Global stability of Minkowski space for Einstein equations with massive scalar fields

Qian Wang

University of Oxford

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Joint work with Dr. Jin-hua Wang from AEI, Golm since March, 2014. The problem itself is suggested by Lars Andersson.

Consider a $(3+1)$ space-time $(\mathbf{M}, \mathbf{g})$, where $\mathbf{M}$ is a 4-dim manifold and $\mathbf{g}$ is a Lorentzian metric of signature $(-,+,+,+)$ satisfying the Einstein equations with massive scalar fields

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = \mathcal{T}_{\mu\nu}$$

with the stress-energy tensor $\mathcal{T}_{\mu\nu}$

$$\mathcal{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m\phi^2), \qquad m = 1.$$

Here **Ric** denotes the Ricci curvature tensor of $\mathbf{g}$, and $\mathbf{R}$ denotes the scalar curvature.

- This is an Einstein Klein-Gordon system.

$$\Box_{\mathbf{g}}\phi = m\phi$$

$$\mathbf{R}_{\alpha\beta} = \partial_\alpha\phi \cdot \partial_\beta\phi + \frac{1}{2} m\, \mathbf{g}_{\alpha\beta}\phi^2$$

- The simplest example is, $\phi = 0$ and the Minkowski space-time $(\mathbb{R}^{3+1}, \mathbf{m})$ with

$$\mathbf{m} = -dt^2 + dx^2 + dy^2 + dz^2.$$

- We consider to construct global in time nontrivial solutions of the Einstein equations with massive scalar fields.

Consider the asymptotic behavior of linear wave equation in $\mathbb{R}^{3+1}$,

$$\Box_{\mathbf{m}}\phi = 0$$

Asymptotic behavior:

$$|\phi| \lesssim \frac{1}{t+1}$$

Three approaches to obtain the decay estimates:

- Representation formula
- Fourier method
- Sobolev inequality by Killing and conformal Killing vector fields (Klainerman)

Commuting vector field approach

Killing and conformal Killing vector fields of Minkowski spacetime:

- The generators of translations: $\partial_\mu, \quad \mu = 0, 1, 2, 3$
- The generators of the Lorentz groups:
  $\Omega_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad \mu, \nu = 0, 1, 2, 3$ where $x_\mu = \mathbf{m}_{\mu\nu} x^\nu$
  Rotation, boost (spacetime rotation)
- The scaling vectorfield: $S = x^\mu \partial_\mu$,
- The four inverted translation vectorfields:
  $K_\mu = -2x_\mu S + \langle x, x \rangle \partial_\mu, \quad \mu = 0, 1, 2, 3$

Commuting vectorfields: Control energies of $Z^{(n)}\phi$ where Z denote the vectorfields of the first three types.
due to the properties

$$[\Box, Z] = 0 \text{ or } 2\Box$$

where the latter occurs only when $Z = S$.
Multiplier X^μ

$$\int_{t=t_0} Q_{\mu\nu} X^\mu \mathbf{T}^\nu.$$

Using $X = K_0$ and with certain modification of energy momentum,

$$\left(\int_{t=t_0} \tilde{Q}_{\mu\nu} K_0^\mu \mathbf{T}^\nu\right)^{\frac{1}{2}} \approx \|\phi\|_{L^2(t=t_0)} + \sum_Z \|Z\phi\|_{L^2(t=t_0)} \qquad (1)$$

Null frame and optical functions:

$$L = \partial_t + \partial_r; \quad \underline{L} = \partial_t - \partial_r, \quad \nabla\!\!\!/_i = \partial_i - \frac{x^i}{r}\partial_r, i = 1, 2, 3$$

$$u = t - r \quad \underline{u} = t + r$$

where $r = \sqrt{\sum_{i=1}^{3} x^{i2}}$.
We use K_0 as a multiplier and the full set of commuting vectorfields to obtain the asymptotic behavior for the free wave

$$(1+t)(1+u)^{\frac{1}{2}}|\phi, Z\phi| \lesssim 1, \text{ due to } |u\underline{L}\phi| + |\underline{u}\bar{\partial}\phi| \lesssim \sum_Z |Z\phi|$$

$$(1+t)(1+u)^{\frac{3}{2}}|\partial\phi| \lesssim 1, \quad (1+t)^2(1+u)^{\frac{1}{2}}|\bar{\partial}\phi| \lesssim 1;$$

where $\bar{\partial} = (L, \nabla\!\!\!/_i)$.

