U_R + \frac{1}{4R}\right)$$

We have

$$\ddot{\xi}^R = -\Big(\eta_R - U_R\Big)\Big((\dot{\xi}^R)^2 + (\dot{\xi}^\theta)^2\Big) - 2\Big(\eta_\theta - U_\theta\Big)\dot{\xi}^\theta\dot{\xi}^R - \Gamma^R_{ab}\dot{\xi}^a\xi^b$$

$$\leqslant \frac{1}{4R}\Big((\dot{\xi}^R)^2 + (\dot{\xi}^\theta)^2\Big) - \Gamma^R_{ab}\dot{\xi}^a\xi^b \leqslant \frac{1}{2R}(\dot{\xi}^R)^2 - \Gamma^R_{ab}\dot{\xi}^a\xi^b$$

since

- the matrix $\begin{pmatrix} \eta_R - U_R + \frac{1}{4R} & \eta_\theta - U_\theta \\ \eta_\theta - U_\theta & \eta_R - U_R + \frac{1}{4R} \end{pmatrix}$ is negative definite
- $(\dot{\xi}^\theta)^2 \leqslant (\dot{\xi}^R)^2$, since the curve is timelike

In view of $\frac{dR}{ds} = \dot{\xi}^R$, we obtain $\ddot{\xi}^R \leqslant \frac{1}{2R}\frac{dR}{ds}\dot{\xi}^R - \Gamma^R_{ab}\dot{\xi}^a\dot{\xi}^b$

Proposition. Evolution equation for $\dot{\xi}^R$

$$\frac{d}{ds}\left(R^{-1/2}\dot{\xi}^R\right) \leqslant -R^{-1/2}\Gamma^R_{ab}\dot{\xi}^a\dot{\xi}^b$$

DERIVATIVE ALONG THE GEODESIC

With

$$g = e^{2(\eta-U)}\left(-\,dR^2 + \,d\theta^2\right) + e^{2U}\left(dx + A\,dy\right)^2 + e^{-2U}R^2\,dy^2$$

and the notation $\mu := \eta - U + \frac{\ln R}{4}$ and $\rho := \eta - U$, so that

$$\mu_R = R\left(U_R - \frac{1}{2R}\right)^2 + R\,U_\theta^2 + \frac{e^{4U}}{4R^2}\left(A_R^2 + A_\theta^2\right)$$

we find

$$\Gamma^R_{ab} = -\frac{1}{2}e^{-2(\eta-U)}g_{ab,R} \leqslant C_0\,e^{-2\rho}\left((\mu_R)^{1/2} + 1\right).$$

Lemma

With $K^2 := -g(\dot\xi, \dot\xi)$, one has:

$$0 \leqslant \mu_R \leqslant \frac{2\dot\xi^R e^{2\rho}}{K^2 + J_X^2 e^{-2U} + (J_Y - AJ_X)^2 e^{2U} R^{-2}}\,\frac{d\mu}{ds}$$

Proof. The length of the tangent vector $K^2 := -g(\dot\xi, \dot\xi)$ is conserved along the geodesic:

$$e^{-2(\eta-U)}\left(K^2 + e^{2U}(\dot\xi^X + A\dot\xi^Y)^2 + e^{-2U}R^2(\dot\xi^Y)^2\right) = (\dot\xi^R)^2 - (\dot\xi^\theta)^2$$

and in terms the conserved momenta J^X, J^Y

$$\begin{aligned}\dot\xi^X &= \left(R^{-2}A^2e^{2U} + e^{-2U}\right)J_X - Ae^{2U}R^{-2}J_Y \\ \dot\xi^Y &= -Ae^{2U}R^{-2}J_X + R^{-2}e^{2U}J_Y\end{aligned}$$

$$\left(\frac{\dot\xi^\theta}{\dot\xi^R}\right)^2 = 1 - \frac{e^{-2\rho}}{(\dot\xi^R)^2}\left(K^2 + J_X^2e^{-2U} + (J_Y - AJ_X)^2e^{2U}R^{-2}\right) =: 1-\chi$$

In other words, $|\dot\xi^\theta| = (1-\chi)^{1/2}\dot\xi^R$ (strictly timelike vector)

First-order derivative $\rho_R = \eta_R - U_R$ and $\mu_R = \rho_R + \frac{1}{4R} \geqslant |\rho_\theta|$ (Einstein constraints):

$$\begin{aligned}\frac{d\rho}{ds} &= (\mu_R - \frac{1}{4R})\dot\xi^R + \rho_\theta\dot\xi^\theta \\ &\geqslant \left(1-(1-\chi)^{1/2}\right)\mu_R\dot\xi^R - \frac{1}{4R}\dot\xi^R\end{aligned}$$

Hence: $\mu_R \leqslant \frac{1}{\dot\xi^R}(1-(1-\chi)^{1/2})^{-1}\frac{d\mu}{ds}$ and we finally use that $1-(1-\alpha)^{1/2} \geqslant \frac{\alpha}{2}$ for $\alpha \in [0,1]$.

Lemma

There exists $C_0, C_1 \geqslant 0$ (depending only on the natural norm of the initial data) such that, along a geodesic $\xi = \xi(s)$,

$$\mu_R \leqslant \frac{2\dot{\xi}^R \, e^{2\rho}}{K^2 + C_1 \, e^{-C_0 R^{1/2}}} \frac{d\mu}{ds}.$$

Moreover, the constant C_1 is stricly positive unless both angular momenta J_X, J_Y vanish.

Proof. If both J_X and J_Y vanish, the inequality is already established.

- If $J_X = 0$ but $J_Y \neq 0$, we use $e^{2U} \leqslant C_0 e^{C_0 R^{1/2}}$.
- If $J_X \neq 0$, we may first drop the positive term containing J_Y to obtain

$$\begin{aligned} \mu_R &\leqslant \frac{2\dot{\xi}^R e^{2\rho}}{K^2 + J_X^2 e^{-2U} + (J_Y - AJ_X)^2 e^{2U} R^{-2}} \frac{d\mu}{ds} \\ &\leqslant \frac{2\dot{\xi}^R e^{2\rho}}{K^2 + J_X^2 e^{-2U}} \frac{d\mu}{ds} \end{aligned}$$

  and we use again $e^{2U} \leqslant C_0 e^{C_0 R^{1/2}}$.

COMPLETION OF THE PROOF

From our analysis so far, there exists a smooth function $F = F(R) > 0$

$$\frac{d}{ds}\left(R(\xi)^{-1/2}\dot{\xi}^R\right) \leqslant R(\xi)^{1/2}|\Gamma^R_{ab}\dot{\xi}^a\dot{\xi}^b| \lesssim R^{1/2}\, e^{-2\rho}\left((\mu_R)^{1/2}+1\right)|\dot{\xi}^a\dot{\xi}^b|$$

$$\lesssim F\left(R(\xi)\right)\left(\dot{\xi}^R + \frac{d\mu}{ds} + 1\right)$$

Lemma

The area goes to infinity on a maximal, future-oriented timelike geodesic

$$R(\xi(s)) \to +\infty \qquad s \to s_1$$

Lemma

The component $\dot{\xi}^R$ grows at most like

$$0 \leqslant \dot{\xi}^R \leqslant C_0\, R(\xi)^{1/2}$$

This is sufficient to conclude, since

$$R(\xi(s)) - R(\xi(s_0)) = \int_{s_0}^{s} \dot{\xi}^R(s')\,ds' \lesssim \int_{s_0}^{s} R(\xi(s')^{1/2}\,ds' \lesssim (s-s_0)R(\xi(s')^{1/2}$$

with $R(\xi(s)) \to +\infty$. This implies that $s_1 = +\infty$, so that *the affinely-parametrized geodesic* ξ *is defined on an unbounded interval.*

Section 4. FUTURE GEODESIC COMPLETENESS OF WEAKLY REGULAR POLARIZED T^2 SYMMETRIC SPACETIMES

- Future expanding, Ricci-flat spacetimes with T^2-symmetry on T^3
- Polarized: two orthogonal Killing fields
- Exclude the Gowdy subclass: non-vanishing twist constants
- Key challenge: the class of Gowdy spacetimes exhibits a **different** dynamical behavior !

Objective

- **Stable late-time asymptotics** within this class (when the initial data set is close to it)
- **Geodesic completeness**: future timelike geodesics are complete

Metric in areal coordinate $[R_0, +\infty)$

$$g = e^{2(\eta - U)} \left(- dR^2 + a^{-2} \, d\theta^2 \right) + e^{2U} \left(dx + A \, dy + (G + AH) \, d\theta \right)^2 + e^{-2U} R^2 \left(dy + H \, d\theta \right)^2$$

- Polarized spacetimes : one can take $A = 0$
- Gowdy spacetimes : $G = H = 0$

Two energy functionals

$$\mathcal{E}(R) := \int_{S^1} E(R,\theta)\, d\theta \qquad E = a^{-1}\, U_R^2 + a\, U_\theta^2$$

$$\mathcal{E}_K(R) := \int_{S^1} E_K(R,\theta)\, d\theta \qquad E_K := E + \frac{K^2}{4R^4}\, a^{-1} e^{2\eta}$$

They are both decreasing in time

$$\frac{d}{dR}\mathcal{E}(R) = -\frac{K^2}{2R^3}\int_{S^1} E\, e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1} a^{-1}\, (U_R)^2 d\theta$$

$$\frac{d}{dR}\mathcal{E}_K(R) = -\frac{K^2}{R^5}\int_{S^1} a^{-1} e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1} a^{-1}\, (U_R)^2\, d\theta$$

Main challenges

- We assume $K > 0$ and thus exclude Gowdy spacetimes —which were treated in Section 3 and exhibit a different late-time asymptotic behavior.
- Only partial dissipation (as for Gowdy spacetimes)
- Singular weighted energies: metric coefficient $a \to 0$ (new feature)

VOLUME-RESCALED ENERGY FUNCTIONALS

Volume of the slice R =constant for the conformal 2-metric $-dR^2 + a^{-2}\, d\theta^2$

$$\mathcal{F}(R) = \mathcal{P}(R)\mathcal{E}(R) \qquad \mathcal{P}(R) = \int_{S^1} \frac{d\theta}{a(R,\theta)}$$

Heuristic leading to the asymptotic behavior

$$\mathcal{P}(R) \sim R^{1/2} \qquad \mathcal{P}_R(R) \sim R^{-1/2}$$
$$e^{2\eta} \sim R^2 \qquad a \sim R^{-1/2}$$

(instead of $\mathcal{P}(R)$ being constant for Gowdy spacetimes !)

Corrected energy

$$\mathcal{G}(R) := \mathcal{P}(R)\big(\mathcal{E}(R) + \mathcal{G}^U(R)\big)$$
$$\mathcal{G}^U(R) := \frac{1}{R}\int_{S^1} \big(U - \big\langle U \big\rangle\big)\, U_R\, a^{-1}\, d\theta$$

$\big\langle U \big\rangle$: volume-weighted average

Formal derivation of an asymptotic model

- From the energies $\mathcal{G} = \mathcal{P}\left(\mathcal{E} + \mathcal{G}^U\right)$ and

$$\mathcal{G}_K = \frac{K^2}{2} \int_{S^1} e^{2\eta} a^{-1} \, d\theta$$

we define the (dimensionless) **rescaled energies**

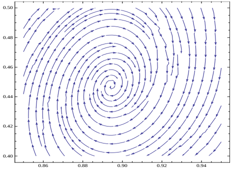
$$c := \frac{\mathcal{P}}{R\sqrt{\mathcal{G}}}, \qquad d := \frac{\mathcal{G}_K}{R^3\sqrt{\mathcal{G}}}$$

- *Neglect* second- and higher-order contributions
- The functions c, d satisfy the **asymptotic ODE model**

$$c_R = \frac{d}{R} - \frac{c}{R} + \frac{c}{2R}$$
$$d_R = \frac{2dc^{-2}}{R} - \frac{3}{R}d + \frac{d}{2R}$$

- Global stable equilibrium and shifted variables

$$c_1 = c - \frac{2}{\sqrt{5}}, \qquad d_1 = d - \frac{1}{\sqrt{5}}$$



GEODEDIC COMPLETENESS PROPERTY

Class of initial data sets

- Fix some initial radius $R_0 > 0$ and a sufficiently small $\epsilon > 0$.
- Consider initial data **sufficiently close to the asymptotic regime** (non-empty class of data)

$$\left|\frac{\mathcal{F}(R_0)}{\mathcal{G}(R_0)} - 1\right| \leqslant \varepsilon$$

$$\left|\frac{\mathcal{P}(R_0)}{R_0\sqrt{\mathcal{G}(R_0)}} - \frac{2}{\sqrt{5}}\right| + \left|\frac{\mathcal{G}_K(R_0)}{R_0^3\sqrt{\mathcal{G}(R_0)}} - \frac{1}{\sqrt{5}}\right| \leqslant \varepsilon$$

Theorem. Weakly regular, T2-symmetric polarized Ricci-flat spacetimes

If the initial data set is sufficiently close to the asymptotic regime on some initial slice of area R_0 then the spacetime asymptotically approaches the asymptotic regime

Volume of a slice	$\mathcal{P}(R) = \frac{2C_0^{1/2}}{\sqrt{5}} R^{1/2} + O(R^{1/4})$
Energy of a slice	$\mathcal{E}(R) = O(R^{-3/2})$
Rescaled energy	$\mathcal{G}(R) = C_0 R^{-1} + O(R^{-3/2})$
Rescaled twist energy	$\mathcal{G}_K(R) = \frac{C_0^{1/2}}{\sqrt{5}} R^{5/2} + O(R^{9/4})$

where $C_0 > 0$ is a constant depending on the initial data set.
Polarized T^2 symmetric, weakly regular, Ricci-flat spacetimes are geodesically complete in future timelike directions.

