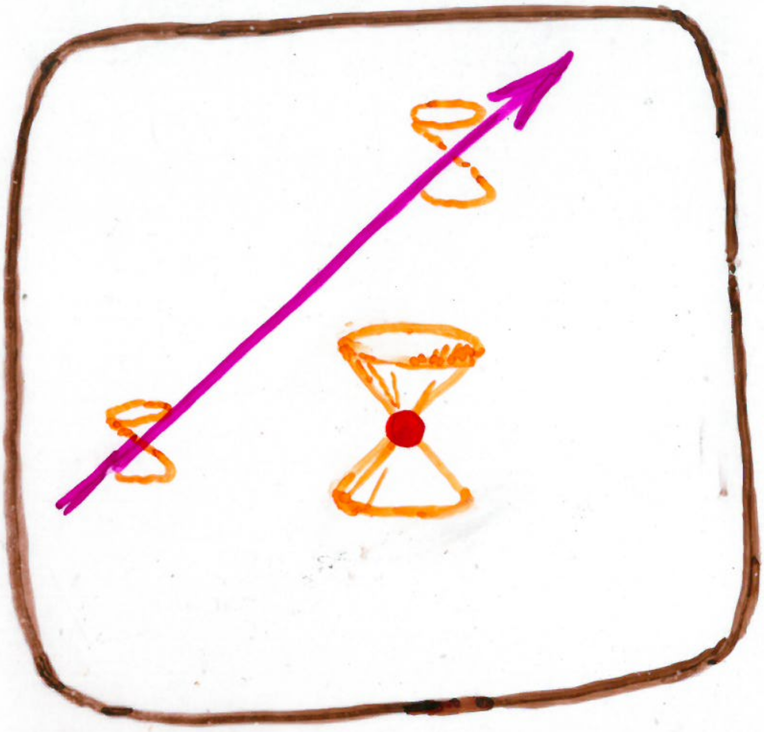
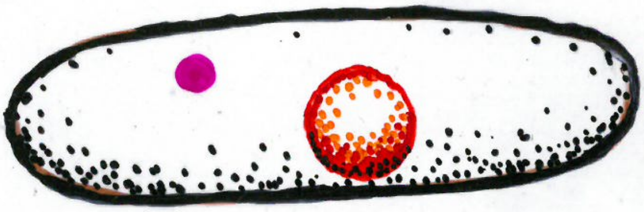
Twistor Theory



