Characterization of (asymptotically) Kerr-de Sitter-like spacetimes at null infinity

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Outline

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2. Kerr-de Sitter metric: local properties and vanishing of $\mathcal{S}^Q$.
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5. Sufficient conditions at $\mathscr{I}$ for $\mathcal{S}^Q = 0$.
6. Characterization of Kerr-de Sitter at null infinity.

Aim and motivation

Motivation:

The Kerr-de Sitter spacetime has a number of interesting properties which make it worth studying.

- Expected to satisfy a stationary black hole uniqueness theorem among vacuum solutions with positive Λ.

From a dynamical perspective:

- It may play some role in the future asymptotic behaviour of matter solutions with positive Λ, when black holes form.
- Also, the dynamical stability of the Kerr-de Sitter spacetime in the expanding region is an interesting problem.

Useful to have a good understanding of the asymptotics of the Kerr-de Sitter metric at null infinity.

Aim:

Characterize uniquely the Kerr-de Sitter spacetime among spacetimes:

(i) solving the $(\Lambda > 0)$-vacuum equations,

(ii) admitting a Killing vector field (KVF),

(iii) admitting a smooth conformal compactification,

in terms of data at infinity.

Kerr-(A) de Sitter spacetime

- The Kerr-de Sitter family of spacetimes depends on three parameters: Λ, a, m.
- Solves the vacuum Einstein equations with Λ:

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0.$$

- Away from fixed points of Killing vectors, the local form of the metric is

$$g = -\frac{\Delta_r - a^2 \Delta_\theta \sin^2\theta}{\rho^2} \left(du - a \sin^2\theta d\varphi \right)^2 + 2 \left(dr - a\Delta_\theta \sin^2\theta d\varphi \right) \times \left(du - a \sin^2\theta d\varphi \right) + \rho^2 \left(\frac{d\theta^2}{\Delta_\theta} + \Delta_\theta \sin^2\theta d\varphi^2 \right)$$

where

$$\rho^2 := r^2 + a^2 \cos^2\theta, \qquad \Delta_\theta := 1 + \frac{\Lambda}{3} a^2 \cos^2\theta$$

$$\Delta_r := \left(r^2 + a^2 \right) \left(1 - \frac{\Lambda}{3} r^2 \right) - 2m\, r.$$

Λ can take any value:

- $\Lambda > 0$ is Kerr-de Sitter, $\Lambda = 0$ is Kerr, $\Lambda < 0$ is Kerr- anti de Sitter.

Admits a generalization with NUT charge ℓ (also Λ-vacuum). Called Kerr-NUT-(A) de Sitter.

Asymptotic properties

The Kerr-NUT-(A) de Sitter metric admits a conformal compactification at infinity.

- Recall:

Definition (Conformal compactification *à la Penrose*)

A spacetime $(\mathscr{M}, g)$ admits a smooth conformal compactification at infinity if

- There exists a spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$ and a conformal embedding ϕ

$$\mathscr{M} \overset{\phi}{\hookrightarrow} \widetilde{\mathscr{M}}, \qquad \phi^{\star}(\Theta^{-2}\widetilde{g}) = g \qquad \Theta \in C^{\infty}(\widetilde{\mathscr{M}}, \mathbb{R}), \quad \Theta|_{\phi(\mathscr{M})} > 0$$

such that $\mathscr{I} := \partial(\phi(\mathscr{M})) \neq \emptyset$ is smooth hypersurface where $\Theta = 0$ and $d\Theta \neq 0$.

- $\mathscr{I}$ is called "null infinity"

If $(\mathscr{M}, g)$ solves the Λ-vacuum field equations then:

$$\mathscr{I} \text{ is null if } \Lambda = 0, \quad \mathscr{I} \text{ is spacelike if } \Lambda > 0, \quad \mathscr{I} \text{ is timelike if } \Lambda < 0.$$

Our aim is to characterize the Kerr-NUT-de Sitter spacetime from data at $\mathscr{I}$.

Two main ingredients:

- Local characterization of Kerr-NUT-de Sitter.
- Friedrich's Cauchy problem for $(\Lambda > 0)$-vacuum equations at $\mathscr{I}$.

Algebraic properties

- The Kerr-NUT-(A) de Sitter metric admits two linearly independent KVF ∂_u, ∂_φ.

The Killing field ∂_u is geometrically privileged:

- For any KVF X in a spacetime $(\mathscr{M}, g)$:

$$F_{\alpha\beta} := \nabla_\alpha X_\beta \quad \text{is a two-form. Called Killing form associated to } X.$$

Assume: $(\mathscr{M}, g)$ orientable with volume form $\boldsymbol{\eta}$.

- Convenient to work with self dual two-forms $\mathcal{U}^\star = -i\,\mathcal{U}$.

$$\mathcal{U} \text{ self-dual} \quad \Longleftrightarrow \quad \mathcal{U} = U + i\,U^\star, \text{ with } U \text{ real.}$$

Space of self-dual two-forms at $p \in \mathscr{M}$ admits a canonical metric $\mathcal{I} = \frac{i}{4}\left(\boldsymbol{\eta} + i\boldsymbol{\eta}^\star\right)$

  - $\mathcal{I}_{\alpha\beta\mu\nu}\mathcal{W}^{\mu\nu} = \mathcal{W}_{\alpha\beta}$.