Open problems

- asymptotics far from the asymptotic regime
- asymptotics of non-polarized T^2 spacetimes

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NEXT (and final) TOPIC

- The global nonlinear stability of Minkowski spacetime
- Einstein-matter system for a *massive* scalar field
- The Hyperboloidal Foliation Method (LeFloch-Ma, 2014)

CHAPTER VI
THE GLOBAL NONLINEAR STABILITY OF MINKOWSKI SPACE for MASSIVE FIELDS

Global existence problem for the Einstein equations

- Self-gravitating massive scalar field
- Choice of gauge (De Donder): wave coordinates $\Box_g x^\alpha = 0$
- Nonlinear stability of Minkowski spacetime

Hyperboloidal Foliation Method

- New method of analysis for quasilinear wave-Klein-Gordon systems
- A framework leading to sharp decay rates
- A $(3+1)$-foliation of Minkowski spacetime by hyperboloids

REFERENCES for this Chapter

- PG. LeFloch and Yue Ma, *The hyperboloidal foliation method,* World Scientific Press, 2014. ArXiv:1411.4910

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  - *Part 1: The wave-Klein-Gordon model* ArXiv:1507.01143
  - *Part 2: Analysis of the Einstein equations* ArXiv:1511.03324

- Further results
  - Qian Wang and Jinhua Wang, *Global existence for the Einstein equations with massive scalar fields,* lecture given at IHP, Paris, November 2015 (independent proof by a different method)
  - D. Fajman, J. Joudioux, and J. Smulevici, *A vector field method for relativistic transport equations with applications,* ArXiv:1510.04939
  - PG. LeFloch and Yue Ma, *The nonlinear stability of Minkowski space for self-gravitating massive fields. Part 3: The f(R) theory of modified gravity,* in preparation.

Section 1. THE GENERAL STRATEGY
Section 1.1 WAVE-KLEIN-GORDON PROBLEMS

- **Systems of coupled nonlinear wave-Klein-Gordon equations** posed on a four-dimensional curved spacetime
- **Simplest wave-Klein-Gordon model**: posed in flat space

$$\Box u = P(\partial u, \partial v)$$
$$\Box v - v = Q(\partial u, \partial v)$$

P, Q: quadratic nonlinearities in $\partial u = (\partial_\alpha u)$ and $\partial v = (\partial_\alpha v)$

- **Many models** arising in mathematical physics
  - Interacting massive/massless fields
  - Massive Einstein equations
  - Modified gravity theory

$$\int_M f(R)\, dv_g \quad \text{with, typically,} \quad f(R) = R + \kappa R^2$$

  - Other models
    - Dirac-Klein-Gordon equations
    - Proca equation (massive spin-1 field in Minkowski spacetime)
- **Objective:** solve the Cauchy problem, global-in-time solutions, small initial data

VECTOR FIELD METHOD

Based on the conformal Killing fields of Minkowski space

- suitably weighted energy estimates (Klainerman, from 1985)
- Klainerman-Sobolev inequalities
- bootstrap arguments
  - nonlinearities satisfying the 'null condition', typically $\nabla_\alpha u \nabla^\alpha v$
  - sufficient time decay
  - derivatives tangential to the light cone

EXTENSIVE LITERATURE

- **Nonlinear wave equations**
  - John, Klainerman, Christodoulou (conformal compactification), Lindblad (blow-up), Bachelot, Machedon, Delort.
  - *Strichartz estimates:* Bahouri, Bourgain, Chemin, Klainerman, Machedon, Tataru.
  - *Extension to the Einstein equations:* Christodoulou-Klainerman (stability of Minkowski), Lindblad, Alinhac, Lindblad-Rodnianski (wave coordinates), Dafermos-Rodnianski, Klainerman-Rodnianski-Szeftel.
- **Nonlinear Klein-Gordon equations**
  - Klainerman, Shatah, Hörmander, Bachelot.

Much less is known on *coupled problems*

COUPLED WAVE-KLEIN-GORDON SYSTEMS

- **Major challenge**
  - Smaller symmetry group: scaling field $t\partial_t + r\partial_r$ is *not conformal Killing* for KG
  - Decay for Klein-Gordon equation $t^{-3/2}$, but t^{-1} for wave equation (in $3+1$ dimensions)
  - Need a robust technique for the coupling of wave equations and Klein-Gordon equations
- **Relying on fewer Killing fields**
  - Georgiev (1990): strong assumption on the nonlinearities
  - Katayama (2012): weaker conditions on the nonlinearities
  - PLF & YM (2014, 2015): The Hyperboidal Foliation Method

Remark. Earlier work based on hyperboloidal foliations

- Friedrich (1981): global existence result for the Einstein equations (conformal transformation, Penrose's compactification)
- Klainerman (1985): global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four spacetime dimensions (pseudo-spherical coordinates, uniform decay rate $t^{5/4}$)
- Christodoulou (1986): existence for nonlinear wave equations (conformal transformation)
- Hörmander (1997): sharp decay for the linear Klein-Gordon equation
- Numerical computations of the Einstein equations: Frauendiener, Moncrief, Rinne, Zenginoglu

Section 1.2 THE HYPERBOLOIDAL FOLIATION

- Family of conformal Killing fields (of Minkowski spacetime) reduced in presence of a massive field
- Scaling vector field not conformal Killing for Klein-Gordon

OUR STRATEGY

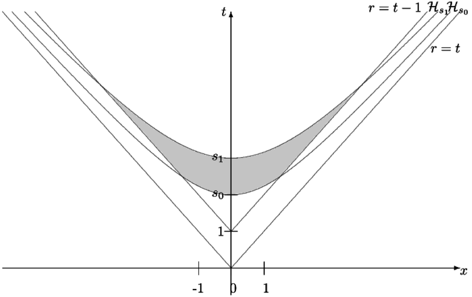
- Foliation (of the interior of a light cone) of Minkowski space by hyperboloidal hypersurfaces (of constant Lorentzian distance)
- Generated by the Lorentz boosts $L_a := x^a \partial_t + t \partial_a$ for $a = 1, 2, 3$
- Revisit standard vector field arguments, but new weighted norms, better adapted to the Minkowski geometry
  - Hyperboloidal energy
  - Sobolev inequality on hyperboloids
  - Hardy-type inequality for the hyperboloidal foliation

Encompass successively broader classes of coupled systems

- Nonlinear wave equations with null forms (Section 3.1)
- Systems with weak metric interactions and null forms (Section 1.3)
- Systems with strong metric interactions and null forms (Section 3.2)
- The Einstein-massive field system: strong metric interactions and quasi-null forms (Section 2)

HYPERBOLOIDAL FOLIATION

- **Hyperboloidal hypersurfaces** $\mathcal{H}_s := \{(t,x) \,/\, t > 0;\ t^2 - |x|^2 = s^2\}$ parametrized by their hyperbolic radius $s > s_0 > 1$
- **Foliation of the future light cone** from $(t,x) = (1,0)$
  $\mathcal{K} := \{(t,x) \,/\, |x| \leqslant t - 1\}$ $\quad (s \leqslant t \leqslant \frac{s^2+1}{2})$



- **Initial data** prescribed on the spacelike hypersurface $t = s_0 > 1$
- **Energy estimate** in domains limited by two hyperboloids

$$\mathcal{K}_{[s_0,s_1]} := \{(t,x) \,/\, |x| \leqslant t - 1, \quad (s_0)^2 \leqslant t^2 - |x|^2 \leqslant (s_1)^2, t > 0\}$$

HYPERBOLOIDAL VECTOR FRAME

- **Lorentz boosts** (hyperbolic rotations) $L_a := x^a\partial_t + t\partial_a$ $(a = 1, 2, 3)$
- **Semi-hyperboloidal frame**
  - Three vectors tangent to the hyperboloids $\underline{\partial}_a := \frac{L_a}{t}$
  - A timelike vector $\underline{\partial}_0 := \partial_t$
- Semi-hyperboloidal decomposition of the wave operator

$$\Box u = -\frac{s^2}{t^2}\underline{\partial}_0\underline{\partial}_0 u - \frac{3}{t}\partial_t u - \frac{x^a}{t}\left(\underline{\partial}_0\underline{\partial}_a u + \underline{\partial}_a\underline{\partial}_0 u\right) + \sum_a \underline{\partial}_a\underline{\partial}_a u$$

HYPERBOLOIDAL ENERGY

- **Weigthed directional derivatives**
- Use full expression of the **energy flux** on the hyperboloids
- For instance, for the linear Klein-Gordon equation $\Box u - c^2\, u = f$ in Minkowski space $s^2 = t^2 - r^2$ and $r^2 = \sum_a (x^a)^2$

$$E[s,u] = E_{m,\sigma}[s,u] := \int_{\mathcal{H}_s} \left(\frac{s^2}{t^2}(\partial_t u)^2 + \sum_{a=1}^{3}\left(\frac{x^a}{t}\partial_t u + \partial_a u\right)^2 + \frac{c^2}{2}\, u^2\right) dx$$

$$= \int_{\mathcal{H}_s} \left(\frac{s^2}{s^2 + r^2}(\underline{\partial}_0 u)^2 + \sum_{a=1}^{3}(\underline{\partial}_a u)^2 + \frac{c^2}{2}\, u^2\right) dx$$

Summary of the features of the method

- **Lorentz invariance.**
  - Foliation of Minkowski space by hyperboloids: level sets of constant Lorentzian distance from some origin
  - Invariance under Lorentz transformations
  - Hyperboloids asymptotic to the *same limiting light cone* and approaching the *same sphere at infinity.*
  - No energy can "escape" through null infinity.