Minkowski Space-Time $\mathbb{M}$

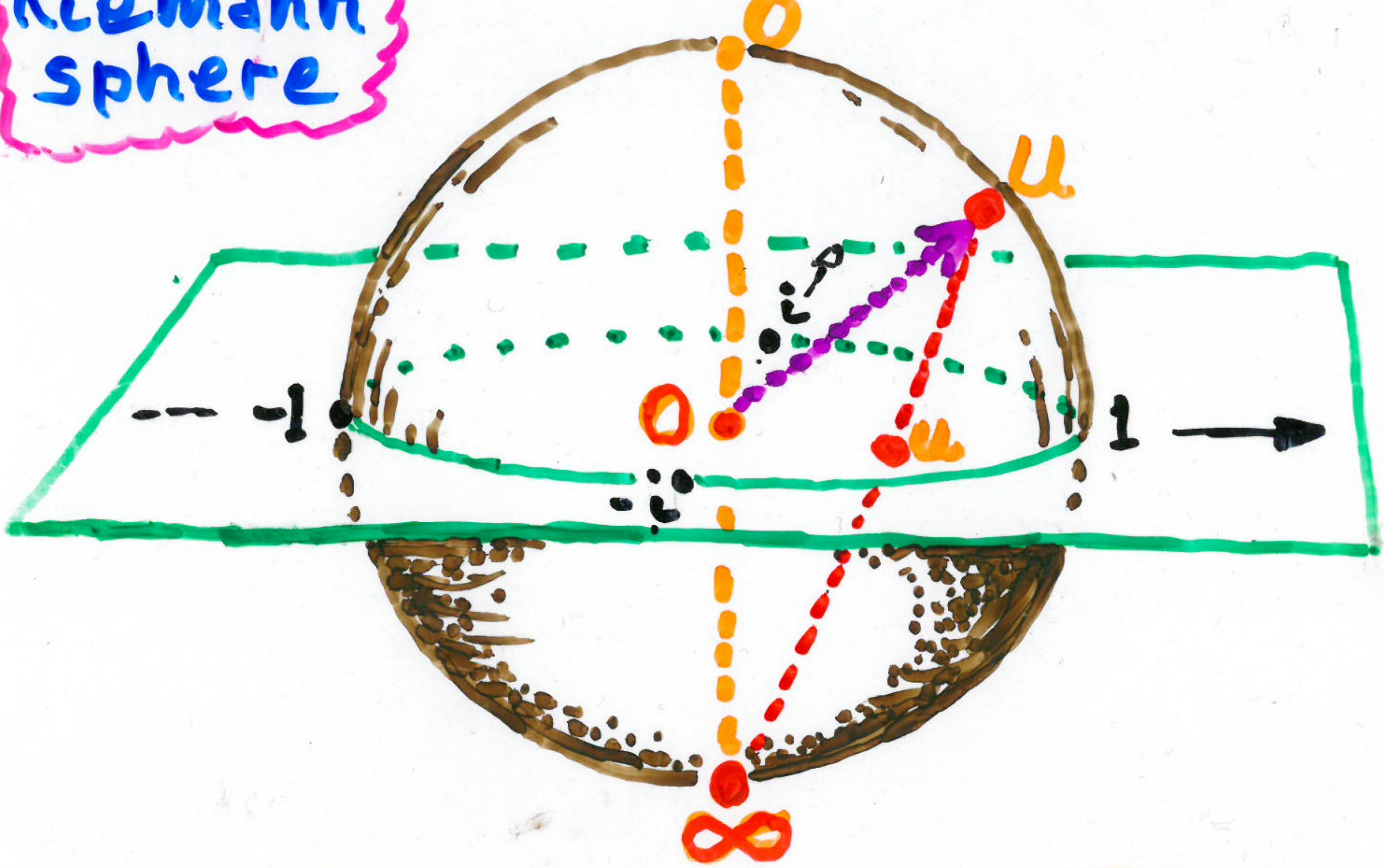


$\mathbb{PN}$

Twistor space

Stereographic projection

Riemann sphere

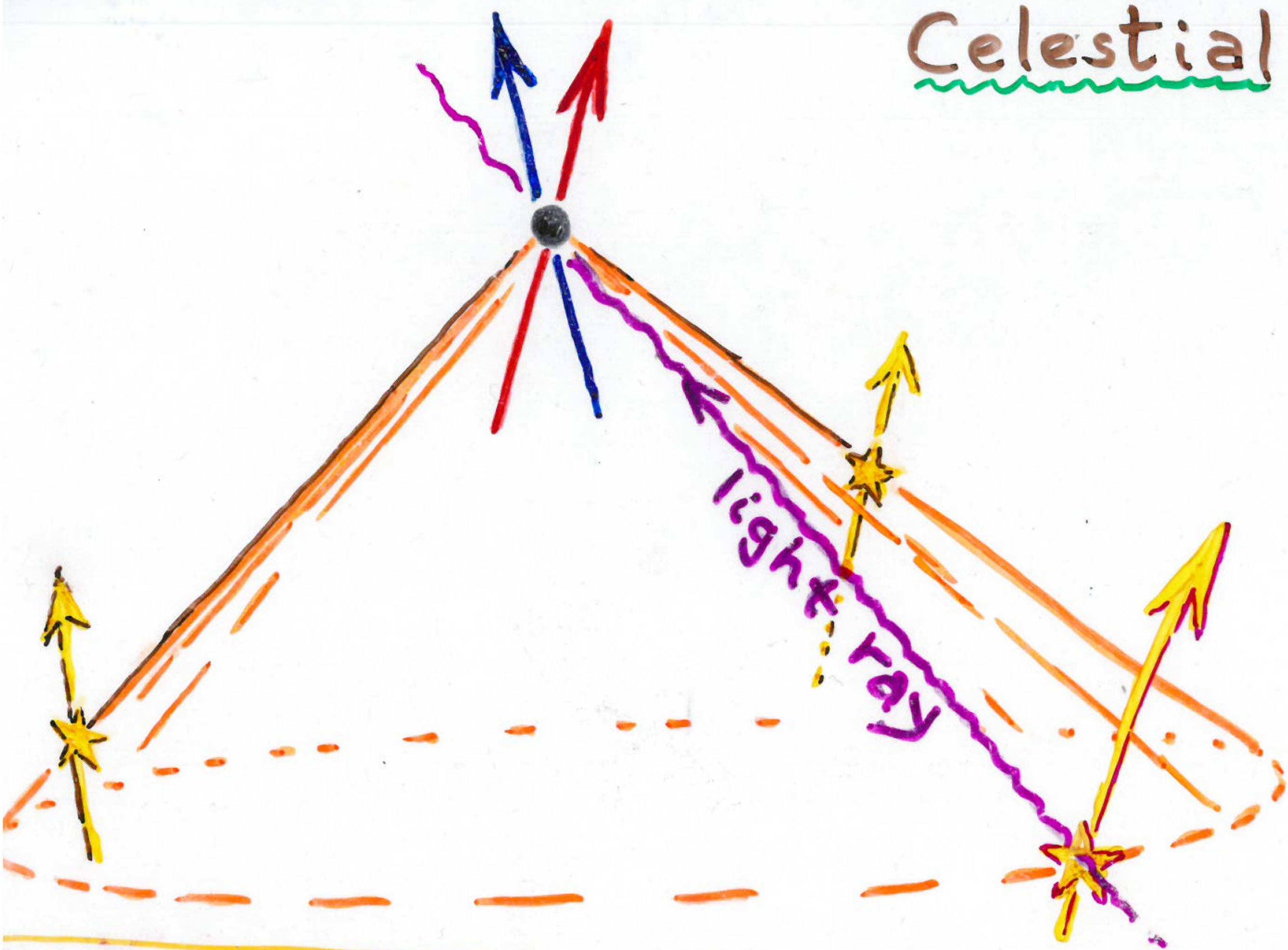