Nonlinear wave

- Global solutions for quasi-linear wave equations for small data in $\mathbb{R}^{n+1}$ with $n \geq 4$, Klainerman, 85
- Null condition, $\mathbb{R}^{3+1}$ (Klainerman, 82)

$$\Box \phi^I = \sum_{J,K=1}^{N} \Gamma^I_{JK}(\phi) B^I_{JK}(\mathbf{D}\phi^J, \mathbf{D}\phi^K) \text{ in } \mathbb{R}^{3+1}$$

where the bilinear form can be written as

$$B^I_{JK}(\mathbf{D}\phi^J, \mathbf{D}\phi^K) = \mathrm{Span}\{Q_0(\mathbf{D}\phi^J, \mathbf{D}\phi^K), Q_{\mu\nu}(\mathbf{D}\phi^J, \mathbf{D}\phi^K)\}$$

where

$$Q_0(\mathbf{D}\phi, \mathbf{D}\psi) = \mathbf{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\psi$$
$$Q_{\mu\nu}(\mathbf{D}\phi, \mathbf{D}\psi) = \partial_\mu\phi\partial_\nu\psi - \partial_\nu\phi\partial_\mu\psi$$

are standard null forms.

- Global solution of quasi-linear wave equations (satisfying null conditions) in $\mathbb{R}^{3+1}$. Christodoulou, Klainerman, 86

• Blow-up: The semilinear and quasilinear nonlinearity can cause blow-up of solution in finite time, even with small data.

1. Ricatti type

$$\Box\phi = (\partial_t\phi)^2$$

2. John's example (Burgers' type)

$$\Box\phi = \phi_t \Delta\phi$$

Weak null condition (Lindblad and Rodnianski 03): Asymptotic equation (system) has global solution, together with some requirement on the asymptotic behavior of the solution and derivatives.

Examples of weak null condition:

- $$\Box u = v_t^2, \Box v = 0$$

   The solution decays slightly weaker, compared with the solution of equations satisfying standard null condition,

   $$|\partial u| \lesssim (t+1)^{-1} \ln(t+2)$$

- The reduced Einstein equations in wave coordinate gauge. (Lindblad and Rodnianski, 2005, 2010)
  1. Multiplier is not in use
  2. The full set of commuting vectorfields are crucially relied on.

Global stability of Minkowski space for Vacuum or with massless scalar fields

1. Christodoulou-Klainerman, Einstein vacuum equations with maximal foliation, 93

$$\mathbf{D}_{[\sigma}W_{\gamma\delta]\alpha\beta} = 0, \quad \mathrm{Tr}\pi = 0 \tag{2}$$

   where $\mathrm{Tr}\pi$ is the mean curvature of the level set of t embedded in the $(3+1)$ Einstein Vacuum spacetime $(\mathcal{M}, \mathbf{g})$.
   - Construct the space-time solutions of the Einstein Vacuum equations, which are geodetically complete, globally asymptotically flat. (Spherical symmetric, or static $\rightarrow$ Flat, due to Lichnerowicz and Birkoff)
   - Solutions produced by a large set of generic data which do not have peeling property. This pointed out some issue of the smooth cementification.
2. Issues with wave coordinates: Choquet-Bruhat, Blanchet and Damour

• Reduced Einstein equations in wave coordinates: Global stability, by Lindblad and Rodnianski, 2005, 2010 (asympotically flat data)
Wave coordinates: $\Box_{\mathbf{g}} x^\mu = 0$

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta \mathbf{g}_{\mu\nu} = \mathcal{N}_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}), \quad \forall\, \mu, \nu = 0, \cdots 3$$

with the nonlinearity $\mathcal{N}(\phi, \psi)$ depending quadratically in ψ.
The system on metric satisfies weakly null condition.

Various works on global stability of Minkowski space $(3+1)$

1. Exterior stability of Einstein Vacuum equation with double null foliation, Klainerman-Nicolo, 2003
2. Einstein vacuum equations with Constant mean curvature foliation, Andersson-Moncrief, 2004
3. Einstein-Maxwell, Zipser (maximal foliation, 2000), Julien Loizelet, (wave coordinates, 2006-2009)
4. Extension of C-K, Bieri, 2008, (avoid the construction of the approximate rotation, using the combination of null Bianchi and S instead)
5. Einstein- Electromagnetic system, Speck, J, (wave coordinates, 2010)

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The method presented in Global stability of Minkowski space by Christodoulou-Klainerman, 93

- Construction of the approximate symmetry in the dynamic Einstein Vaccum spacetime:
  Rotation, Scaling, Time translation, Morawetz
- The first systematic treatment of null hypersurfaces in Lorentzian spacetime.