- **Smaller set of Killing fields.**
  - Avoid using the scaling vector field $S := r\partial_r + t\partial_t$
  - The key in order to be able to handle Klein-Gordon equations
  - Apply to the Einstein-matter system when the matter model is not conformally invariant

- **Sharp decay rates in time.**
  - Need to control interaction terms related to the curved geometry
  - Sharp pointwise bounds for nonlinear wave equations and nonlinear Klein-Gordon equations

Section 1.3 WAVE-KLEIN-GORDON SYSTEMS WITH WEAK METRIC INTERACTIONS

$$\Box w_i + G_i^{j\alpha\beta}(w,\partial w)\partial_\alpha\partial_\beta w_j + c_i^2 w_i = F_i(w,\partial w)$$

$$w_i(t_0,\cdot) = w_{i0} \qquad \partial_t w_i(t_0,\cdot) = w_{i1}$$

with unknowns $(w_i)_{1\leqslant i\leqslant n}$ defined on Minkowski space $(\mathbb{R}^{3+1}, m)$

- **Wave-Klein-Gordon structure**
  - $c_i = 0$ $\quad 1 \leqslant i \leqslant n'$
  - $c_i \geqslant \sigma > 0$ $\quad n'+1 \leqslant i \leqslant n$
- **Quadratic nonlinearities**
  - Curved metric terms
    $$G_i^{j\alpha\beta}(w,\partial w) = A_i^{j\alpha\beta\gamma k}\partial_\gamma w_k + B_i^{j\alpha\beta k} w_k$$
  - Nonlinear coupling
    $$F_i(w,\partial w) = P_i^{\alpha\beta jk}\partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha jk} w_k \partial_\alpha w_j + R_i^{jk} w_j w_k$$
- **Symmetry properties**
  - $G_i^{j\alpha\beta} = G_i^{j\beta\alpha}$ existence of an energy for the curved metric
  - $G_i^{j\alpha\beta} = G_j^{i\alpha\beta}$ existence of an energy for the coupled system

Index convention

- Wave-Klein-Gordon structure
  - $c_i = 0, \qquad 1 \leqslant i \leqslant n'$
  - $c_i \geqslant \sigma > 0, \qquad n' + 1 \leqslant i \leqslant n$
- Wave components $u_{\hat{\imath}} := w_{\hat{\imath}} \qquad \hat{\imath} = 1, \ldots, n'$
- Klein-Gordon components $v_{\check{\imath}} := w_{\check{\imath}} \qquad \check{\imath} = n' + 1, \ldots, n$

MAIN ASSUMPTIONS

- **Null condition** for wave components: for $(\xi_0)^2 - \sum_{a=1,2,3}(\xi_a)^2 = 0$

$$A_{\hat{\imath}}^{\hat{\jmath}\alpha\beta\gamma\hat{k}}\xi_\alpha\xi_\beta\xi_\gamma = B_{\hat{\imath}}^{\hat{\jmath}\alpha\beta\hat{k}}\xi_\alpha\xi_\beta = P_{\hat{\imath}}^{\alpha\beta\hat{\jmath}\hat{k}}\xi_\alpha\xi_\beta = 0$$

  - Imposed only on the quadratic forms associated with wave components
  - No such restriction for Klein-Gordon components
- **Structural condition** on the sources (essentially to avoid blow-up) (excludes terms like $u\partial u, u\partial v$, and u^2)

$$Q_i^{\alpha j\hat{k}} = 0, \qquad R_i^{j\hat{k}} = R_i^{\hat{\jmath}k} = 0$$

- **Weak metric interactions** (excludes metric terms like $u\partial\partial v$)

$$B_i^{\check{\jmath}\alpha\beta\hat{k}} = 0$$

Theorem. (2014) Global existence theory for nonlinear wave-KG systems with weak metric interactions

Consider the initial value problem

- coupled nonlinear wave-Klein Gordon systems with weak metric interactions
- smooth and localized (compactly supported) initial data posed on a spacelike hypersurface of constant time $t = t_0$.

Then, there exists $\varepsilon > 0$ such that

- provided the initial data satisfy $\|w_{i0}\|_{\mathbf{H}^6(\mathbb{R}^3)} + \|w_{i1}\|_{\mathbf{H}^5(\mathbb{R}^3)} < \varepsilon$
- the Cauchy problem admits a unique global-in-time solution (w_i).

Proof.

- A complex bootstrap argument based on three levels of regularity
- A family of L^2 norms based on the collection of **admissible vector fields** $Z \in \{\partial_\alpha, L_a\}$

BOOTSTRAP FORMULATION Given $C_1 > 1$ and $\varepsilon > 0$ and $\delta \in (0, 1/6)$, we formulate our bootstrap assumptions within a given interval $[s_0, s_1]$ of hyperbolic time.

- **Hierarchy of energy bounds** ($s \in [s_0, s_1]$ and all admissible fields):
  - High-order energy $\quad |I^\sharp| \leqslant 5$
    $$E[s, Z^{I^\sharp} u_{\hat\imath}]^{1/2} \leqslant C_1 \varepsilon s^\delta \qquad 1 \leqslant \hat\imath \leqslant n'$$
    $$E[s, Z^{I^\sharp} v_{\check\jmath}]^{1/2} \leqslant C_1 \varepsilon s^\delta \qquad n'+1 \leqslant \check\jmath \leqslant n$$
  - Intermediate-order energy $\quad |I^\dagger| \leqslant 4$
    $$E[s, Z^{I^\dagger} u_{\hat\imath}]^{1/2} \leqslant C_1 \varepsilon s^{\delta/2} \qquad 1 \leqslant \hat\imath \leqslant n'$$
    $$E[s, Z^{I^\dagger} v_{\check\jmath}]^{1/2} \leqslant C_1 \varepsilon s^{\delta/2} \qquad n'+1 \leqslant \check\jmath \leqslant n$$
  - Low-order energy (uniform, specific to wave components) $\quad |I| \leqslant 3$
    $$E[s, Z^{I} u_{\hat\imath}]^{1/2} \leqslant C_1 \varepsilon \qquad 1 \leqslant \hat\imath \leqslant n'$$
- **Hierarchy of bounds for the curved metric and sources**

From these assumptions, we are able to deduce a **hierarchy of enhanced energy bounds** with C_1 replaced by $C_1/2$

Observations.

- Algebraic (slow) growth factors
- Uniform bound for the low-order energy of the wave components

Section 2. THE EINSTEIN-MASSIVE FIELD SYSTEM

Section 2.1 STATEMENT OF THE NONLINEAR STABILITY

New approach for proving the nonlinear stability of Minkowski space

- Covers self-gravitating massive fields
- Previous works concern vacuum spacetimes or massless scalar fields
    - Christodoulou-Klainerman 1993: null foliation / maximal foliation (fully geometric proof, Bianchi identities, geometry of null cones)
    - Lindblad-Rodnianski 2010: wave coordinates (standard foliation)
    - Massless models: Bieri-Zipser (2009, weaker decay assumptions), Speck (2014)

Einstein equations for a spacetime (M, g) $\qquad R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = T_{\alpha\beta}$

- Stress-energy tensor $\quad T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2} \nabla_\gamma \phi \nabla^\gamma \phi + V(\phi) \Big) g_{\alpha\beta}$
- Potential $\qquad V(\phi) := \frac{c^2}{2} \phi^2$ with $c > 0$
- **Klein-Gordon equation** $\qquad \Box_g \phi = V'(\phi) = c^2 \phi$
- Perturbation of a spacelike hypersurface in Minkowski space
- Wave gauge $\Gamma^\gamma := g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} = 0$ (see Chapter 1)

CAUCHY DEVELOPMENTS

Initial data set.

- Riemannian 3-manifold $(\overline{M}, \overline{g})$
- Symmetric 2-covariant tensor field $\overline{k}$
- Einstein's constraint equations

$$\overline{R} + \left(\mathrm{Tr}\,\overline{k}\right)^2 - |\overline{k}|^2 = 16\pi\, T_{00} \qquad (\text{Hamiltonian } G_{00} = 8\pi\ T_{00})$$

$$\overline{\nabla}_b \overline{k}_a^{\ b} - \overline{\nabla}_a(\mathrm{Tr}\,\overline{k}) = 8\pi\, T_{0a} \qquad (\text{Momentum } T_{0a} = 8\pi\ T_{0a})$$

- Matter fields (components $T_{00},\ T_{0a}$)
- Notation
  - $\overline{g}_{ab}$, $\overline{k}_{ij}$ (with $a, b = 1, 2, 3$)
  - $\overline{\nabla}$: connection associated with $\overline{g}$ $\quad$ $\overline{R}$: scalar curvature of $\overline{g}$
  - Trace $\mathrm{Tr}\,\overline{k} = \overline{k}_a^{\ a} = \overline{g}^{ab}\overline{k}_{ab}$ $\quad$ norm $|\overline{k}|^2 = \overline{k}_{ab}\overline{k}^{ab}$

INITIAL VALUE PROBLEM. Future development of the initial data set

- Lorentzian manifold satisfying the Einstein equations (M, g)
- Embedding $\psi : \overline{M} \to \mathcal{H} \subset M$
- Induced metric $\psi^\star g = \overline{g}$
- Second fundamental form k $\psi^\star k = \overline{k}$
- Matter fields

FORMULATION AS PARTIAL DIFFERENTIAL EQUATIONS

- choice of coordinates / diffeomorphism invariance
- **wave gauge**
  - coordinate functions such that
    $$\Box_g x^\alpha := \nabla^\alpha \nabla_\alpha x^\alpha = 0$$
  - system of coupled nonlinear wave equations for the metric
    $$\Box_g g_{\alpha\beta} = Q_{\alpha\beta}(g, \partial g)$$
  - hyperbolic-elliptic system of PDE's

Theorem. The global nonlinear stability of Minkowski spacetime for self-gravitating massive fields. The geometric formulation

$$R_{\alpha\beta} = 8\pi\big(\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\, g_{\alpha\beta}\big) \qquad \Box_g\phi = V'(\phi)$$

with an initial data set $(\overline{M}, \overline{g}, \overline{k}, \phi_0, \phi_1)$ satisfying the constraint equations

$$\overline{R} + \big(\mathrm{Tr}\,\overline{k}\big)^2 - |\overline{k}|^2 = 16\pi T_{00} \qquad \text{(Hamiltonian } G_{00} = 8\pi\, T_{00})$$

$$\overline{\nabla}_b \overline{k}_a^b - \overline{\nabla}_a(\mathrm{Tr}\,\overline{k}) = 8\pi T_{0a} \qquad \text{(Momentum } T_{0a} = 8\pi\, T_{0a})$$

- is close to an asymptotically flat slice of the (vacuum) Minkowski spacetime (suitable decay at spatial infinity)
- more precisely, in a neighborhood of spacelike infinity coincides with a spacelike slice of Schwarzschild spacetime with sufficiently small ADM mass *(Compact Matter Perturbation Problem)*

Then, the corresponding initial value problem admits a globally hyperbolic Cauchy development

- which is an asymptotically flat and future geodesically complete spacetime.
- In other words, every affinely parametrized, timelike geodesic can be extended indefinitely toward the future.

Theorem. The global nonlinear stability of Minkowski space for self-gravitating massive fields. The formulation in wave gauge

Einstein-massive field system in wave coordinates $\Box_g x^\alpha = 0$

$$\widetilde{\Box}_g g_{\alpha\beta} = Q_{\alpha\beta}(g;\partial g) + P_{\alpha\beta}(g;\partial g) - 2\big(\partial_\alpha\phi\partial_\beta\phi + V(\phi)g_{\alpha\beta}\big)$$

$$\widetilde{\Box}_g \phi - V'(\phi) = 0 \qquad\qquad V(\phi) = \frac{c^2}{2}\phi^2$$

$\widetilde{\Box}_g := g^{\alpha\beta}\partial_\alpha\partial_\beta$ reduced wave operator

$Q_{\alpha\beta}$: standard null terms

$P_{\alpha\beta}$: quasi-null terms

together with an initial data set $(\overline{M}, \overline{g}, \overline{k})$

coincides, in a neighborhood of spacelike infinity, with a spacelike slice of Schwarzschild space in wave coordinates (t, x)

$$g_{S,00} = -\frac{1-\frac{m_S}{r}}{1+\frac{m_S}{r}}$$

$$g_{S,ab} = \frac{1+\frac{m_S}{r}}{1-\frac{m_S}{r}}\omega_a\omega_b + \left(1-\frac{m_S}{r}\right)^2(\delta_{ab} - \omega_a\omega_b)$$

m_S being the ADM mass with $r := |x|$ and $\omega^a := x^a/r$

That is, consider an initial data set prescribed on the hypersurface $\{t = 2\}$:

$$g_{\alpha\beta}(2,\cdot) = g_{0,\alpha\beta} \qquad \partial_t g_{\alpha\beta}(2,\cdot) = g_{1,\alpha\beta}$$
$$\phi(2,\cdot) = \phi_0 \qquad \partial_t \phi(2,\cdot) = \phi_1$$

which coincides with g_S outside the ball $\{r < 1\}$, i.e. for all $r \geqslant 1$

$$g_{\alpha\beta}(2,\cdot) = g_{S,\alpha\beta} \qquad \partial_t g_{\alpha\beta}(2,\cdot) = 0 \qquad \phi(2,\cdot) = \partial_t\phi(2,\cdot) = 0$$