- Define $\mathcal{F}_{\alpha\beta} := F_{\alpha\beta} + i\,F^\star_{\alpha\beta}$. The tensor

$$\mathcal{U}_{\alpha\beta\mu\nu} := -\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} + \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}, \qquad \mathcal{F}^2 := \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$$

  is a symmetric, double, self-dual, two-form and trace-free.

- Same properties as the self-dual Weyl tensor: $\mathcal{C}_{\alpha\beta\mu\nu} := \mathrm{C}_{\alpha\beta\mu\nu} + i\,\mathrm{C}^\star_{\alpha\beta\mu\nu}$

The Kerr-NUT-(A) de Sitter metric has the following property:

- Construct $\mathcal{F}$ and $\mathcal{U}$ associated to $X = \partial_u$ in the Kerr-NUT-(A) de Sitter metric.

Algebraic property

There exists a function $Q \in C^\infty(\mathscr{M}, \mathbb{C})$ satisfying $\mathcal{S}^Q_{\alpha\beta\mu\nu} := \mathcal{C}_{\alpha\beta\mu\nu} + Q\,\mathcal{U}_{\alpha\beta\mu\nu} = 0$.

In the $\Lambda = 0$ case, this property was know to characterize the Kerr-NUT spacetime:

Theorem (M., 2001)

Let $(\mathscr{M}^4, g)$ be Ricci-flat with a KVF X. Assume $(\mathscr{M}, g)$ is not locally flat and $\mathcal{F}^2$ not identically zero. If $\mathcal{S}^Q$ vanishes for some $Q \in C^\infty(\mathscr{M}, \mathbb{C})$ then

- *$\mathcal{F}^2$ vanishes nowhere.*
- *There exist constants $\mathcal{A} \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{R}$ such that*

$$Q = -\frac{6\mathcal{A}}{(\mathcal{F}^2)^{\frac{1}{4}}} \qquad -g(X,X) + Re\left(\frac{(\mathcal{F}^2)^{\frac{1}{4}}}{\mathcal{A}}\right) = c.$$

If $c > 0$ then $(\mathscr{M}, g)$ is locally isometric to Kerr-NUT with mass m and NUT parameter ℓ

$$m - i\ell = -\frac{i}{2c\sqrt{c}\mathcal{A}^2}.$$

A similar local characterization exists for all Λ. [M & Senovilla '2014]

Useful intermediate result:

Proposition (M. & Senovilla)

Let $(\mathscr{M}, g)$ be a Λ-vacuum spacetime with a KVF X such that $\mathcal{S}^Q = 0$ for some $Q \in C^\infty(\mathscr{M}, \mathbb{C})$. There are four exclusive cases:

- (a) *If $Q = 0$ everywhere then $(\mathscr{M}, g)$ is locally isometric to (A) de Sitter or Minkowski.*
- (b) *If $Q \not\equiv 0$ and $\mathcal{F}^2 = 0$ on a non-empty open set then $\Lambda \leq 0$.*
- (c) *If exists $p \in \mathscr{M}$ where $\mathcal{F}^2 Q|_p \neq 0$, then $\mathcal{F}^2$ vanishes nowhere. Moreover, either $Q\mathcal{F}^2 - 4\Lambda$ vanishes everywhere (c.1) or nowhere (c.2)*
  - (c.1) *The metric is locally $g = h_- + h_+$ with $h_\pm$ of constant curvature.*

Consequence:

To study spacetimes with $\mathcal{S}^Q = 0$ admitting a conformal compactification, we can assume $\mathcal{F}^2 \neq 0$ and $Q\mathcal{F}^2 - 4\Lambda \neq 0$ everywhere.

Conformal Λ-vacuum field equations and Cauchy problem at $\mathscr{I}$

The Λ-vacuum field equations are conformally regular (Friedrich 1986):

- The Einstein equations $R(g)_{\alpha\beta} = \Lambda g_{\alpha\beta}$ on $\mathscr{M}$ are equivalent to a set of equations on $(\mathscr{M}, \widetilde{g} := \Theta^2 g)$ called Conformal Field Equations.
- Fundamental property: the equations are regular at $\Theta = 0$.

Assume $\Lambda > 0$. The conformal Λ-vacuum equations have well-posed initial data at $\mathscr{I}$.

Theorem (H. Friedrich, 1986)

Let (Σ, h, D) be a Riemannian 3-manifold (Σ, h) endowed with a symmetric tensor D_{ij}. There exists a unique (up to conformal diffeomorphism) maximal globally hyperbolic development $(\widetilde{\mathscr{M}}, \widetilde{g})$ of the conformal $(\Lambda > 0)$-vacuum field equations and an isometric embedding $\iota : (\Sigma, h) \longrightarrow (\widetilde{\mathscr{M}}, \widetilde{g})$ satisfying

- $\iota^\star(\Theta) = 0$
- $\iota^\star(\Theta^{-1}\widetilde{C}(n, \cdot, n, \cdot))_{ij} = D_{ij}$, *where n is the unit future normal to $\mathscr{I} := \iota(\Sigma)$,*

if and only if D is a TT-tensor (i.e. $tr_h D = 0$ and $div_h D = 0$).