- Inherit and further develop both the aspects of regime:
  - Christodoulou: The formation of shocks in 3-dimensional fluids, EMS, 2007
  - Christodoulou: The formation of black holes in general relativity, EMS, 2009
- Analyze null hypersurface (with limited regularity):
  - L^2 curvature conjecture for Einstein-Vacuum equations, Klainerman, Rodnianski and Szeftel, 2012-2013

KLein Gordon equation $\Box_{\mathbf{m}}\phi = \mathfrak{m}\phi$.

• Commuting vector field approach for nonlinear-Klein-Gordon (Klainerman 85)

- Scaling can not be used for K-G as the commuting vector fields. $[(\Box_{\mathbf{m}} - m), S] = 2\Box$
- Sobolev embedding on hyperboloids (Klainerman, 85) which crucially relies on the lie algebra of Lorentz boosts.

Remark:

- Multiplier of Morawetz or Scaling can not be used for K-G.
- Run the scheme by Lindblad and Rodnianski without using scaling vector field S.

- E-K-G under wave coordinates gauge : Model problem

$$\Box\psi_1 = \phi^2 + Q((\partial\psi)_1, \partial\psi) + \cdots \quad (3)$$

$$\Box\psi_2 = \phi^2 + |(\partial\psi)_1|^2 + \cdots \quad (4)$$

$$\Box\phi = \psi \cdot \partial^2\phi + m\phi \quad (5)$$

where $\psi = (\psi_1, \psi_2)$, represent components of $\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta}$.

- $\phi[0]$ is compacted supported within a ball of radius 1, Einstein data is schwarzschild ($0 < M \leq \epsilon$) outside of the ball of radius 2.
- The model system is summarized by using the framework of Lindblad and Rodnianski.
- If $\psi \cdot \partial^2\phi$ is changed to $\partial\psi \cdot \partial^2\phi$, solved in a known result by Katayama, 2012!

Error integral ($\Rightarrow \|\partial Z^{(n)}\phi, Z^{(n)}\phi\|_{L^2_x}$)

$$\int_1^t \int_{\mathbb{R}^3} Z\psi * Z^{(n-1)}\partial^2\phi \cdot \mathbf{D_T} Z^{(n)}\phi dx dt' \tag{6}$$

1. "Expectation" on $Z\psi$ from Lindblad-Rodnianski, 05,

$$|Z\psi| \lesssim \epsilon \frac{u}{t}$$

   Far from enough if the $Z^{(n-1)}\partial^2\phi$ does not take weight.
   - Does $Z^{(n-1)}\partial^2\phi$ carry any weight of u?

2. By using the $L^\infty \to L^\infty$ estimate (see Katayama 2012)

$$|Z\psi| \lesssim \epsilon t^{-1+\delta}$$

   Lindblad-Rodnianski and the estimate in Katayama $\Rightarrow$ always losing, but close.

Vector fields in Minkowski space

- Hyperboloids: $H_\varrho = \{t^2 - (x_1^2 + x_2^2 + x_3^2) = \varrho^2\}$.
- Scaling :$S = t\partial_t + r\partial_r$ is orthogonal to hyperboloids.
- Boosts $\{{}^{(i)}R = t\partial_i + x_i\partial_t, i = 1, 2, 3\}$ are tangent to hyperboloids.
- $\Omega = \sum_{i=1}^3 \frac{x^i}{r}{}^{(i)}R = t\partial_r + r\partial_t$.
- $|u\underline{L}f|^2 + |\underline{u}Lf|^2 \approx |Sf|^2 + |\Omega f|^2$.
- If Sf is missing, no weight can be put on Lf and $\underline{L}f$. (Given the fact that multiplier of S or Morawetz can not be applied to K-G)
- $\sum_{i=1}^3 |{}^{(i)}Rf|^2 \approx |t\nabla\!\!\!/ f|^2 + |\varrho\underline{\nabla} f|^2$ where $\nabla\!\!\!/$ is angular derivative and $\underline{\nabla}$ is the covariant derivative on H_ϱ.

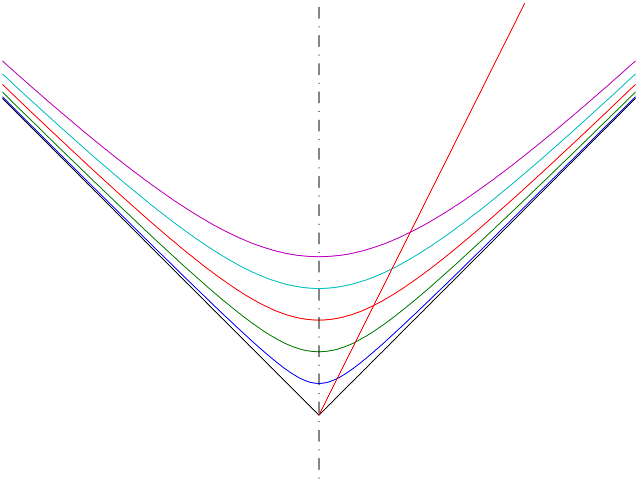
Hyperboloidal frame

Consider Z to be boosts R and rewrite the error integral

$$\int_1^t \int {}^{(R)}\pi^{\alpha\beta} R^{(n-1)} \partial^2_{\alpha\beta}\phi \cdot \mathbf{D_T} R^{(n)}\phi dx dt' \quad (\Rightarrow \|\partial R^{(n)}\phi, R^{(n)}\phi\|_{L^2_x})$$