Then, there exist constants $\epsilon_0, \delta > 0$ and $C_1 > 0$ and an integer N such that provided (with $h_{\alpha\beta} = g_{\alpha\beta} - m_{\alpha\beta}$)

$$\sum_{\alpha,\beta,\gamma} \|\partial_\gamma h_{0,\alpha\beta}\|_{H^N(\{r<1\})} + \|\phi_0\|_{H^{N+1}(\{r<1\})} + +\|\phi_1\|_{H^N(\{r<1\})} + m_S \leqslant \epsilon \leqslant \epsilon_0,$$

then the initial value problem for the Einstein-massive field system

- admits a global-in-time solution in wave coordinates $(g_{\alpha\beta}, \phi)$
- satisfying the following energy bounds for all $s \geqslant 2$

$$E_M(s, \partial^I L^J h_{\alpha\beta})^{1/2} \leqslant C_1 \epsilon s^\delta \qquad |I| + |J| \leqslant N$$
$$E_{M,c^2}(s, \partial^I L^J \phi)^{1/2} \leqslant C_1 \epsilon s^{\delta+1/2} \qquad |I| + |J| \leqslant N$$
$$E_{M,c^2}(s, \partial^I L^J \phi)^{1/2} \leqslant C_1 \epsilon s^\delta \qquad |I| + |J| \leqslant N - 4$$

FOR THE PROOF

Hyperboloidal foliation

- Killing fields: hyperbolic rotations
- Lorentz-invariant norm
- commutators

Tensorial structure

- wave coordinate condition $\Box_g x^\alpha = 0$
- null terms, quasi-null terms (refered to as "weak null" terms by Lindblad-Rodnianski)

Sharp pointwise estimates

- explicit integration along characteristics or rays
- L^∞-L^∞ estimate for wave equations on curved space
- L^∞-L^∞ estimate for Klein-Gordon equations on curved space

Hierarchy of energy bounds

- several levels of regularity
- algebraic growth in the hyperboloic time s
- proof based on successive improvements of the estimates

Section 2.2 THE QUASI-NULL STRUCTURE OF THE EINSTEIN-MASSIVE FIELD SYSTEM

Proposition

$$R_{\alpha\beta} = -\frac{1}{2} g^{\lambda\delta} \partial_\lambda \partial_\delta g_{\alpha\beta} + \frac{1}{2} \left(\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\beta \right) + \frac{1}{2} F_{\alpha\beta}$$

where $F_{\alpha\beta} := P_{\alpha\beta} + Q_{\alpha\beta} + W_{\alpha\beta}$ is a sum of (i) null terms

$$\begin{aligned} Q_{\alpha\beta} :=& g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\lambda'} \partial_{\delta'} g_{\beta\lambda} - g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'} \right) \\ &+ g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'} \right) \\ &+ \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta} \right) \\ &+ g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha} - \partial_\beta g_{\lambda\alpha} \partial_\delta g_{\lambda'\delta'} \right) \\ &+ \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda\alpha} \partial_{\lambda'} g_{\delta\delta'} - \partial_\beta g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\alpha} \right) \end{aligned}$$

(ii) "quasi-null terms"

$$P_{\alpha\beta} := -\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\delta\lambda'} \partial_\beta g_{\lambda\delta'} + \frac{1}{4} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\beta g_{\delta\delta'} \partial_\alpha g_{\lambda\lambda'}$$

and (iii) terms vanishing in the wave gauge $W_{\alpha\beta} := g^{\delta\delta'} \partial_\delta g_{\alpha\beta} \Gamma_{\delta'} - \Gamma_\alpha \Gamma_\beta$.

CONSEQUENCES.

- **Vacuum Einstein equation** $R_{\alpha\beta} = 0$

$$\tilde{\Box}_g g_{\alpha\beta} = P_{\alpha\beta} + Q_{\alpha\beta} + W_{\alpha\beta} + \left(\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha\right).$$

Under the wave condition $g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} = 0$, we have $\Gamma_\beta = 0$ and therefore:

$$\tilde{\Box}_g g_{\alpha\beta} = P_{\alpha\beta}(g;\partial g) + Q_{\alpha\beta}(g;\partial g)$$

- **Einstein-massive field system**

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2} g_{\alpha\beta}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + c^2\phi^2\right)$$

therefore

$$R_{\alpha\beta} = 8\pi\left(\nabla_\alpha\phi\nabla_\beta\phi + \frac{c^2}{2}\phi^2 g_{\alpha\beta}\right)$$

and the system in wave coordinates reads

$$\tilde{\Box}_g g_{\alpha\beta} = P_{\alpha\beta}(g;\partial g) + Q_{\alpha\beta}(g;\partial g) - 16\pi\partial_\alpha\phi\partial_\beta\phi - 8\pi c^2\phi^2 g_{\alpha\beta}$$

$$\tilde{\Box}_g \phi - c^2\phi = 0$$

DERIVATION OF THE RICCI DECOMPOSITION

$$R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\delta_{\lambda\delta} - \Gamma^\lambda_{\alpha\delta}\Gamma^\delta_{\beta\lambda}$$

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\lambda'}\left(\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'} - \partial_{\lambda'} g_{\alpha\beta}\right)$$

Only the first two terms in the expression $R_{\alpha\beta}$ involves second-order derivatives of the metric. In view of

$$\begin{aligned} 2\,\partial_\lambda \Gamma^\lambda_{\alpha\beta} =& - g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + g^{\lambda\delta}\partial_\lambda\partial_\alpha g_{\beta\delta} + g^{\lambda\delta}\partial_\lambda\partial_\beta g_{\alpha\delta} \\ &+ \partial_\lambda g^{\lambda\delta}\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right) \\ 2\,\partial_\alpha \Gamma^\lambda_{\beta\lambda} =& \partial_\alpha\partial_\beta g_{\lambda\delta} + \partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta}, \end{aligned}$$

the **second-order terms in the Ricci curvature** read

$$\begin{aligned} &2\left(\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}\right) \\ &= -g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + g^{\lambda\delta}\partial_\alpha\partial_\lambda g_{\delta\beta} + g^{\lambda\delta}\partial_\beta\partial_\lambda g_{\delta\alpha} - g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} \\ &\quad - \partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \partial_\lambda g^{\lambda\delta}\partial_\alpha g_{\beta\delta} + \partial_\lambda g^{\lambda\delta}\partial_\beta g_{\alpha\delta} - \partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta} \end{aligned}$$

- second-order terms in the metric
- products of first-order terms in the metric

On the other hand, consider the term $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$ (which appears in our decomposition):

- **Christoffel symbols**

$$\Gamma^\gamma = g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta}\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right)$$
$$= g^{\gamma\delta} g^{\alpha\beta}\partial_\alpha g_{\beta\delta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta}\partial_\delta g_{\alpha\beta}$$

  and, therefore, $\Gamma_\lambda = g_{\lambda\gamma}\Gamma^\gamma = g^{\alpha\beta}\partial_\alpha g_{\beta\lambda} - \frac{1}{2} g^{\alpha\beta}\partial_\lambda g_{\alpha\beta}$.

- **Derivatives of Christoffel symbols**

$$\partial_\alpha \Gamma_\beta = \partial_\alpha\left(g^{\delta\lambda}\partial_\delta g_{\lambda\beta}\right) - \frac{1}{2}\partial_\alpha\left(g^{\lambda\delta}\partial_\beta g_{\lambda\delta}\right)$$
$$= g^{\delta\lambda}\partial_\alpha\partial_\delta g_{\lambda\beta} - \frac{1}{2} g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta} + \partial_\alpha g^{\delta\lambda}\partial_\delta g_{\lambda\beta}.$$

The term of interest is thus:

$$\begin{aligned}\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha =& g^{\lambda\delta}\partial_\alpha\partial_\lambda g_{\delta\beta} + g^{\lambda\delta}\partial_\beta\partial_\lambda g_{\delta\alpha} - g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} \\ &+ \partial_\alpha g^{\lambda\delta}\partial_\delta g_{\lambda\beta} + \partial_\beta g^{\lambda\delta}\partial_\delta g_{\lambda\alpha} \\ &- \frac{1}{2}\partial_\beta g^{\lambda\delta}\partial_\alpha g_{\lambda\delta} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta}\end{aligned}$$

Observe that

- The second-order terms in $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$ are exactly three of the (four) second-order terms arising in the expression of $\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}$.
- The last term in $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$ coincides with the last term in $\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}$.

$$\begin{aligned}2\left(\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}\right) &= -g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + \left(\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha\right) \\ &\quad - \partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \partial_\lambda g^{\lambda\delta}\partial_\alpha g_{\beta\delta} + \partial_\lambda g^{\lambda\delta}\partial_\beta g_{\alpha\delta} \\ &\quad - \partial_\alpha g^{\lambda\delta}\partial_\delta g_{\lambda\beta} - \partial_\beta g^{\lambda\delta}\partial_\delta g_{\lambda\alpha} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta} + \frac{1}{2}\partial_\beta g^{\lambda\delta}\partial_\alpha g_{\lambda\delta}\end{aligned}$$

$$\begin{aligned}2\left(\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}\right) &= -\partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \left(\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha\right) \\ &\quad + g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\delta g_{\alpha\beta} - g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\alpha g_{\beta\delta} \\ &\quad - g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\beta g_{\alpha\delta} + \underline{\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\beta g_{\lambda\delta}} \\ &\quad + g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} + g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha} \\ &\quad - \underline{\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\alpha g_{\lambda\delta}}\end{aligned}$$

- We have used the identity $\partial_\alpha g^{\lambda\delta} = -g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda'\delta'}$.
- The two underlined terms above are opposite to each other.

Second-order terms in the Ricci curvature

$$
\begin{aligned}
2\left(\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda}\right) = & -\partial_\lambda g^{\lambda\delta} \partial_\delta g_{\alpha\beta} + \left(\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha\right) \\
& + g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\delta g_{\alpha\beta} - g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\alpha g_{\beta\delta} \\
& - g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\beta g_{\alpha\delta} \\
& + g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} + g^{\lambda\lambda'} g^{\delta\delta'} \partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha}
\end{aligned}
$$

We need to deal with the five terms above.

Quadratic terms in the expression of the Ricci curvature

$$4\,\Gamma^{\lambda}_{\alpha\beta}\Gamma^{\delta}_{\lambda\delta} = g^{\lambda\lambda'}g^{\delta\delta'}\Big(\partial_\lambda g_{\delta\delta'}\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'}\partial_\lambda g_{\delta\delta'} - \partial_{\lambda'} g_{\alpha\beta}\partial_\lambda g_{\delta\delta'}\Big)$$

$$\begin{aligned}4\,\Gamma^{\lambda}_{\alpha\delta}\Gamma^{\delta}_{\beta\lambda} = g^{\lambda\lambda'}g^{\delta\delta'}\Big(&\partial_\alpha g_{\delta\lambda'}\partial_\beta g_{\lambda\delta'} + \partial_\alpha g_{\delta\lambda'}\partial_\lambda g_{\beta\delta'} - \partial_\alpha g_{\delta\lambda'}\partial_{\delta'} g_{\beta\lambda}\\ &+ \partial_\delta g_{\alpha\lambda'}\partial_\beta g_{\lambda\delta'} + \partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\alpha\lambda'}\partial_{\delta'} g_{\beta\lambda}\\ &- \partial_{\lambda'} g_{\alpha\delta}\partial_\beta g_{\lambda\delta'} - \partial_{\lambda'} g_{\alpha\delta}\partial_\lambda g_{\beta\delta'} + \partial_{\lambda'} g_{\alpha\delta}\partial_{\delta'} g_{\beta\lambda}\Big)\end{aligned}$$

We deduce that

$$\begin{aligned}&4\left(\Gamma^{\lambda}_{\alpha\beta}\Gamma^{\delta}_{\lambda\delta} - \Gamma^{\lambda}_{\alpha\delta}\Gamma^{\delta}_{\beta\lambda}\right)\\ &= -g^{\lambda\lambda'}g^{\delta\delta'}\partial_{\lambda'} g_{\alpha\beta}\partial_\lambda g_{\delta\delta'} + g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_{\delta'} g_{\beta\lambda} + g^{\lambda\lambda'}\partial_{\lambda'} g_{\alpha\delta}\partial_\lambda g_{\beta\delta'}\\ &\quad - g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\delta\lambda'}\partial_\beta g_{\lambda\delta'}\\ &\quad + \underline{g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\delta\delta'}\partial_\alpha g_{\beta\lambda'}} + \underline{g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\delta\delta'}\partial_\beta g_{\alpha\lambda'}} - 2\,g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}.\end{aligned}$$

Observe that:

- The first three terms are “null terms”.
- The fourth term is a quasi-null term.
- The two underlined terms will be the opposite to two other (underlined) terms derived below.