Define $\mathscr{M}$ as the connected component of $\{\Theta > 0\}$ with $\mathscr{I} \subset \overline{\mathscr{M}}$.

- Then $(\mathscr{M}, g := \Theta^{-2}\widetilde{g})$ is Λ-vacuum with a smooth conformal compactification.

(Σ, h, D) defines the Cauchy data at $\mathscr{I}$ for the $(\Lambda > 0)$-vacuum field equations.

Killing initial data at $\mathscr{I}$

Assume: $(\mathscr{M}, g)$ admits a conformal compactification and KVF X.

- $\phi_\star(X)$ is a conformal Killing vector (CKV) of $(\mathscr{M}, \widetilde{g})$ which extends smoothly as a tangential vector to $\mathscr{I}$.

Let Y be the restriction of $\phi_\star(X)$ to $\mathscr{I}$ and (Σ, h, D) the Cauchy data at $\mathscr{I}$:

- Y is a CKV of (Σ, h).
- The TT-tensor D satisfies the reduced KID equation:

$$\mathscr{L}_Y D + \frac{1}{3}(\mathrm{div}_h Y)\, D = 0.$$

Converse also true:

Theorem (Paetz, 2014)

Let (Σ, h, D) be the Cauchy data at $\mathscr{I}$ for the conformal $(\Lambda > 0)$-vacuum field equations. The maximal development $(\widetilde{\mathscr{M}}, \widetilde{g})$ of this data admits a CKV X with the properties

- *X is tangential to $\mathscr{I}$ and $X|_{\mathscr{I}} = \iota_\star(Y)$,*
- *X is Killing in $(\mathscr{M}, g := \Theta^{-2}\widetilde{g})$,*

if and only if Y is a CKV of (Σ, h) satisfying the reduced KID equations.

- Call such data Killing initial data at $\mathscr{I}$.
- We want to find necessary conditions for Killing data at $\mathscr{I}$ of Kerr-NUT-de Sitter.

Strategy:

- Study the family of tensors $\mathcal{S}^Q$ at $\mathscr{I}$ for $(\Lambda > 0)$-vacuum solutions admitting a conformal compactification.

Assumptions:

- $(\mathscr{M}, g)$ admits a conformal compactification $(\widetilde{\mathscr{M}}, \widetilde{g})$ with data at $\mathscr{I}$: (Σ, h, D, Y).
- Denote by X the CKV at $(\widetilde{\mathscr{M}}, \widetilde{g})$ generated by Y.

We are not assuming that $\mathcal{S}^Q := \mathcal{C} + Q\mathcal{U}$ vanishes for some Q.

- Conformal invariance of the Weyl tensor $C_{\alpha\beta\mu}{}^{\nu} = \widetilde{C}_{\alpha\beta\mu}{}^{\nu}$ implies $\mathcal{C}_{\alpha\beta\mu}{}^{\nu} \xrightarrow{\mathscr{I}} 0$.

At $\mathscr{I}$:

$$\mathcal{S}^Q_{\alpha\beta\mu}{}^{\nu}|_{\mathscr{I}} = Q\,\mathcal{U}_{\alpha\beta\mu}{}^{\nu}|_{\mathscr{I}}.$$

- Need to find the behaviour of $\mathcal{U} := -\mathcal{F} \otimes \mathcal{F} + \frac{1}{3}\mathcal{F}^2\mathcal{I}$ near $\mathscr{I}$.

Lemma

The tensor $\Theta^4\mathcal{U}_{\alpha\beta\mu}{}^{\nu}$ admits a smooth extension at $\mathscr{I}$ which vanishes at $p \in \mathscr{I}$ if and only if $Y|_p = 0$.

Consequences:

- $\mathcal{S}^Q$ is regular at $\mathscr{I}$ if and only if $\Theta^{-4}Q$ admits a smooth extension to $\mathscr{I}$.
- $\mathcal{S}^Q$ vanishes at $\mathscr{I}$ if and only if Q has a zero of order five at null infinity.

It turns out there is a natural choice of Q for which this is always true.

- Whenever $\mathcal{F}^2 \neq 0$ a convenient way to define Q is by the condition

$$(\mathcal{S}^Q)_{\alpha\beta\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}^{\mu\nu} = 0 \quad \Longleftrightarrow \quad Q = Q_0 := \frac{3}{2}\frac{1}{\mathcal{F}^4}\mathcal{C}_{\alpha\beta\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}^{\mu\nu}.$$

Asymptotics of $\mathcal{F}^2$ and Q_0

$$\mathcal{F}^2 = -\frac{4}{3}\Theta^{-2}|Y|^2 + O(\Theta^{-1}), \quad \text{and} \quad Q_0 = \frac{\Theta^5}{|Y|^4}(\text{smooth expression at } \mathscr{I}) + O(\Theta^6).$$

Consequence: $\mathcal{S}^{(0)} := \mathcal{S}^{Q_0}$ vanishes at $\mathscr{I}$.