Minkowskian hyperboloidal frame:

- $\varrho B = S; \quad te_m = {}^{(m)}R$, with $\varrho = \sqrt{t^2 - r^2}$. ($B$ is the time like unit normal of hyperboloids).
- If decompose ${}^{(R)}\pi$ relative to the frame $(B, e_m, m = 1, 2, 3)$, we need ${}^{(R)}\pi^{BB} = 0$.
- Since $BB\phi \approx \phi$ by using K-G, then we go back to (6).



- We construct the foliation of intrinsic hyperboloids in the Lorentzian spacetime and a set of **boost** vector fields $\mathcal{R} = ({}^{(i)}\mathcal{R}, i = 1, 2, 3)$, whose deformation tensor, ${}^{(\mathcal{R})}\pi$, verifies
$$\mathcal{L}_{\mathcal{R}}^{(n)} {}^{(\mathcal{R})}\pi_{\mathfrak{B}\mu} = 0, \quad n \geq 0 \tag{7}$$
where $\mathfrak{B}$ is the timelike unit normal of $\mathcal{H}_{\varrho}$.
- Geometrically, we are ready to consider Bianchi equation under maximal foliation gauge.
- **Warning:** As a multiplier approach, it is incompatible with Klein-Gordon equations.
- This issue causes the fundamental difficulty for controlling energies of Weyl tensor as well.

Set-up

Let $(\mathbf{M}, \mathbf{g})$ be globally hyperbolic, i.e., it can be foliated by the level surfaces Σ_t of a time function t.
Let $\mathbf{T}$ be the future directed unit normal to Σ_t. Then

$$\partial_t = n\mathbf{T} + Y,$$

where n is the lapse function and $Y \in T\Sigma_t$ is the shift vector field.
Let g be the induced metric of $\mathbf{g}$ on Σ_t. Let π be the second fundamental form of Σ_t in $\mathbf{M}$ defined by

$$\pi(X, Z) := -\mathbf{g}(\mathbf{D}_X \mathbf{T}, Z), \quad X, Z \in T\Sigma_t,$$

where $\mathbf{D}$ denotes the covariant differentiation of $\mathbf{g}$ in $\mathbf{M}$.

Assume $Y = 0$, then the metric $\mathbf{g}$ can be written as

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j, \tag{8}$$

and the Einstein equations are equivalent to the evolution equations

$$\partial_t g_{ij} = -2n\pi_{ij}, \tag{9}$$

$$\partial_t \pi_{ij} = -\nabla_i \nabla_j n + n(-\mathbf{R}_{ij} + R_{ij} + \mathrm{Tr}\pi\, \pi_{ij} - 2\pi_{ia}\pi_j^a) \tag{10}$$

together with the constraint equations

$$R - |\pi|^2 + (\mathrm{Tr}\pi)^2 = 2\mathbf{R}_{\mathbf{TT}} + \mathbf{R}, \qquad \nabla^j \pi_{ji} - \nabla_i \mathrm{Tr}\pi = \mathbf{R}_{\mathbf{T}i}, \tag{11}$$

where $\mathrm{Tr}\pi := g^{ij}\pi_{ij}$ is the mean curvature of Σ_t in $\mathbf{M}$, ∇ denotes the covariant differentiation of g, R_{ij} and R are the Ricci curvature and the scalar curvature of g on Σ_t.

The Maximal foliation gauge imposes

$$Y = 0 \quad \text{and} \quad \mathrm{Tr}\pi = 0 \quad \text{on } \Sigma_t.$$

This implies n satisfies the elliptic equation

$$\Delta_g n - |\pi|^2 n = n\mathbf{R}_{\mathbf{TT}} \qquad \text{on } \Sigma_t.$$

And the second fundamental form π satisfies Codazzi equation

$$(\operatorname{div}\pi)_i = \mathbf{R}_{\mathbf{T}i}, \quad \operatorname{curl}\pi = H.$$

Main result

Theorem 1

Consider the Einstein Klein-Gordon system under maximal foliation gauge. With any maximal data set $(g_0, \pi_0, \phi[0])$*, which is smooth and satisfying (11),*

(i) *the metric* $(g_0)_{ij}$ *coincides with the spatial part of the Schwartzschild metric* g_s

$$(g_0)_{ij} = \frac{r+2M}{r-2M}dr^2 + (r+2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad r > 2$$

and $\pi = 0$ *for* $r > 2$.