Hence, only the last term remains to be dealt with.

In summary, in the Ricci expression, we need to deal with six terms:

$$\Omega_1 := \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\delta g_{\alpha\beta} \qquad \Omega_2 := -\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\alpha g_{\beta\delta}$$

$$\Omega_3 := -\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\lambda g_{\lambda'\delta'} \partial_\beta g_{\alpha\delta} \qquad \Omega_4 := \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta}$$

$$\Omega_5 := \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha} \qquad \Omega_6 := -\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'}$$

General identities for the metric

$$g^{\alpha\beta} \partial_\alpha g_{\beta\delta} - \frac{1}{2} g^{\alpha\beta} \partial_\delta g_{\alpha\beta} = \Gamma_\delta \qquad g_{\beta\delta} \partial_\alpha g^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \partial_\delta g^{\alpha\beta} = \Gamma_\delta$$

First three terms

$$\Omega_1 = \frac{1}{2} g^{\delta\delta'} \partial_\delta g_{\alpha\beta} \Gamma_{\delta'} + \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\beta} \partial_{\delta'} g_{\lambda\lambda'}$$

$$\Omega_2 = -\frac{1}{2} g^{\delta\delta'} \partial_\alpha g_{\beta\delta} \Gamma_{\delta'} - \underline{\frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_{\delta'} g_{\lambda\lambda'} \partial_\alpha g_{\beta\delta}}$$

$$\Omega_3 = -\frac{1}{2} g^{\delta\delta'} \partial_\beta g_{\alpha\delta} \Gamma_{\delta'} - \underline{\frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_{\delta'} g_{\lambda\lambda'} \partial_\beta g_{\alpha\delta}}$$

- The last term in Ω_1 is one of the quasi-null term.
- The two underlined terms are opposite to the two underlined terms above.

Fourth term

$$\begin{aligned}\Omega_4 =& \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\right) + \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'} \\
=& \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\right) \\
& + \frac{1}{2} g^{\lambda\lambda'} \partial_\alpha g_{\lambda\beta} \Gamma_{\lambda'} + \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} \\
=& \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\right) \\
& + \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta}\right) \\
& + \frac{1}{2} g^{\lambda\lambda'} \partial_\alpha g_{\lambda\beta} \Gamma_{\lambda'} + \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta}\end{aligned}$$

$$\begin{aligned}\Omega_4 =& \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'}\right) \\
& + \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta}\right) \\
& + \frac{1}{2} g^{\lambda\lambda'} \partial_\alpha g_{\lambda\beta} \Gamma_{\lambda'} + \frac{1}{4} g^{\delta\delta'} \partial_\alpha g_{\delta\delta'} \Gamma_\beta + \frac{1}{8} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\alpha g_{\delta\delta'} \partial_\beta g_{\lambda\lambda'}\end{aligned}$$

Fifth term

$$\Omega_5 = \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha} - \partial_\beta g_{\lambda\alpha} \partial_\delta g_{\lambda'\delta'}\right)$$
$$+ \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda\alpha} \partial_{\lambda'} g_{\delta\delta'} - \partial_\beta g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\alpha}\right)$$
$$+ \frac{1}{2} g^{\lambda\lambda'} \partial_\beta g_{\lambda\alpha} \Gamma_{\lambda'} + \frac{1}{4} g^{\delta\delta'} \partial_\beta g_{\delta\delta'} \Gamma_\alpha + \frac{1}{8} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\beta g_{\delta\delta'} \partial_\alpha g_{\lambda\lambda'}$$

Sixth and last term Ω_6

$$- \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\right) - \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}$$
$$= - \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\right) - \frac{1}{2} g^{\lambda\lambda'} \partial_\lambda g_{\alpha\lambda'} \Gamma_\beta - \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\beta g_{\delta\delta'} \partial_\lambda g_{\alpha\lambda'}$$
$$= - \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\right) - \frac{1}{2} g^{\lambda\lambda'} \partial_\lambda g_{\alpha\lambda'} \Gamma_\beta - \frac{1}{4} g^{\delta\delta'} \partial_\beta g_{\delta\delta'} \Gamma_\alpha$$
$$- \frac{1}{8} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda\lambda'} \partial_\beta g_{\delta\delta'}$$

$$\Omega_6 = - \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'}\right) - \frac{1}{2} \Gamma_\alpha \Gamma_\beta$$
$$- \frac{1}{4} g^{\delta\delta'} \partial_\alpha g_{\delta\delta'} \Gamma_\beta - \frac{1}{4} g^{\delta\delta'} \partial_\beta g_{\delta\delta'} \Gamma_\alpha$$
$$- \frac{1}{8} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\lambda\lambda'} \partial_\beta g_{\delta\delta'}$$

In conclusion the **quadratic terms in the Ricci curvature** $R_{\alpha\beta}$ read:

$$
\begin{aligned}
&\frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\delta g_{\alpha\lambda'} \partial_{\delta'} g_{\beta\lambda} - \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\delta g_{\alpha\lambda'} \partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'} \partial_\lambda g_{\alpha\lambda'} \right) \\
&+ \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda'\delta'} \partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta} \partial_\delta g_{\lambda'\delta'} \right) \\
&+ \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\alpha g_{\lambda\beta} \partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\beta} \right) \\
&+ \frac{1}{2} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda'\delta'} \partial_\delta g_{\lambda\alpha} - \partial_\beta g_{\lambda\alpha} \partial_\delta g_{\lambda'\delta'} \right) \\
&+ \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \left(\partial_\beta g_{\lambda\alpha} \partial_{\lambda'} g_{\delta\delta'} - \partial_\beta g_{\delta\delta'} \partial_{\lambda'} g_{\lambda\alpha} \right) \\
&- \frac{1}{4} g^{\lambda\lambda'} g^{\delta\delta'} \partial_\alpha g_{\delta\lambda'} \partial_\beta g_{\lambda\delta'} + \frac{1}{8} g^{\delta\delta'} g^{\lambda\lambda'} \partial_\beta g_{\delta\delta'} \partial_\alpha g_{\lambda\lambda'} \\
&+ \frac{1}{2} g^{\delta\delta'} \partial_\delta g_{\alpha\beta} \Gamma_{\delta'} - \frac{1}{2} \Gamma_\alpha \Gamma_\beta .
\end{aligned}
$$

By collecting all the terms above, we arrive at the desired identity.

Section 3. THE HYPERBOLOIDAL FOLIATION METHOD

3.1 WAVE EQUATIONS WITH NULL INTERACTIONS

The simplest model

$$\Box u = P^{\alpha\beta} \partial_\alpha u \, \partial_\beta u \qquad u|_{\mathcal{H}_{s_0}} = u_0, \quad \partial_t u|_{\mathcal{H}_{s_0}} = u_1 \qquad (\star)$$

- initial data u_0, u_1 compactly supported in the intersection of the spacelike hypersurface $\mathcal{H}_{s_0}$ and the cone $\mathcal{K} = \big\{(t,x) \,/\, |x| < t-1\big\}$ with $s_0 > 1$
- standard null condition: $P^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ for all $\xi \in \mathbb{R}^4$ satisfying $-\xi_0^2 + \sum_a \xi_a^2 = 0$
- hyperboloidal energy $E_M = E_{M,0}$: Minkowski metric and zero K-G mass
- admissible vector fields $Z \in \mathscr{Z}$: spacetime translations ∂_α, boosts L_a

Theorem. Global existence theory for wave equations with null interactions

There exist $\varepsilon_0 > 0$ and $C_1 > 1$ such that for all initial data satisfying

$$\sum_{|I|\leqslant 3} \sum_{Z \in \mathscr{Z}} E_M(s_0, Z^I u)^{1/2} \leqslant \varepsilon \leqslant \varepsilon_0$$

the Cauchy problem $(\star)$ admits a global-in-time solution, satisfying the uniform energy bound

$$\sum_{|I|\leqslant 3} \sum_{Z \in \mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant C_1 \varepsilon$$

and the uniform decay estimate $\big|\partial_\alpha u(t,x)\big| \leqslant \frac{C_1 \varepsilon}{t\,(t-|x|)^{1/2}}$.

Bootstrap strategy. For some (sufficiently large) constant $C_1 > 1$, let us assume that the local-in-time solution satisfies within some time interval $[s_0, s_1]$

$$\sum_{|I| \leqslant 3, Z \in \mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant C_1 \varepsilon, \qquad s \in [s_0, s_1]. \qquad (\star\star)$$

More precisely:

- let s_1 be the *largest* hyperboloidal time such that $(\star\star)$ holds true.
- Since $C_1 > 1$, by the local existence theory and by a continuity argument, we do have $s_1 > s_0$.

Suppose s_1 is finite. We are going to prove that

- for suitably chosen constants ε_0 and $C_1 > 1$ and for all $\varepsilon \leqslant \varepsilon_0$

$$\sum_{|I| \leqslant 3, Z \in \mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant \frac{C_1}{2} \varepsilon, \qquad s \in [s_0, s_1].$$

- Consequently, s_1 cannot be the largest time and we get a contradiction
- unless $s_1 = +\infty$.

Proposition

Let u be a solution defined in $[s_0, s_1]$ whose initial data satisfies

$$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s_0, Z^I u)^{1/2} \leqslant \varepsilon.$$

Then, there exist constants $\varepsilon_0 > 0$ and $C_1 > 1$ such that if

$$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant C_1\varepsilon, \qquad s \in [s_0, s_1]$$

for $\varepsilon \leqslant \varepsilon_0$, then the following stronger estimate holds

$$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant \frac{C_1}{2}\varepsilon, \qquad s \in [s_0, s_1].$$

Recall our notation

- The semi-hyperboloidal frame consists of three vectors tangent to the hyperboloids $\underline{\partial}_a := \frac{L_a}{t} = \frac{x^a}{t}\partial_t + \partial_a$ and the timelike vector $\underline{\partial}_0 := \partial_t$.
- Admissible vector fields $Z \in \{\partial_\alpha, L_a\}$

Lemma. The energy estimate for hyperboloids

$$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s, Z^I u)^{1/2}$$

$$\leqslant \sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s_0, Z^I u)^{1/2} + \int_{s_0}^{s} \sum_{|I|\leqslant 3, Z\in\mathscr{Z}} \left\| Z^I\left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right)\right\|_{L^2_f(\mathcal{H}_{s'})} ds'$$

Notation. $\|u\|_{L^2_f(\mathcal{H}_s)} := \int_{\mathcal{H}_s} |u|\, dx = \int |u(\sqrt{s^2+|x|^2}, x)|\, dx$

Proof. Using the product Z^I (with $|I| \leqslant 3$) and the commutation property $[Z^I, \Box] = 0$:

$$\Box(Z^I u) = Z^I\left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right).$$

By multiplication by $\partial_t Z^I u$, we see that $\tilde{u} := Z^I u$ satisfies

$$\frac{1}{2}\partial_t\Big((\partial_t\tilde{u})^2 + \sum_a(\partial_a\tilde{u})^2\Big) - \partial_a\left(\partial_a\tilde{u}\partial_t\tilde{u}\right) = \partial_t\tilde{u}\, Z^I\left(P^{\alpha\beta}\partial_\alpha\tilde{u}\partial_\beta\tilde{u}\right).$$

By integrating in the region $\mathcal{K}_{[s_0,s]}$ limited by two hyperboloids

$$\int_{\mathcal{K}_{[s_0,s]}} \Big(\frac{1}{2}\partial_t\Big((\partial_t\tilde{u})^2 + \sum_a(\partial_a\tilde{u})^2\Big) - \partial_a\left(\partial_a\tilde{u}\partial_t\tilde{u}\right)\Big)\, dtdx = \int_{\mathcal{K}_{[s_0,s]}} \partial_t\tilde{u}\, Z^I\left(P^{\alpha\beta}\partial_\alpha\tilde{u}\partial_\beta u\right) dtdx.$$