- The rescaled tensor $\mathcal{T}^{(0)}_{\alpha\beta\mu}{}^{\nu} := \Theta^{-1}\mathcal{S}^{(0)}_{\alpha\beta\mu}{}^{\nu}$ is well-defined and regular near any non-fixed point of Y at $\mathscr{I}$.

For any Q: Regularity of $\mathcal{T}^Q$ at $\mathscr{I}$ holds as long as $Q = O(\Theta^5)$.

Necessary conditions at $\mathscr{I}$ for the vanishing of $\mathcal{T}^{(0)}$

- Use $\widehat{\ }$ to denote objects associated to $(\mathscr{I}, h)$:

Schouten tensor: $$\widehat{L}_{ij} = \widehat{R}_{ij} - \frac{1}{4}\widehat{R}h_{ij}$$

Cotton-York tensor: $$\widehat{C}_{ij} = -\frac{1}{2}\widehat{\eta}_i{}^{kl}\left(\widehat{\nabla}_k\widehat{L}_{lj} - \widehat{\nabla}_j\widehat{L}_{lk}\right)$$

- $\widehat{C}$ is a TT-tensor. Vanishes everywhere iff (Σ, h) is locally conformally flat.

Proposition

Let $(\mathscr{M}, g)$ be a $(\Lambda > 0)$-vacuum spacetime with a smooth conformal compactification and a KVF X. Let $(\mathscr{I}, h, D, Y)$ be the KID data at $\mathscr{I}$ and assume $|Y|^2 > 0$. Then

$$\mathcal{T}^{(0)}(n,\cdot,n,\cdot)_{ij}|_{\mathscr{I}} = \mathcal{D}_{ij} - \mathit{scalar} \times (Y \otimes Y)^{\mathrm{tf}}_{ij}$$

where $\mathcal{D}_{ij} := D_{ij} - i\sqrt{\frac{3}{\Lambda}}\widehat{C}_{ij}$ and $(Y \otimes Y)^{\mathrm{tf}}_{ij} := Y_i Y_j - \frac{1}{3}|Y|^2 h_{ij}$.
Moreover, $\mathcal{T}^{(0)}|_{\mathscr{I}}$ vanishes if and only if

$$D_{ij} = \frac{A_D}{|Y|^5}(Y \otimes Y)^{\mathrm{tf}}_{ij} \qquad \widehat{C}_{ij} = \frac{A_{\widehat{C}}}{|Y|^5}(Y \otimes Y)^{\mathrm{tf}}_{ij},$$

where A_D and $A_{\widehat{C}}$ are constants on each connected component of $\mathscr{I}$.

- Condition $Y \neq 0$ is superfluous except in de Sitter $(\Sigma, h, D) = (\mathbb{S}^3, h \in [\gamma_{\mathbb{S}^3}], D = 0)$.

If we are not in de Sitter, then:

- either $\widehat{C}_{ij} \neq 0$ somewhere (metric not in the conformal class of $\gamma_{\mathbb{S}^3}$)
- or $D_{ij} \not\equiv 0$ (rescaled electric part of Weyl tensor not zero at $\mathscr{I}$), or both.

Since $|Y|^{-5}(Y \otimes Y)^{\mathrm{tf}}$ is singular at $Y = 0$ $\Longrightarrow$ fixed points of Y are not part of $\mathscr{I}$.

Combining the Proposition with the property that Kerr-NUT-de Sitter has $\mathcal{S}^Q = 0$:

Theorem (Necessary conditions at $\mathscr{I}$ [M., Paetz, Senovilla, Simon], 2015)

Let $(\mathscr{M}, g)$ be the Kerr-NUT-de Sitter spacetime or, more generally, a $(\Lambda > 0)$-vacuum spacetime satisfying

(i) *$(\mathscr{M}, g)$ admits a smooth conformal compactification.*

(ii) *there is a function $Q \in C^\infty(\mathscr{M}, \mathbb{C})$ such that $\mathcal{S}^Q = 0$.*

Then, the asymptotic Killing data (Σ, h, D, Y) at each connected component of $\mathscr{I}$ satisfies

$$D_{ij} = \frac{A_D}{|Y|^5}(Y \otimes Y)^{\mathrm{tf}}_{ij} \qquad \widehat{C}_{ij} = \frac{A_{\widehat{C}}}{|Y|^5}(Y \otimes Y)^{\mathrm{tf}}_{ij},$$

where A_D and $A_{\widehat{C}}$ are constants.

What about sufficient conditions?

Strategy:

(i) Prove that $\mathcal{T}^Q|_{\mathscr{I}} = 0$ implies $\mathcal{T}^Q = 0$ (i.e. $\mathcal{S}^Q = 0$) everywhere.

(ii) Identify Kerr-de Sitter among all spacetimes with $\mathcal{S}^Q = 0$.