(* The existence of such data set is due to Corvino-Schoen.)

(ii) $n^2(r) = \frac{r-2M}{r+2M}$ for $r > 2$, and $\phi[0]$ is compactly supported in $r \leq 1$

(iii) The data $(g_0, \pi_0, \phi[0])$ satisfies smallness condition

$$\epsilon = \sqrt{E_3(W)(0)} + \sqrt{E_4(\phi)(0)} + M < \epsilon_0$$

where $E_3(W)$ represents various types of Bel-Robinson energy upto three order derivatives and $E_4(\phi)(0)$ is the canonical energy of K-G data upto the fifth order.
*The subscript represents the order of commuting vector fields.

Then there exists a unique, globally hyperbolic, smooth and geodetically complete solution $(\mathcal{M}, \mathbf{g}, \phi)$ foliated with level set of a maximal time function t with the properties that with some constant C

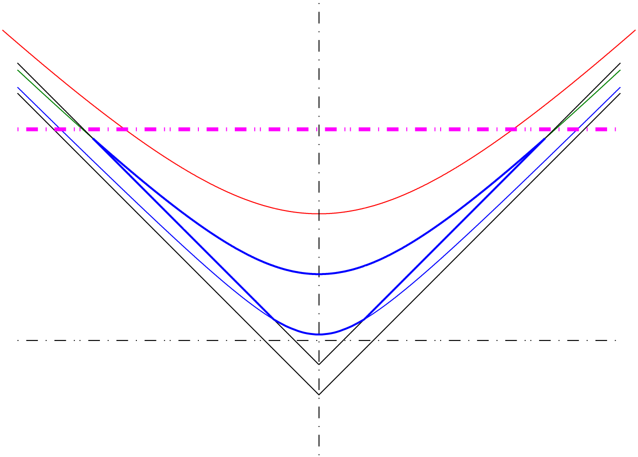
$$E_3(W)(\varrho) + E_4(\phi)(\varrho) \lesssim \epsilon^2 \langle \varrho \rangle^{C\epsilon}$$

$$\overset{\circ}{E}_2 (W)(\varrho) + \overset{\circ}{E}_3 (\phi)(\varrho) \lesssim \epsilon^2$$

$$E_1(W)(t) + E_2(\phi)(t) \lesssim \epsilon^2 \langle t \rangle^{C\epsilon}$$

$$\overset{\circ}{E}_1 (W)(t) + \overset{\circ}{E}_1 (\phi)(t) \lesssim \epsilon^2$$

where ϱ is the intrinsic replacement of "$\sqrt{t^2 - r^2}$". $\overset{\circ}{E} (W)$ and $\overset{\circ}{E} (\phi)$ are some energies in the set of $E(W)$ and $E(\phi)$ respectively.



Decay for Weyl components, (all brackets are dropped)

$$
\begin{aligned}
\underline{\alpha}: &\quad \epsilon u^{-\frac{3}{2}} t^{-1} \\
\underline{\beta}: &\quad \epsilon t^{-\frac{3}{2}} u^{-1}, \quad \epsilon t^{-2} u^{-\frac{1}{2}} \varrho^{\frac{C\epsilon}{2}} \\
(\rho, \sigma): &\quad \epsilon t^{-2} u^{-\frac{1}{2}}, \quad \epsilon t^{-\frac{5}{2}} \varrho^{\frac{C\epsilon}{2}} \\
\beta: &\quad \epsilon t^{-\frac{5}{2}}, \\
\alpha: &\quad \epsilon t^{-\frac{5}{2}}, \quad \epsilon t^{-3} \varrho^{\frac{C\epsilon}{2}}
\end{aligned}
$$

Decay of ϕ, $\mathcal{R}\phi$: $\epsilon t^{-\frac{3}{2}} \cdots$
where $u(t, \rho)$ is a natural replacement of the true optical function.

C-K

- $\mathbf{T}$ is the time-like unit normal of the maximal foliation.
- $\mathcal{O}, K_0, S$ are the approximate rotation, Morawetz vector field and the scaling vector field
- $\hat{\mathcal{L}}$ denotes the normalized Lie derivative.