Left-hand side. By Stokes' formula:

$$\frac{1}{2}\int_{\mathcal{H}_s}\Big(\big(\partial_t\widetilde{u}\big)^2+\sum_a\big(\partial_a\widetilde{u}\big)^2, 2\partial_t\widetilde{u}\partial_a\widetilde{u}\Big)\cdot n\,d\sigma$$

$$-\frac{1}{2}\int_{\mathcal{H}_{s_0}}\big(|\partial_t\widetilde{u}|^2+\sum_a|\partial_a\widetilde{u}|^2, 2\partial_t\widetilde{u}\partial_a\widetilde{u}\big)\cdot n d\sigma$$

- $n=\big(t^2+|x|^2\big)^{-1/2}(t,-x^a)$: future oriented, unit normal to the hyperboloids
- $d\sigma=\frac{\big(t^2+|x|^2\big)^{1/2}}{t}dx$: induced measure on the hyperboloids

$$\frac{1}{2}\int_{\mathcal{H}_s}\Big(|\partial_t\widetilde{u}|^2+\sum_a|\partial_a\widetilde{u}|^2+2\frac{x^a}{t}\partial_a\widetilde{u}\partial_t\widetilde{u}\Big)\,dx$$

$$-\frac{1}{2}\int_{\mathcal{H}_{s_0}}\Big(|\partial_t\widetilde{u}|^2+\sum_a|\partial_a\widetilde{u}|^2+2\frac{x^a}{t}\partial_a\widetilde{u}\partial_t\widetilde{u}\Big)\,dx$$

$$=\frac{1}{2}E_M(s,Z^I u)-\frac{1}{2}E_M(s_0,Z^I u)$$

in view of the expression of the hyperboloidal energy
$E_{M,c}(s,u)=\int_{\mathcal{H}_s}\Big(\frac{s^2}{s^2+r^2}(\underline{\partial}_0 u)^2+\sum_{a=1}^3\big(\underline{\partial}_a u\big)^2+\frac{c^2}{2}\,u^2\Big)\,dx$ but here $c=0$.

Right-hand side. With the change of variable $s' = (t^2 - |x|^2)^{1/2}$ and the identity $dtdx = (s'/t)ds'dx$:

$$\int_{s_0}^{s} \int_{\mathcal{H}_{s'}} (s'/t)\partial_t u Z^I \left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right) dxds'$$

Consequently

$$\frac{1}{2}E_M(s, Z^I u) - \frac{1}{2}E_M(s_0, Z^I u) = \int_{s_0}^{s} \int_{\mathcal{H}_{s'}} (s'/t)\partial_t u Z^I \left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right) ds'dx$$

After differentiation in s:

$$\begin{aligned} E(s, Z^I u)^{1/2} \frac{d}{ds} E(s, Z^I u)^{1/2} &= \int_{\mathcal{H}_s} (s'/t)\partial_t u Z^I \left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right) dx \\ &\leqslant \|(s/t)\partial_t u\|_{L^2_f(\mathcal{H}_s)} \left\|Z^I \left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right)\right\|_{L^2_f(\mathcal{H}_s)}. \end{aligned}$$

By the definition of the hyperboloidal energy: $E(s, u)^{1/2} \geqslant \|(s/t)\partial_t u\|_{L^2_f(\mathcal{H}_s)}$ and therefore

$$\frac{d}{ds} \sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E(s, Z^I u)^{1/2} \leqslant \sum_{|I|\leqslant 3, Z\in\mathscr{Z}} \left\|Z^I \left(P^{\alpha\beta}\partial_\alpha u \partial_\beta u\right)\right\|_{L^2_f(\mathcal{H}_s)}.$$

We conclude by integrating over the hyperbolic time interval $[s_0, s]$.

Lemma. L^2 estimate based on the hyperboloidal energy

For all $s \in [s_0, s_1]$: $$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} \|\underline{\partial}_a Z^I u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)\underline{\partial}_0 Z^I u\|_{L^2_f(\mathcal{H}_s)} \leqslant CC_1\varepsilon$$

Lemma. L^2 estimate based on commutators

For all $s \in [s_0, s_1]$:

$$\sum_{|I_1|+|I_2|\leqslant 3} \sum_{Z\in\mathscr{Z}} \|Z^{I_1}\underline{\partial}_a Z^{I_2} u\|_{L^2_f(\mathcal{H}_s)} + \|Z^{I_1}\big((s/t)\underline{\partial}_0 Z^{I_2} u\big)\|_{L^2_f(\mathcal{H}_s)} \leqslant CC_1\varepsilon$$

Lemma. Commutator estimates

$$\big|[Z^I, \underline{\partial}_\alpha]u\big| + \big|[Z^I, \partial_\alpha]u\big| \lesssim \sum_\beta \sum_{|J|<I} \big|\partial_\beta Z^J u\big|$$

$$\big|[Z^I, \underline{\partial}_a]u\big| \lesssim \sum_b \sum_{|J_1|<|I|} \big|\underline{\partial}_b Z^{J_1} u\big| + \frac{1}{t}\sum_\gamma \sum_{J_2|<|I|} \big|\partial_\gamma Z^{J_2} u\big|$$

$$\big|[Z^I, \partial_\alpha\partial_\beta]u\big| \lesssim \sum_{\gamma,\gamma'} \sum_{|J|<|I|} \big|\partial_\gamma\partial_{\gamma'} Z^J u\big|$$

$$\big|[Z^I, \underline{\partial}_a\underline{\partial}_\beta]u\big| + \big|[Z^I, \underline{\partial}_\alpha\underline{\partial}_b]u\big| \lesssim \sum_{c,\gamma} \sum_{|J_1|\leqslant|I|} \big|\underline{\partial}_c\underline{\partial}_\gamma Z^{J_1} u\big| + \frac{1}{t}\sum_\gamma \sum_{|J_2|\leqslant|I|} \big|\partial_\gamma Z^{J_2} u\big|$$

Proposition. The Sobolev inequality on hyperboloids

For all functions u defined on the hyperboloid $\mathcal{H}_s$ $\qquad (s^2 = t^2 - r^2)$

$$\sup_{(t,x)\in\mathcal{H}_s} t^{3/2}|u(t,x)| \lesssim \sum_{|I|\leqslant 2} \|L^I u\|_{L^2_f(\mathcal{H}_s)} \qquad s \geqslant s_0 > 1$$

with summation over all boosts $L \in \{L_a = x^a \partial_t + t\partial_a\}$

Lemma. Decay estimate in the sup-norm

For all $s \in [s_0, s_1]$:

$$\|t^{3/2}\underline{\partial}_a Z^J u\|_{L^\infty(\mathcal{H}_s)} + \|t^{1/2} s \underline{\partial}_0 Z^J u\|_{L^\infty(\mathcal{H}_s)} \leqslant CC_1\varepsilon \qquad |J| \leqslant 1$$

Recall that $s \leqslant t$

We now decompose the null forms on the semi-hyperboloidal frame. The following lemma follows from the identity (and similar identities)

$$(\partial_t u)^2 - \sum_a (\partial_a u)^2 = \frac{t^2 - r^2}{t^2} (\underline{\partial}_0 u)^2 + 2\frac{x^a}{t} \underline{\partial}_0 u \underline{\partial}_a u - \sum_a (\underline{\partial}_a u)^2$$

in which we have used $\underline{\partial}_0 = \partial_t$ and $\underline{\partial}_a = \frac{x^a}{t}\partial_t + \partial_a$ $(a = 1, 2, 3)$.

Lemma. Algebraic structure of null quadratic forms

For all null quadratic form $T^{\alpha\beta}\partial_\alpha u \partial_\beta v$

$$\left|Z^I \left(T^{\alpha\beta}\partial_\alpha u \partial_\beta v\right)\right| \lesssim \sum_{a,\beta} \sum_{|I_1|+|I_2|\leqslant|I|} \left(\left|Z^{I_1}\underline{\partial}_a u \, Z^{I_2}\underline{\partial}_\beta v\right| + \left|Z^{I_1}\underline{\partial}_\beta u \, Z^{I_2}\underline{\partial}_a v\right|\right)$$
$$+ (s/t)^2 \sum_{|I_1|+|I_2|\leqslant|I|} \left|Z^{I_1}\underline{\partial}_0 u Z^{I_2}\underline{\partial}_0 v\right|$$

Observations.

- The derivatives $\underline{\partial}_a$ enjoy better L^∞ and L^2 decay estimates.
- The derivative $\underline{\partial}_0$
    - not enough decay
    - but a favorable factor $(s/t)^2$ in front of $Z^{I_1}\underline{\partial}_0 u Z^{I_2}\underline{\partial}_0 v$

Lemma. Decay estimate for the interaction term

$$\left\| Z^I \left(P^{\alpha\beta} \partial_\alpha u \partial_\beta u \right) \right\|_{L^2_f(\mathcal{H}_s)} \leqslant C \left(C_1 \varepsilon \right)^2 s^{-3/2}$$

$C > 0$ being a universal constant

Proof.

$$\begin{aligned}
\left| Z^I \left(P^{\alpha\beta} \partial_\alpha u \partial_\beta u \right) \right| \leqslant{} & C (s/t)^2 \sum_{|I_1|+|I_2| \leqslant |I|} \left| Z^{I_1} \partial_t u Z^{I_2} \partial_t u \right| \\
& + C \sum_{\substack{a, \beta, \\ |I_1|+|I_2| \leqslant |I|}} \left(\left| Z^{I_1} \underline{\partial}_a u \, Z^{I_2} \underline{\partial}_\beta u \right| + \left| Z^{I_1} \underline{\partial}_\beta u \, Z^{I_2} \underline{\partial}_a u \right| \right) \\
=:{} & T_1 + T_2
\end{aligned}$$

We now combine the L^2 and L^∞ estimates together.

Terms in T_1

Observe that $|I_1| + |I_2| \leqslant |I| \leqslant 3$ implies that $|I_1| \leqslant 1$ or $|I_2| \leqslant 1$. Without loss of generality, we can assume $|I_2| \leqslant 1$ and write

$$\begin{aligned}\left\|(s/t)^2 Z^{I_1}\partial_t u Z^{I_2}\partial_t u\right\|_{L^2_f(\mathcal{H}_s)} &\leqslant \left\|(s/t) Z^{I_1}\partial_t u\right\|_{L^2_f(\mathcal{H}_s)} \left\|t^{-3/2}\left(t^{1/2} s Z^{I_2}\partial_t u\right)\right\|_{L^\infty(\mathcal{H}_s)} \\ &\leqslant C(C_1\varepsilon)^2 s^{-3/2} \qquad \text{since } t \geqslant s\end{aligned}$$

Terms in T_2

When $|I_1| \leqslant 1$ we write

$$\left\|Z^{I_1}\underline{\partial}_a u\, Z^{I_2}\underline{\partial}_\beta u\right\|_{L^2_f(\mathcal{H}_s)} \leqslant s^{-3/2}\left\|t^{3/2} Z^{I_1}\underline{\partial}_a u\right\|_{L^\infty(\mathcal{H}_s)} \left\|(s/t) Z^{I_2}\underline{\partial}_\beta u\right\|_{L^2_f(\mathcal{H}_s)} \leqslant C(C_1\varepsilon)^2 s^{-3/2}$$

When $|I_2| \leqslant 1$, we write

$$\left\|Z^{I_1}\underline{\partial}_a u\, Z^{I_2}\underline{\partial}_\beta u\right\|_{L^2_f(\mathcal{H}_s)} \leqslant s^{-3/2}\left\|Z^{I_1}\underline{\partial}_a u\right\|_{L^2_f(\mathcal{H}_s)} \left\|t^{1/2} s Z^{I_2}\underline{\partial}_\beta u\right\|_{L^\infty(\mathcal{H}_s)} \leqslant C(C_1\varepsilon)^2 s^{-3/2}$$

CONCLUSION.