Start with item (i):

- $\mathcal{T}^{(0)}$ does not seem to satisfy any useful evolution equation.
- Need a better choice of Q to deal with sufficiency:

On $(\mathscr{M}, g)$ with a KVF X: Ernst one-form: $\sigma_\mu := 2\mathcal{F}_{\nu\mu}X^\nu$

- In Λ-vacuum, σ_μ is closed. Locally exists an Ernst potential σ, $(\nabla_\mu \sigma = \sigma_\mu)$.
- Defined up to an additive complex constant (called σ-constant).

Proposition (Further properties of spacetimes with $\mathcal{S}^Q = 0$ [M. & Senovilla, 2014])

Let $(\mathscr{M}, g)$ be a $(\Lambda > 0)$-vacuum spacetime with a Killing vector X. Assume $\mathcal{S}^Q = 0$ for some $Q \in C^\infty(\mathscr{M}, \mathbb{C})$ and $Q\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ are both not identically zero.

Then the Ernst potential σ exists globally. Moreover, the σ-constant can be chosen so that σ has no zeros and

$$Q = \frac{3}{\sigma}\left(1 - \sqrt{1 + \frac{4\Lambda\sigma}{\mathcal{F}^2}}\right) + \frac{4\Lambda}{\mathcal{F}^2} := Q^{\text{ev}}.$$

- We can compute Q^{ev} near $\mathscr{I}$ for any $(\mathscr{M}, g)$ with a smooth conformal compactification and a KVF X.

Requires four terms in the expansion of the Ernst potential σ near $\mathscr{I}$.

Proposition

Let $(\mathscr{M}, g)$ be $(\Lambda > 0)$-vacuum with a smooth conformal compactification and a KVF X. The function Q^{ev} vanishes to order Θ^4 at $\mathscr{I}$ near any non- fixed point of Y. Moreover, there exists a (unique) choice of σ-constant such that Q^{ev} has a zero of order 5 if and only if

$$D_{ij} Y^j \propto Y_i, \qquad \widehat{C}_{ij} Y^j \propto Y_i, \tag{1}$$

In that case the leading order terms of Q_{ev} and Q_0 coincide, which implies

$$\mathcal{T}^{(\mathrm{ev})\,\rho}_{\mu\nu\sigma}|_{\mathscr{I}} = \mathcal{T}^{(0)}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}} \, .$$

- $\mathcal{T}^{(0)}$ is always finite at $\mathscr{I}$. $\mathcal{T}^{(\mathrm{ev})}$ is finite only under (1), which is much weaker than

$$D_{ij} - i\sqrt{\frac{3}{\Lambda}}\widehat{C}_{ij} = \mathrm{Const} \times \left(Y_i Y_j - \frac{|Y|^2}{3} h_{ij} \right) \iff \mathcal{T}^{(0)}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}} = 0 = \mathcal{T}^{\mathrm{ev}}_{\mu\nu\sigma}{}^{\rho}|_{\mathscr{I}} .$$

Makes sense to put forward a definition:

Asymptotically Kerr-de Sitter like spacetimes

Definition (Asymptotically Kerr-de Sitter like spacetimes)

Let $(\mathscr{M}, g)$ be a $(\Lambda > 0)$-vacuum space-time admitting smooth conformal compactification and corresponding null infinity $\mathscr{I}$. $(\mathscr{M}, g)$ is called asymptotically Kerr-de Sitter-like at a connected component $\mathscr{I}_0$ of $\mathscr{I}$ if it admits a (non-trivial) KVF X which induces a CKV Y on $\mathscr{I}_0$ such that

- The rescaled electric part of the Weyl tensor $D_{ij} := \Theta^{-1}\widetilde{C}(n, \cdot, n, \cdot)_{ij}$
- The Cotton-York tensor $\widehat{C}_{ij}$ of $(\mathscr{I}_0, h)$

have Y as a common eigenvector.

Equivalently: The rescaled tensor $\mathcal{T}_{\alpha\beta\mu}^{(\mathrm{ev})\nu}$ is regular at $\mathscr{I}_0$.

- Generically: $\mathcal{T}_{\alpha\beta\mu}^{(\mathrm{ev})\nu}$ is singular at $\mathscr{I}$, with a leading order term diverging as Θ^{-1}.
- Cannot be expected to satisfy an evolution equation with regular coefficients at $\mathscr{I}$.

However, it does satisfy a symmetric hyperbolic system of PDE with a Fuchsian term.

Linear evolution equation for $\mathcal{T}^{(\mathrm{ev})}$

- In the Ricci flat case, the tensor $\mathcal{S}^{(\mathrm{ev})}$ satisfies a linear wave equation [Ionescu, Klainerman, 2007].
    - Useful to study uniqueness of the Kerr black hole without assuming analyticity.
- For Λ-vacuum spacetimes, Q^{ev} also satisfies a useful evolution equation.

Lemma

Let $(\mathscr{M}, g)$ be a Λ-vacuum spacetime with a Killing vector X. On a neighborhood of a point where $\mathcal{F}^2$ and the Ernst-potential do not vanish, $\mathcal{S}^{(\mathrm{ev})}$ satisfies the equation

$$\nabla_\rho \mathcal{S}^{(\mathrm{ev})}{}_{\alpha\beta\mu}{}^{\rho} = \mathcal{J}(\mathcal{S}^{(\mathrm{ev})})_{\alpha\beta\mu}$$

where $\mathcal{J}(\mathcal{S}^{(\mathrm{ev})})_{\alpha\beta\mu}$ is linear and homogeneous in $\mathcal{S}^{(\mathrm{ev})}{}_{\alpha\beta\mu}{}^{\nu}$.