Bell-Robinson energy and flux

$$\mathcal{Q}_{\alpha\beta\gamma\delta} = W_{\alpha\mu\beta\nu} W_{\gamma\ \delta}^{\ \mu\ \nu} + {}^{\star}W_{\alpha\mu\beta\nu} {}^{\star}W_{\gamma\ \delta}^{\ \mu\ \nu}$$

Energy momentum

$$\begin{aligned} P_0 &= \mathcal{Q}(W)(\cdot, \bar{K}, \mathbf{T}, \mathbf{T}) \\ P_1 &= \mathcal{Q}(\hat{\mathcal{L}}_{\mathcal{O}} W)(\cdot, \bar{K}, \bar{K}, \mathbf{T}) + \mathcal{Q}(\hat{\mathcal{L}}_{\mathbf{T}} W)(\cdot, \bar{K}, \bar{K}, \bar{K}) \\ P_2 &= \mathcal{Q}(\hat{\mathcal{L}}_{\mathcal{O}}^2 W)(\cdot, \bar{K}, \bar{K}, \mathbf{T}) + \mathcal{Q}(\hat{\mathcal{L}}_{\mathcal{O}} \hat{\mathcal{L}}_{\mathbf{T}} W)(\cdot, \bar{K}, \bar{K}, \bar{K}) \\ &\quad + \mathcal{Q}(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_{\mathbf{T}} W)(\cdot, \bar{K}, \bar{K}, \bar{K}) + Q(\hat{\mathcal{L}}_{\mathbf{T}}^2 W)(\cdot, \bar{K}, \bar{K}, \bar{K}) \end{aligned}$$

where $\bar{K} = K_0 + \mathbf{T}$.

Energy on Σ_t and flux on light cones C_u:

$$\int_{\Sigma} P_i^{\mu} \mathbf{n}_{\mu} \text{ where } i = 0, 1, 2$$

where $\Sigma = (\Sigma_t, C_u)$.

- Highly weighted B-R energy for $\hat{\mathcal{L}}_{\mathbf{T}} W$, $\hat{\mathcal{L}}_S W$ (interior)+ Rotation (exterior)
- Due to K-G,
  1. We attach less weighted multiplier to energy momentum, then the commuting vector fields have to take weights.
  2. Rotation by itself is not strong enough.
  3. Avoid S to be commuting vector fields. We need the vector fields non-vanish in the radial direction and taking weights.
  4. Boost vector fields solve all the issues.

- **Null condition**
  Weyl currents provide null structure: The three tensor $J_{\beta\gamma\delta}$ is called Weyl current if

$$J_{\beta\gamma\delta} + J_{\gamma\delta\beta} + J_{\delta\beta\gamma} = 0$$
$$J_{\beta\gamma\delta} + J_{\beta\delta\gamma} = 0$$
$$\mathrm{tr} J_\delta = \mathbf{g}^{\beta\gamma} J_{\beta\gamma\delta} = 0.$$

- **No null hypersurfaces**
  - C-K: Σ_t, construct null cones C_u.
  - For E-K-G: Σ_t, intrinsic hyperboloids $\mathcal{H}_\varrho$.
    1. $\mathcal{T}_{\mu\nu}\mathbf{T}^\nu$ is the correct energy current for ϕ, which needs the control of the decay of ${}^{(\mathbf{T})}\pi$.
    2. We need energy estimates on Σ_t for controlling ${}^{(\mathbf{T})}\pi$.

Frames of the two foliations

In Minkowski space, we denote by the normal and the radial normal of a standard hyperboloid by B and $\underline{N}$. There hold

$$\varrho B = t\partial_t + r\partial_r; \quad \varrho(B - \underline{N}) = (t - r)\underline{L} = u\underline{L}$$
$$\varrho \underline{N} = t\partial_r + r\partial_t; \quad \varrho(B + \underline{N}) = (t + r)L = \underline{u}L$$

where $L = \partial_t + \partial_r$ and $\underline{L} = \partial_t - \partial_r$.

We do have null frames without introducing null hypersurfaces.

- Weyl tensor is decomposed in null frames.
- Null forms are exhibited in terms of null frames.

"Clash"

Loss due to incompatibility of frames, since

$$\partial_t = \frac{tB - r\underline{N}}{\varrho}, \quad \partial_r = \frac{t\underline{N} - rB}{\varrho}. \tag{12}$$

- Example: Consider the $L^2(\mathcal{H}_\varrho)$ of $\mathbf{D_T}{}^{(\mathcal{R})}\pi_{ab}$, with $\{e_a\}$ orthonormal on $\mathcal{H}_\varrho$.