We combine the null form estimate with the energy estimate:

$$\sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant \sum_{|I|\leqslant 3, Z\in\mathscr{Z}} E_M(s_0, Z^I u)^{1/2} + C(C_1\varepsilon)^2 \int_{s_0}^{s} (s')^{-3/2}\, ds'$$

$$\leqslant \varepsilon + C(C_1\varepsilon)^2 \leqslant \frac{1}{2} C_1\varepsilon$$

by choosing $C_1 > 2$ and $\varepsilon \leqslant \varepsilon_0 := \frac{C_1 - 2}{2CC_1^2}$.

Section 3.2 WAVE-KLEIN-GORDON SYSTEMS WITH STRONG METRIC INTERACTIONS

From the Einstein equations in wave gauge, we can formally derive the following model:

$$
\begin{aligned}
-\Box u &= P^{\alpha\beta}\partial_\alpha v \partial_\beta v + R v^2 \\
-\Box v + u\, H^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v &= 0
\end{aligned}
$$

with arbitrary $P^{\alpha\beta}, H^{\alpha\beta}, R$

Theorem. Global existence theory for wave-Klein-Gordon systems with strong metric interactions

Consider the nonlinear wave-Klein-Gordon model with given $P^{\alpha\beta}, R, H^{\alpha\beta}$ and $c > 0$. Given any $N \geqslant 8$, there exists $\epsilon = \epsilon(N) > 0$ such that

- if the initial data satisfy $\|(u_0, v_0)\|_{H^{N+1}(\mathbb{R}^3)} + \|(u_1, v_1)\|_{H^N(\mathbb{R}^3)} < \epsilon$
- then the Cauchy problem for the model problem admits a global-in-time solution.

Bootstrap based on a hierarchy of energy bounds $(k := |J|)$

$E_M[s, \partial^I L^J u]^{1/2} \leqslant C_1 \epsilon s^{k\delta}$	$\|I\| + \|J\| \leqslant N$	wave/high-order
$E_M[s, \partial^I L^J v]^{1/2} \leqslant C_1 \epsilon s^{1/2+k\delta}$	$\|I\| + \|J\| \leqslant N$	K-G/high-order
$E_M[s, \partial^I L^J u]^{1/2} \leqslant C_1 \epsilon$	$\|I\| + \|J\| \leqslant N-4$	wave/low-order
$E_M[s, \partial^I L^J v]^{1/2} \leqslant C_1 \epsilon s^{k\delta}$	$\|I\| + \|J\| \leqslant N-4$	K-G/low-order

Energy bounds

- Wave component u
    - High-order energy "quasi-conserved"
    - Low-order energy "conserved"
- Klein-Gordon component v
    - High-order energy in $s^{1/2+}$: specific to strong metric interactions
    - Low-order energy "quasi-conserved"
- The proof uses also the energy functional associated with the curved metric.

NEW TECHNICAL LEMMAS (Hardy, $L^\infty - L^\infty$)

Proposition. Hardy-type estimates for the hyperboloidal foliation

For all functions u defined on a hyperboloid $\mathcal{H}_s$:

$$\left\| \frac{u}{r} \right\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} \qquad \text{with } \underline{\partial}_a = t^{-1} L_a$$

For all functions defined on the hyperboloidal foliation

$$\left\| \frac{u}{s} \right\|_{L^2_f(\mathcal{H}_s)} \lesssim \left\| \frac{u}{s_0} \right\|_{L^2_f(\mathcal{H}_{s_0})} + \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}$$
$$+ \sum_a \int_{s_0}^{s} \Big(\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_{s'})} + \|(s'/t)\partial_a u\|_{L^2_f(\mathcal{H}_{s'})} \Big) \frac{ds'}{s'}$$

Proof. Compute the divergence of the vector field $W = \Big(0, \frac{t\, x^a}{(1+r^2)s^2} \chi(r/t) u^2\Big)$ for some smooth cut-off function $\chi(r) = \begin{cases} 0 & 0 \leqslant r \leqslant 1/3 \\ 1 & 2/3 \leqslant r \end{cases}$

Remark. Similar to $\partial_r(u^2/r)$ for the classical Hardy inequality

Lemma. Decomposition for Klein-Gordon in curved space

v being a solution to $-\widetilde{\Box}_g v + c^2 v = f$

- metric $g^{\alpha\beta} = m^{\alpha\beta} - h^{\alpha\beta}$
- $\sup |\overline{h}^{00}| \leqslant 1/2$ (component in the hyperboloidal frame)

The function $w_{t,x}(\lambda) := \lambda^{3/2} v(\lambda t/s, \lambda x/s)$ (with $s = \sqrt{t^2 - r^2}$) satisfies the second-order ODE in λ

$$\frac{d^2}{d\lambda^2} w_{t,x} + \frac{c^2}{1 + \overline{h}^{00}} w_{t,x} = k_{t,x}$$

with $\overline{h}^{00} = \overline{h}^{00}(\lambda t/s, \lambda x/s)$ and a right-hand side $k_{t,x} = k_{t,x}(\lambda)$ defined below

- We do this analysis in **hyperboloidal coordinates** $(\overline{x}^0, \overline{x}^a) = (s, x^a)$
- $\overline{h}^{\alpha\beta}$: components
- in the **hyperboloidal frame**

$$\overline{\partial}_0 := \partial_s = \frac{s}{t}\partial_t = \frac{\sqrt{t^2 - r^2}}{t}\partial_t$$

$$\overline{\partial}_a := \partial_{\overline{x}^a} = \frac{\overline{x}^a}{t}\partial_t + \partial_a = \frac{x^a}{t}\partial_t + \partial_a$$

$k_{t,x} := \frac{R_1[v]+R_2[v]+R_3[v]+s^{3/2}f}{1+\overline{h}^{00}}$ $\overline{\Psi}^{\alpha}_{\beta}$: matrix of change of frame

$$R_1[v] := s^{3/2}\overline{\partial}_a\overline{\partial}^a v + \frac{1}{s^{1/2}}\left(x^a x^b \overline{\partial}_a\overline{\partial}_b v + \frac{3}{4}v + 3x^a\overline{\partial}_a v\right)$$

$$R_2[v] := 3\,\overline{h}^{00}\left(\frac{1}{4s^{1/2}}v + s^{1/2}\overline{\partial}_0 v\right) - s^{3/2}\left(2\overline{h}^{0b}\overline{\partial}_0\overline{\partial}_b v + \overline{h}^{ab}\overline{\partial}_a\overline{\partial}_b v + h^{\alpha\beta}\partial_\alpha\overline{\Psi}^{\beta'}_{\beta}\,\overline{\partial}_{\beta'} v\right)$$

$$R_3[v] := \overline{h}^{00}\left(2x^a s^{1/2}\overline{\partial}_0\overline{\partial}_a v + \frac{2x^a}{s^{1/2}}\overline{\partial}_a v + \frac{x^a x^b}{s^{1/2}}\overline{\partial}_a\overline{\partial}_b v\right)$$

Lemma. Technical ODE estimate

Given $G : [s_0, s_1) \to [-1/2, 1/2]$ and $k : [s_0, s_1) \to \mathbb{R}$ with $s_1 \in [s_0, +\infty)$ and G', k integrable, the solution z to $z''(\lambda) + \frac{c^2}{1+G(\lambda)}z(\lambda) = k(\lambda)$ with prescribed initial data $z(s_0) = z_0$ and $z'(s_0) = z_1$ satisfies for all $s \in [s_0, s_1)$

$$|z(s)| + |z'(s)| \lesssim |z_0| + |z_1| + K(s) + \int_{s_0}^{s}\left(|z_0| + |z_1| + K(s')\right)|G'(s')|e^{C\int_{s'}^{s}|G'(\lambda)|d\lambda}\,ds'$$

with $K(s) := \int_{s_0}^{s}|k(s')|\,ds'$ and $C > 0$.

Proposition. L^∞-L^∞ estimate for Klein-Gordon equations on curved space

For the Klein-Gordon equation on a curved background $-\widetilde{\Box}_g v + c^2 v = f$

- metric $g^{\alpha\beta} = m^{\alpha\beta} - h^{\alpha\beta}$ perturbation of the Minkowski metric
- data prescribed on a hyperboloid $v|_{\mathcal{H}_{s_0}} = v_0, \quad \partial_t v|_{\mathcal{H}_{s_0}} = v_1$

Then, in the future of $\mathcal{H}_{s_0}$, one has

$$s^{3/2}|v(t,x)| + t\,s^{1/2}|\underline{\partial}_\perp v(t,x)| \lesssim V(t,x)$$

with V defined below. $\underline{\partial}_\perp := \partial_t + \frac{x^a}{t}\partial_a$

Notation. $h_{t,x}(\lambda) := \overline{h}^{00}(\lambda t/s, \lambda x/s)$ $s^2 = t^2 - r^2$

The derivative in λ reads $\underline{\partial}_\perp = \partial_t + \frac{x^a}{t}\partial_a$

$$\begin{aligned} h'_{t,x}(\lambda) &= \frac{t}{s}\partial_t \overline{h}^{00}(\lambda t/s, \lambda x/s) + \frac{x^a}{s}\partial_a \overline{h}^{00}(\lambda t/s, \lambda x/s) \\ &= \frac{t}{s}\,(\underline{\partial}_\perp \overline{h}^{00})(\lambda t/s, \lambda x/s) \end{aligned}$$

Function F defined from the right-hand side f of the K-G equation

$$F(s') := \int_{s_0}^{s'} \Big(\big(R_1[v]+R_2[v]+R_3[v]\big)(\lambda t/s, \lambda x/s)+\lambda^{3/2} f(\lambda t/s, \lambda x/s)\Big)\, d\lambda$$

Function V defined by distinguishing between two regions

- "Far" from the light cone $\qquad 0 \leqslant r/t \leqslant \frac{1+s_0^2}{s_0^2-1}$

$$V(t,x) := \Big(\|v_0\|_{L^\infty(\mathcal{H}_{s_0})} + \|v_1\|_{L^\infty(\mathcal{H}_{s_0})} \Big) \Big(1 + \int_{s_0}^{s} |h'_{t,x}(s')| e^{C \int_{s'}^{s} |h'_{t,x}(\lambda)| d\lambda}\, ds' \Big)$$
$$+ F(s) + \int_{s_0}^{s} F(s') |h'_{t,x}(\lambda)| e^{C \int_{s'}^{s} |h'_{t,x}(\lambda)| d\lambda}\, ds'$$

- "Near" the light cone $\qquad \frac{1+s_0^2}{s_0^2-1} < r/t < 1$

$$V(t,x) := F(s) + \int_{s_0}^{s} F(s') |h'_{t,x}(s')| e^{C \int_{s'}^{s} |h'_{t,x}(\lambda)| d\lambda}\, ds'$$

Proposition. L^∞-L^∞ **estimate for the wave equation with source**

u being a spatially compactly supported to the wave equation $-\Box u = f$ in the cone $\mathcal{K}$

- assume vanishing initial data
- the right-hand side satisfies $|f| \lesssim \frac{1}{t^{2+\nu}(t-r)^{1-\mu}}, \quad t \geqslant 2$

for some exponents $0 < \mu \leqslant 1/2$ and $0 < |\nu| \leqslant 1/2$.