- The rescaled $\mathcal{T}^{(\mathrm{ev})} := \Theta^{-1}\mathcal{S}^{(\mathrm{ev})}$ satisfies the PDE

$$\widetilde{\nabla}_\rho \mathcal{T}^{(\mathrm{ev})}{}_{\alpha\beta\mu}{}^{\rho} = \mathcal{J}(\mathcal{T}^{(\mathrm{ev})})_{\alpha\beta\mu} \,.$$

- Need to understand $\mathcal{J}(\mathcal{T}^{(\mathrm{ev})})_{\alpha\beta\mu}$ near $\mathscr{I}$.

Structure of the PDE

- Given the symmetries of $\mathcal{T}_{\alpha\beta\mu}^{(\mathrm{ev})\nu}$, $\mathcal{E}_{\alpha\mu} := n^\beta n_\nu \mathcal{T}_{\alpha\beta\mu}^{(\mathrm{ev})\nu}$ contains all independent components.
- The system of equations satisfied by $\mathcal{T}_{\alpha\beta\mu}^{(\mathrm{ev})\nu}$ contains the following subsystem

$$n^\rho \widetilde{\nabla}_\rho \mathcal{E}_{\alpha\beta} - i\widehat{\eta}_{\mu(\alpha}{}^\rho \widetilde{\nabla}_\rho \mathcal{E}_{\beta)}{}^\mu + \frac{1}{\Theta}\sqrt{\frac{\Lambda}{3}} N_0(\mathcal{E})_{\alpha\beta} = N_1(\mathcal{E})_{\alpha\beta}, \qquad (2)$$

  where $\widehat{\eta}_{\mu\alpha\beta} := n^\nu \eta_{\nu\mu\alpha\beta}$.
- $N_0(\mathcal{E})_{\alpha\beta}$, $N_1(\mathcal{E})_{\alpha\beta}$ are linear homogeneous in $\mathcal{E}_{\alpha\beta}$ and regular at $\mathscr{I}$.

In Gaussian coordinates $\{t, x^i\}$ adapted to $\mathscr{I}$ and $n = \partial_t$

- $\Theta = \sqrt{\frac{\Lambda}{3}} t + O(t^2)$
- $\mathcal{E}_{ij}$ are the only non-zero components. Decompose $\mathcal{E}_{ij} = E_{ij} + i\, B_{ij}$.

The system (2) has the form

$$A^0 \partial_t u + A^i \partial_i u + \frac{1}{t} N_0(u) = N_1(u), \qquad u = (E_{ij}, B_{ij})$$

- A^0, A^i are self-adjoint w.r.t the scalar product $\langle u, u\rangle = E_{ij}E^{ij} + B_{ij}B^{ij}$.
- A^0 is positive definite.

The system is symmetric hyperbolic with a zeroth-order, divergent term $\frac{1}{t}$.

Uniqueness of solutions to Fuchsian equations

- System of PDEs with divergence terms $\frac{1}{t}$ are called Fuchsian.

- Fuchsian system of PDE have been analyzed mainly in the analytic case.
- In the smooth case, there are results by [Claudel & Newman, 1997], [Rendall, 2000] and recently [Ames. Beyer, Eisenberg & LeFloch, 2012].

Adapting ideas of Ames *et. al.* we can prove a localized uniqueness theorem for symmetric hyperbolic, linear, homogeneous Fuchsian system.

Lemma (Uniqueness of solutions)

Consider a manifold $\mathcal{N}^n$ and a smooth hypersurface $\Sigma := \{t = 0\}$. Suppose that $u := \mathscr{M} \mapsto \mathbb{R}^m$ satisfies the PDE

$$A^0 \partial_t u + A^i \partial_i u + \frac{1}{t} N_0(u) = N_1(u),$$

with N_0, N_1 linear, homogeneous and regular and A^0, A^i self-adjoint w.r.t a positive definite scalar product $\langle u, u \rangle|_p$, at each $p \in \mathcal{N}$.

If a C^1 solution u vanishes on a domain $\Omega \subset \Sigma$ and A^0 and $A^0 + N_0$ are positive definite, then $u \equiv 0$ on the domain of dependence of Ω.

Characterization result at $\mathscr{I}$

- Concerning the PDE satisfied by $\mathcal{T}^{(\mathrm{ev})\mu}_{\alpha\beta\mu}$ it turns out that $A^0 + N_0$ is positive definite.

We conclude that if $\mathcal{T}^{(\mathrm{ev})\mu}_{\alpha\beta\mu}$ vanishes at $\mathscr{I}$, then it vanishes in a neighbourhood of $\mathscr{I}$.