We use (12) to decompose $\mathbf{T}$ in terms of $\mathfrak{B}$ and $\underline{\mathcal{N}}$,

$$\mathbf{D_T}{}^{(\mathcal{R})}\pi_{ab} = \text{``}\tfrac{t}{\varrho}\text{''}\,\mathbf{D}_{\mathfrak{B}}{}^{(\mathcal{R})}\pi_{ab} - \text{``}\tfrac{r}{\varrho}\text{''}\,\mathbf{D}_{\underline{\mathcal{N}}}{}^{(\mathcal{R})}\pi_{ab}$$

the factors $\frac{t}{\rho}$ and $\frac{r}{\rho}$ add $t^{\frac{1}{2}}$ growth. (exterior, $\rho^2 \approx tu$)

Framework of Einstein part

We will use Sobolev inequalities and weighted energies to obtain L^∞ decay for

1. Deformation tensors: ${}^{(\mathbf{T})}\pi$, ${}^{(\mathcal{R})}\pi$ and k_{ab} where with $\mathfrak{B}$ be the unit normal of a hyperboloidal level set $\mathcal{H}_\varrho$,

$$k_{ab} := \langle \mathbf{D}_a \mathfrak{B}, e_b \rangle, \check{k}_{ab} := k_{ab} - \frac{1}{\varrho} \underline{g}_{ab}, \tag{13}$$

   where $\{e_a\}$ is a frame on $T\mathcal{H}_\varrho$, and $\underline{g}$ is the induced metric on $\mathcal{H}_\varrho$.

2. Weyl component
3. massive scalar field

- Sobolev inequalities

$$\sup_{\mathcal{H}_\varrho}(t\rho^{\frac{1}{2}}|f|) \lesssim \Big(\int_{\mathcal{H}_\rho} \sum_{a=1}^{3}\sum_{i=0}^{2} \|^{(a)}\mathcal{R}^{(i)} f\|^2_{L^2(\mathcal{H}_\rho)}\Big)^{\frac{1}{2}}; \quad d\mu_{\mathcal{H}_\varrho} \approx \frac{\varrho}{t} d\mu_\Sigma$$

$$\sup_{\Sigma_t}(r^{\frac{3}{2}}|f|) \lesssim \Big(\int_{\Sigma_t} |f|^2 + r^2|\nabla\!\!\!/ f|^2 + r^2|\nabla\!\!\!/_N f|^2 + r^4|\nabla\!\!\!/^2 f|^2 + r^4|\nabla\!\!\!/ \nabla\!\!\!/_N f|^2\Big)^{\frac{1}{2}}$$

$$\sup_{\Sigma_t}(r\langle u\rangle^{\frac{1}{2}}|f|) \lesssim \Big(\int_{\Sigma_t} |f|^2 + r^2|\nabla\!\!\!/ f|^2 + \langle u\rangle^2|\nabla\!\!\!/_N f|^2 + r^4|\nabla\!\!\!/^2 f|^2 + r^2\langle u\rangle^2|\nabla\!\!\!/ \nabla\!\!\!/_N f|^2\Big)^{\frac{1}{2}}$$

For $\check{k}_{ab}$ and ${}^{(\mathcal{R})}\pi_{ab}$

1. Transport equations: ${}^{(\mathcal{R})}\pi_{ab} \overset{\mathfrak{B}}{\dashrightarrow} \mathcal{L}_{\mathcal{R}}\check{k}$, $\check{k} \overset{\mathfrak{B}}{\dashrightarrow}$ Riemann
2. Codazzi equations:

$$\text{div}\,\check{k} = \underline{\nabla}\text{tr}k + \mathbf{R}_{\mathfrak{B}a},\ \text{curl}\,\check{k} = \underline{H} \text{ on } \mathcal{H}_\rho$$

To get L^∞ decay for ${}^{(\mathcal{R})}\pi_{ab}$,

$$\mathcal{L}_{\mathcal{R}}^{(j)}{}^{(\mathcal{R})}\pi_{ab} \overset{\mathfrak{B}}{\dashrightarrow} \mathcal{L}_{\mathcal{R}}^{(j+1)}\check{k}_{\alpha\beta} \overset{\mathfrak{B}}{\dashrightarrow} Q(\mathcal{L}_{\mathcal{R}}^{(j+1)}W)$$

$$\mathbf{D}\mathcal{L}_{\mathcal{R}}^{(j)}{}^{(\mathcal{R})}\pi_{ab} \overset{\mathfrak{B}}{\dashrightarrow} \overset{Codazzi}{\dashrightarrow} Q(\mathcal{L}_{\mathcal{R}}^{(j+1)}W), Q(\mathcal{L}_{\mathcal{R}}^{(j+1)}\phi), \cdots$$

$$Q(\mathcal{L}_{\mathcal{R}}^{(i)}W) \leftarrow\dashrightarrow Q(\mathcal{L}_{\mathcal{R}}^{(i)}\phi), \cdots, i = 0, 1, 2, 3$$

- It is very difficult to close the energy estimate for $i \geq 1$.
- Einstein equations create two "miracles" to close the energy argument.