Then, one has the decay property

$$t\,|u(t,x)| \lesssim \begin{cases} \dfrac{1}{\nu\mu}\dfrac{1}{(t-r)^{\nu-\mu}} & 0 < \nu \leqslant 1/2 \\ \dfrac{1}{|\nu|\mu}(t-r)^{\mu}t^{\nu} & -1/2 \leqslant \nu < 0 \end{cases}$$

Proof based on the solution formula for the wave equation

OUTLINE OF THE BOOTSTRAP ARGUMENTS

FIRST PART: energy, commutators, Sobolev

- **Basic L^2 estimates** (bootstrap assumptions) $|I| + |J| \leqslant N$

$$\|(s/t)\partial_\alpha \partial^I L^J u\|_{L^2_f(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{k\delta}$$

$$\|(s/t)\partial_\alpha \partial^I L^J v\|_{L^2_f(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{1/2+k\delta}$$

- **Consequence based on commutators** $|I| + |J| \leqslant N$

$$\|(s/t)\partial^I L^J \partial_\alpha u\|_{L^2_f(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{k\delta}$$

$$\|(s/t)\partial^I L^J \partial_\alpha v\|_{L^2_f(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{1/2+k\delta}$$

- **Basic sup-norm estimates** (Sobolev ineq. on hyperboloids) $|I| + |J| \leqslant N - 2$

$$\|t^{1/2} s \partial_\alpha \partial^I L^J u\|_{L^\infty(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{(k+2)\delta}$$

$$\|t^{1/2} s \partial_\alpha \partial^I L^J v\|_{L^\infty(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{1/2+(k+2)\delta}$$

- **Consequence based on commutators** $|I| + |J| \leqslant N - 2$

$$\|t^{1/2} s \partial^I L^J \partial_\alpha u\|_{L^\infty(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{(k+2)\delta}$$

$$\|t^{1/2} s \partial^I L^J \partial_\alpha v\|_{L^\infty(\mathcal{H}_s)} + \ldots \lesssim C_1 \epsilon s^{1/2+(k+2)\delta}$$

SECOND PART: sharp pointwise bounds $(|I| + |J| \leqslant N - 4)$

$$t\,|L^J u| \lesssim C_1 \epsilon s^{k\delta}$$
$$t^{3/2}|\partial^I L^J v| \lesssim C_1 \epsilon s^{k\delta}(s/t)^{1/2-4\delta}$$
$$t^{3/2}|\underline{\partial}_\perp \partial^I L^J v| \lesssim C_1 \epsilon s^{k\delta}(s/t)^{3/2-4\delta}$$

- **First bound for the wave component** : *low-order* $\partial^I L^J$

(L^∞–L^∞ for the wave component)

$|I| + |J| \leqslant N - 7$

$$|\partial^I L^J u| \lesssim C_1 \epsilon t^{-3/2} + (C_1\epsilon)^2 (t/s)^{-(k+4)\delta} t^{-1} s^{(k+4)\delta}$$

- **Second/first bounds for wave/KG components** *at zero-order*

(L^∞–L^∞ for the wave and K-G equations)

$$t\,|u(t,x)| \lesssim C_1 \epsilon$$
$$s^{3/2}\,|v| + ts^{1/2}|\underline{\partial}_\perp v(t,x)| \lesssim C_1 \epsilon (t/s)^{-2+7\delta}$$

(uniform integrability for the coefficients of the ODE)

- **Second bound for the KG component**: *only* ∂^I and again L^∞–L^∞
  $|I| \leqslant N-4$

$$|\underline{\partial}_\perp \partial^I v(t,x)| \lesssim C_1\epsilon (t/s)^{-3/2+4\delta} t^{-3/2}$$
$$|\partial^I v(t,x)| \lesssim C_1\epsilon (t/s)^{-1/2+4\delta} t^{-3/2}$$

- **Final estimate for wave/KG components**: *applying now* L^J
  $|I|+|J| \leqslant N-4$

$$t\,|L^J u| \lesssim C_1\epsilon s^{k\delta}$$
$$\left((t/s)^{3-7\delta} s^{3/2} |\underline{\partial}_\perp \partial^I L^J v|\right) + \left((t/s)^{2-7\delta} s^{3/2} |\partial^I L^J v|\right) \lesssim C_1\epsilon s^{k\delta}$$
$$\left((t/s)^{1-7\delta} s^{3/2} |\partial_\alpha \partial^I L^J v|\right) \lesssim C_1\epsilon s^{k\delta}$$

To conclude the bootstrap argument we return to the system differentiated with $\partial^I L^J$ ($|I|+|J| \leqslant N$ and $|J| =: k$)

$$-\Box \partial^I L^J u = \partial^I L^J \left(P^{\alpha\beta} \partial_\alpha v \partial_\beta v\right) + \partial^I L^J \left(R v^2\right)$$
$$-\Box \partial^I L^J v + u\, H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^I L^J v + c^2 \partial^I L^J v = -[\partial^I L^J, u\, H^{\alpha\beta} \partial_\alpha \partial_\beta] v$$

ESTIMATES for the NONLINEAR TERMS

$$-\Box\partial^I L^J u = \partial^I L^J \left(P^{\alpha\beta}\partial_\alpha v \partial_\beta v\right) + \partial^I L^J \left(Rv^2\right)$$

$$-\Box\partial^I L^J v + u\, H^{\alpha\beta}\partial_\alpha\partial_\beta\partial^I L^J v + c^2\partial^I L^J v = -[\partial^I L^J, u\, H^{\alpha\beta}\partial_\alpha\partial_\beta]v$$

with arbitrary $P^{\alpha\beta}, H^{\alpha\beta}, R$.

We investigate the decay in the hyperboloid variable s for the key terms

$$T_1^{I,J}(s) := \left\|\partial^I L^J \left(P^{\alpha\beta}\partial_\alpha v \partial_\beta v\right)\right\|_{L^2_f(\mathcal{H}_s)}$$

$$T_2^{I,J}(s) := \left\|\partial^I L^J \left(Rv^2\right)\right\|_{L^2_f(\mathcal{H}_s)}$$

$$T_3^{I,J}(s) := \left\|[\partial^I L^J, u\, H^{\alpha\beta}\partial_\alpha\partial_\beta]v\right\|_{L^2_f(\mathcal{H}_s)}$$

- **Low-order derivatives** $|I| + |J| \leqslant N - 4$**:** more decay and we can control $T_1^{I,J}(s)$ and $T_2^{I,J}(s)$ from the bootstrap assumption and Sobolev inequality on hyperboloids $T_1^{I,J}(s) + T_2^{I,J}(s) \lesssim s^{-3/2+(k+2)\delta}$
- **Higher-order derivatives:** The basic decay rate is not sufficient and we need to use our sharp pointwise estimates.
- **Third term** $T_3^{I,J}(s)$ (for arbitrary $|I| + |J|$): again we need our sharp pointwise estimates.

The sharp L^∞–L^∞ estimates have allowed us to improve the basic pointwise bounds and get $|I| + |J| \leqslant N - 4$

$$t\,|L^I u| \lesssim C_1\epsilon\, s^{k\delta}$$

$$s^{3/2}|\partial^I L^J v| \lesssim C_1\epsilon\,(s/t)^{2-4\delta} s^{k\delta} \lesssim C_1\epsilon\, s^{k\delta}$$

$$s^{3/2}|\partial^I L^J \partial_\alpha v| \lesssim C_1\epsilon\,(s/t)^{1-7\delta} s^{k\delta} \lesssim C_1\epsilon\, s^{k\delta}$$

Controling $P^{\alpha\beta}$ and R

- For $|I| + |J| \leqslant N$ and by assuming, without loss of generality, $|I_1| + |J_1| \leqslant N - 4$ i the following calculation

$$\left\|\partial^I L^J \left(\partial_\alpha v \partial_\beta v\right)\right\|_{L^2_f(\mathcal{H}_s)} \simeq \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J}} \left\|\partial^{I_1} L^{J_1} \partial_\alpha v\, \partial^{I_2} L^{J_2} \partial_\beta v\right\|_{L^2_f(\mathcal{H}_s)}$$

$$\lesssim C_1\epsilon s^{-3/2} s^{|J_1|\delta} \left\|\partial^{I_2} L^{J_2} \partial_\beta v\right\|_{L^2_f(\mathcal{H}_s)}$$

$$\lesssim (C_1\epsilon)^2 s^{-1+k\delta}$$

- Similarly, we obtain $\|\partial^I L^J \left(v^2\right)\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\epsilon)^2 s^{-1+k\delta}$.

We thus control $T_1^{I,J}(s)$ and $T_2^{I,J}(s)$ (for all relevant I, J) and we arrive at the desired improved energy bounds for the wave component.

Term $T_3^{I,J}(s) = \big\|[\partial^I L^J, u\, H^{\alpha\beta}\partial_\alpha\partial_\beta]v\big\|_{L_f^2(\mathcal{H}_s)}$ arising in the Klein-Gordon equation.

It is a linear combination of

$$T_{3,1}^{I,J}(s) \quad (\partial^{I_1} L^{J_1} u)\,\partial^{I_2} L^{L_2}\partial_\alpha\partial_\beta v \qquad I_1 + I_2 = I, \quad J_1 + J_2 = J, \quad |I_1| \geqslant 1$$

$$T_{3,2}^{I,J}(s) \quad (L^{J_1'} u)\,\partial^{I} L^{J_2'}\partial_\alpha\partial_\beta v \qquad J_1' + J_2' = J, \quad J_1' \geqslant 1$$

$$T_{3,3}^{I,J}(s) \quad u\,\partial_\alpha\partial_\beta\partial^I L^{J'} v \qquad J' \leqslant J - 1$$

- Terms $T_{3,1}^{I,J}(s)$: again controled with the sharp decay estimates
- Terms $T_{3,2}^{I,J}(s)$ and $T_{3,3}^{I,J}(s)$: due to the presence of $L^J u$, we need both the sharp decay estimates and Hardy inequality, as follows.

Let us for instance treat here $T_{3,2}^{I,J}(s)$ ($T_{3,3}^{I,J}(s)$ being treated similarly).

Higher-order derivatives

- For all $|J_1'| \leqslant N-4$, we use the sharp decay bound $t\,|L^I u| \lesssim C_1 \epsilon s^{k\delta}$ combined with the energy bound on $\partial_\alpha \partial_\beta v$ (implied by our bootstrap assumption).
- For $|I| + |J_2'| \leqslant N-4$, we use the sharp bound $|\partial^I L^J \partial_\alpha v| \lesssim C_1 \epsilon\, (s/t)^{1-7\delta} s^{-3/2+k\delta}$ and Hardy's inequality for $L^{J_1'} u$.

We are led to the slow growth (after integration in s)

$$\big\| [\partial^I L^J, H^{\alpha\beta} \partial_\alpha \partial_\beta] v \big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1 \epsilon)^2 s^{-1/2+k\delta}.$$

Lower-order derivatives: this is easier

- For $|J_1'| \leqslant |I| + |J| \leqslant N-4$, we apply directly the sharp bound $t\,|L^I u| \lesssim C_1 \epsilon\, s^{k\delta}$
- together with the energy bound given by our bootstrap assumption.

This leads us to the decay

$$\big\| [\partial^I L^J, H^{\alpha\beta} \partial_\alpha \partial_\beta] v \big\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1 \epsilon)^2 s^{-1+k\delta}.$$

DEALING WITH THE EINSTEIN EQUATIONS

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This requires a number of additional ideas:

- The wave gauge condition
- The quasi-null terms
- Much involved algebraic structure
- Specific hierarchy of (low-order, high-order) energy bounds
- Integration along the characteristics of the curved metric
- Non-compact solutions (Schwarzschild at spatial infinity)

FINAL REMARKS

- **Dynamically unstable solutions** to the Einstein equations do not exist for self-gravitating massive fields with *sufficiently small mass* and spacetimes *sufficiently close to flat space*.
- **The $f(R)$ theory of modified gravity**: nonlinear stability of Minkowski spacetime for $f(R) \simeq R + \kappa R^2$ with $\kappa > 0$.

OPEN PROBLEMS

- PROBLEM 1: **Appli. of the Hyperboloidal Foliation Method**
  - Encompass a large class of nonlinear wave-Klein-Gordon systems with *quasi-null coupling*

    Rely on the earlier work: S. Alinhac, H. Lindblad
- PROBLEM 2: **Investigate the growing rate $s^{1/2}$**
  - Investigate whether it is necessary in the wave gauge under consideration (by constructing explicit examples)
  - Improve this rate by introducing additional geometric arguments (hyperboloidal foliation based on the curved metric)

    Ongoing work by Q. Wang and J. Wang
- PROBLEM 3: **Extend our method to other massive fields**
  - Kinetic models (density fc.), Vlasov eq. (collisionless), Boltzmann eq.

    Rely on: D. Fajman, J. Joudioux, and J. Smulevici for the Vlasov eq.
- PROBLEM 4: **Establish Penrose's peeling estimates**
  - Precise asymptotics for the spacetime curvature along timelike directions (D. Christodoulou and S. Klainerman's geometric method)
  - Very challenging in wave coordinates (H. Lindblad and I. Rodnianski)

The Hyperboloidal Foliation Method provides sharp bounds (for instance a uniform energy bound in presence of null forms and without metric coupling) *and could allow one to establish the peeling estimates in wave gauge.*