- It is the easy to conclude that $\mathcal{T}^{(\mathrm{ev})\mu}_{\alpha\beta\mu}$ and hence $\mathcal{S}^{(\mathrm{ev})\,\nu}_{\alpha\beta\mu}$ vanish everywhere.

We conclude:

Theorem (M., Paetz, Senovilla, Simon, 2015)

Let $(\mathscr{M}, g)$ be a $(\Lambda > 0)$-vacuum smooth spacetime admitting a smooth conformal compactification at $\mathscr{I}$ and a KVF X. Let $\mathscr{I}_0$ be a connected component of $\mathscr{I}$. Let h be the induced by $\widetilde{g} = \Theta^2 g$ on $\mathscr{I}_0$ and Y the CKV induced by X on $\mathscr{I}_0$.

Then, there exists a smooth function Q_0 for which the tensor $\mathcal{S}^{(0)}_{\mu\nu\sigma}{}^{\rho}$ associated to X vanishes in the domain of dependence of $\mathscr{I}_0$ if and only if:

(i) *The Cotton-York tensor $\widehat{C}_{ij}$ of h has the form*

$$\widehat{C}_{ij} = \frac{A_{\widehat{C}}}{|Y|^5}\left(Y_i Y_j - \frac{1}{3}|Y|^2 h_{ij}\right), \qquad B \in \mathbb{R}.$$

(ii) *The electric part of the rescaled Weyl tensor at $\mathscr{I}_0$ $D_{ij} = \Theta^{-1}\widetilde{C}(n,\cdot,n,\cdot)_{ij}|_{\mathscr{I}_0}$ is*

$$D_{ij} = \frac{A_D}{|Y|^5}\left(Y_i Y_j - \frac{1}{3}|Y|^2 h_{ij}\right), \qquad A \in \mathbb{R}.$$

- It remains to identify Kerr-NUT-de Sitter at $\mathscr{I}$.
- Apply a local characterization result among spacetimes with $\mathcal{S}^Q = 0$.

Theorem (Local characterization of Kerr-NUT-(A) de Sitter [M. & Senovilla, 2014])

Let $(\mathscr{M}, g)$ be a $(\Lambda > 0)$-vacuum spacetime with a KVF X such that $\mathcal{S}^Q = 0$ for some $Q \in C^\infty(\mathscr{M}, \mathbb{C})$. Assume $\mathcal{F}^2$ and $Q\mathcal{F}^2 - 4\Lambda$ vanish nowhere.

Define $W := \frac{6\sqrt{\mathcal{F}^2}}{Q\mathcal{F}^2 - 4\Lambda} := y + iZ$. There exist four real constants $b_1, b_2, c, k \in \mathbb{R}$ such that

$$b_2 - i\, b_1 = -\frac{1}{6} Q\mathcal{F}^2 W^3, \qquad \frac{3}{\Lambda} c = -|X|^2_g - Re\left(\frac{1}{6} W^2 (Q\mathcal{F}^2 + 2\Lambda)\right)$$

$$\left(\frac{3}{\Lambda}\right)^3 k = |W|^2 |\nabla Z|^2_g - b_2 Z + cZ^2 + \frac{\Lambda}{3} Z^4.$$

Moreover, $(\mathscr{M}, g)$ is locally isometric to the Kerr-de Sitter spacetime if and only if $b_2 = 0$ and

$$k > 0 \qquad or \qquad (k = 0, \quad c > 0).$$

The Kerr-de Sitter parameters are

$$m = \sqrt{2} \left(\frac{\Lambda}{3}\right)^{\frac{3}{2}} \frac{b_1}{\left(c + \sqrt{c^2 + 4k}\right)^{\frac{3}{2}}} \qquad a = \sqrt{\frac{3}{\Lambda}} \frac{2\sqrt{k}}{c + \sqrt{c^2 + 4k}}.$$

- We want to identify these constants in terms of the asymptotic data $(\mathscr{I}, h, D, Y)$.

Using $Q = Q_0$, the functions b_1, b_2, c, k can be evaluated at $\mathscr{I}$ for any $(\Lambda > 0)$-vacuum spacetime admitting a conformal compactification.

Proposition (The "constants" at $\mathscr{I}$,)

Let $(\mathscr{M}, g)$ be $(\Lambda > 0)$-vacuum spacetime $(\mathscr{M}, g)$ with KVF X and admitting a conformal compactification.

- *The functions b_1, b_2, c, k defined with $Q = Q_0$ extend smoothly to $\mathscr{I}$.*
- *Their expressions at $\mathscr{I}$ in terms of the asymptotic Killing data $(\mathscr{I}, h, D, Y)$ are*

$$b_2 - ib_1|_{\mathscr{I}} = \left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} |Y| \left(\sqrt{\frac{3}{\Lambda}}\widehat{C}_{ij} + i\, D_{ij}\right) Y^i Y^j \; \left(= \frac{6}{\Lambda^2}\left(A_{\widehat{C}} + i\sqrt{\frac{\Lambda}{3}} A_D\right) \;\text{ if } \mathcal{S}^{(0)}|_{\mathscr{I}} = 0\right)$$

$$c|_{\mathscr{I}} = \text{explicit expression in terms of } Y \text{ and the geometry } (\mathscr{I}, h) := c(Y).$$

$$k|_{\mathscr{I}} = \text{explicit expression in terms of } Y \text{ and the geometry } (\mathscr{I}, h) := k(Y).$$

- $c(Y)$ and $k(Y)$ depend only depend on (Σ, h, Y) (not on D).