Strategy of decoupling in energy estimates

- Suppose y is a set of energies,

$$\frac{dy}{ds} \leq \frac{\epsilon}{s} y + l.o.t$$

Then $y \lesssim (y(1) + \cdots) s^{\epsilon}$

- If decay rate $\frac{\epsilon}{s}$ can only be derived in terms of y via Sobolev inequality, then the energy estimate fails.
    1. Obtain the decay by the bounded lower order energy. $Q(\mathcal{L}_{\mathcal{R}}^{(\leq i-1)} W) \leftarrow\text{-----}\rightarrow Q(\mathcal{L}_{\mathcal{R}}^{(i)} \phi), \cdots, i = 1, 2, 3.$
    2. Obtain the decay from other energies defined by different multipliers.

Energy hierarchy for Weyl part

- Overweighted energy
- Morawetz energy
- Standard weighted energies (upto top order)

1. The more hyperboloidal frames are used in an energy, the easier it is to close the energy.
2. All the above energies have ϵ-growth except the lower-order standard weighted energies.

Some difficulties

I Extensive comparison between intrinsic boosts with the Minkowskian boosts.
Such comparison is much more complicated than for rotation in C-K.
In Minkowsi space,

$$[\mathcal{O}_i, \underline{L}] = 0, \quad [R_i, \underline{L}] = \frac{x^i}{r}(\partial_t + \frac{t}{r}\partial_r) - \frac{\underline{u}}{r}\partial_i$$

II Hyperboloids are very singular near the causal boundary. The geometry of the hyperboloids needs to be justified throughout the open cone.

III Recover the picture of wave zone by Klainerman 85, in the Lorentzian spacetime by intrinsic foliations.

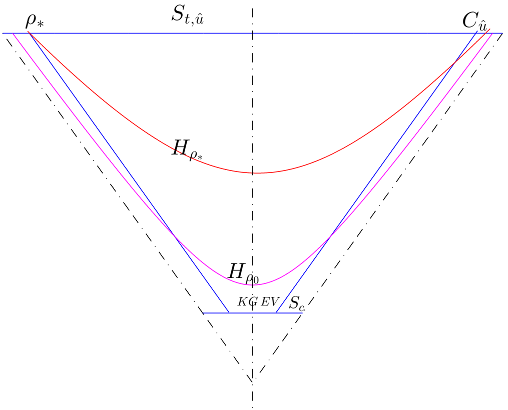


Figure: Let Σ_{t_*} be the last slice. The cuts created by schwarzschild cone $C_{\hat{u}}$ do not coincide, i.e. $S_{t_*,\hat{u}}$, S_{t_*,ρ_*} and $S_{\hat{u},\rho_*}$ are three different two spheres.

(1) In order to guarantee all possible elliptic estimates on Codazzi equations to have reasonable boundaries, we slightly extend the wave zone.

(2) In the modified wave zone, prove

$$u \gtrsim 1 \tag{14}$$

where the function $u(t, \rho)$, which measures the distance to the Schwarzschild cone $C_{\hat{u}}$,

- u, not optical relative to any metric, being closer to Minkowskian optical function would fail to achieve (14).
- u can be proved to be almost optical in the Schwartzchild zone, and increasing inward.
- Inspired by the comparison technique for controlling the rotation in "The Formation of shocks in 3-dimentional fluids."

- We need better decay of the radial components of tensors on hyperboloids, such as of ${}^{(\mathcal{R})}\pi$ or k.
- Connection coefficients of the radial foliation on hyperboloids are involved.
- Connection coefficients of frames are completely representable.

$$\check{k}_{ab} : \epsilon t^{-1}\rho^{-1+\epsilon}; \quad \check{k}_{a\underline{\mathcal{N}}} : \epsilon t^{-\frac{3}{2}}$$

$$^{(\mathbf{T})}\pi_{ij} : \epsilon t^{-1+\epsilon}u^{-\frac{1}{2}}; \quad ^{(\mathbf{T})}\pi_{iN} : \epsilon t^{-\frac{3}{2}+\epsilon}$$

$$^{(\mathcal{R})}\pi_{ab} : \epsilon t^{-1/2}\rho^{\epsilon}; \quad ^{(\mathcal{R})}\pi_{a\underline{\mathcal{N}}} : \epsilon t^{-1}\rho^{\epsilon+\frac{1}{2}}$$