Kerr-de Sitter data is:

- Y CKV in $(\mathbb{S}^3, [\gamma_{\mathbb{S}^3}])$, $\mathscr{I}$ domain of $\mathbb{S}^3 \setminus \{Y = 0\}$, $D = A_D |Y|^{-5} (Y \otimes Y)^{\text{tf}}, Y)$,
- Constant A_D is directly related to the mass parameter.

- The algebra of CKV in $(\mathbb{S}^3, [\gamma_{\mathbb{S}^3}])$ is ten dimensional.
- Need to classify the asymptotic data
    - $c(Y)$ and $k(Y)$ play an essential role.

In $(\mathbb{S}^3, [\gamma_{\mathbb{S}^3}], Y)$: $c(Y), k(Y)$ are necessarily constant.

- The data (Σ, h, D, Y) is conformally invariant, i.e. $(\Sigma, \Omega^2 h, \Omega^{-1} D, Y)$ gives rise to the same spacetime.
- Fixing the representative in $[\gamma_{\mathbb{S}^3}]$ as $h = \gamma_{\mathbb{S}^3}$ leaves as freedom the conformal group of the sphere.

$$\mathrm{Conf}(\mathbb{S}^3) = \{\Phi \in \mathrm{Diff}(\mathbb{S}^n) \quad s.t. \quad \Phi^\star(\gamma_{\mathbb{S}^n}) = \Omega^2 \gamma_{\mathbb{S}^n}, \Omega \in C^\infty(\mathbb{S}^3, \mathbb{R}^+)\}$$

Consequence:

Conformal invariance

Let $\mathscr{I}$ be a domain in $\mathbb{S}^3$ and $\Phi \in \mathrm{Conf}(\mathbb{S}^3)$. Then

$$(\mathscr{I}, \gamma_{\mathbb{S}^3}, D(Y) := A_D |Y|^{-5} (Y \otimes Y)^{\mathrm{tf}}, Y) \quad \sim \quad (\Phi(\mathscr{I}), \gamma_{\mathbb{S}^3}, D(\Phi_\star(Y)), \Phi_\star(Y))$$

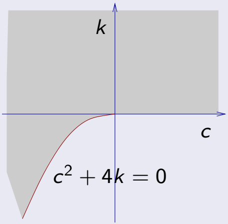
Define the equivalence relation $Y_1 \sim Y_2$ iff $Y_1 = \Phi_\star(Y_2)$.

- Denote by $\bar{Y}$ the equivalence class of Y.

Theorem (Behaviour under conformal group, [M., Paetz, Senovilla, 2015])

Let Y denote a CKV in $(\mathbb{S}^3, \gamma_{\mathbb{S}^3})$. The following properties hold:

- *The constants $c(Y)$ and $k(Y)$ depend only on the conformal class $[Y]$.*
- *The range of $(c(Y), k(Y))$ is*
- *Given constants (c, k) in this range there exists precisely one equivalence class $\bar{Y}$ with $(c(\bar{Y}) = c, k(\bar{Y}) = k)$, except if $c < 0$ and $k = 0$ where there are precisely two classes.*



- *The equivalence classes with $\{k(Y) > 0\}$ or $\{k(Y) = 0, c(Y) > 0\}$ (corresponding to the Kerr-de Sitter data) are characterized by the property that Y has precisely two isolated zeros.*

Conclusions and outlook

Conclusions:

- All ($\Lambda > 0$)-vacuum spacetimes with a KVF satisfying $\mathcal{S}^Q = 0$ and admitting a smooth conformal compactification ($\star$) can be characterized from data at $\mathscr{I}$.
- In particular, the Kerr-de Sitter spacetime corresponds to data

$$\Sigma = \mathbb{S}^3 \setminus \{\mathrm{N}, \mathrm{S}\}, \qquad h \in [\gamma_{\mathbb{S}^3}]$$
$$Y = \text{ is any CKV of } (\mathbb{S}^3, \gamma_{\mathbb{S}^3}) \text{ vanishing precisely at N and } S$$
$$D_{ij} = A_D |Y|^{-5} \Big(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \Big).$$

- All spacetimes satisfying ($\star$) with $(\Sigma, h \in [\gamma_{\mathbb{S}^3}])$ can be completely identified.
- A much larger class of data for so-called asymptotically Kerr-de Sitter like spacetimes arises naturally.

Future work:

- Identify all three-geometries (Σ, h) with a CKV Y and Cotton-York tensor $\widehat{C}_{ij}$ satisfying

$$\widehat{C}_{ij} = A_{\widehat{C}} |Y|^{-5} \Big(Y_i Y_j - \frac{1}{3} |Y|^2 h_{ij} \Big).$$

- Identify all data (Σ, h, D, Y) at infinity for asymptotically Kerr-de Sitter like spacetimes.