Spin $\frac{1}{2}$ particle (e.g. electron
proton
neutron
quark)

$$u = z/w$$

Riemann sphere

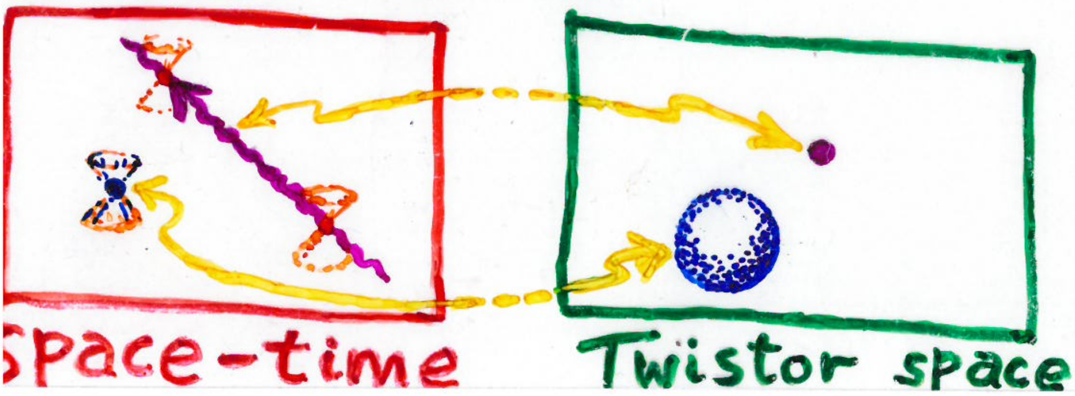
Celestial Sphere



Blue observer's sky

Red observer's sky

Twistor theory:

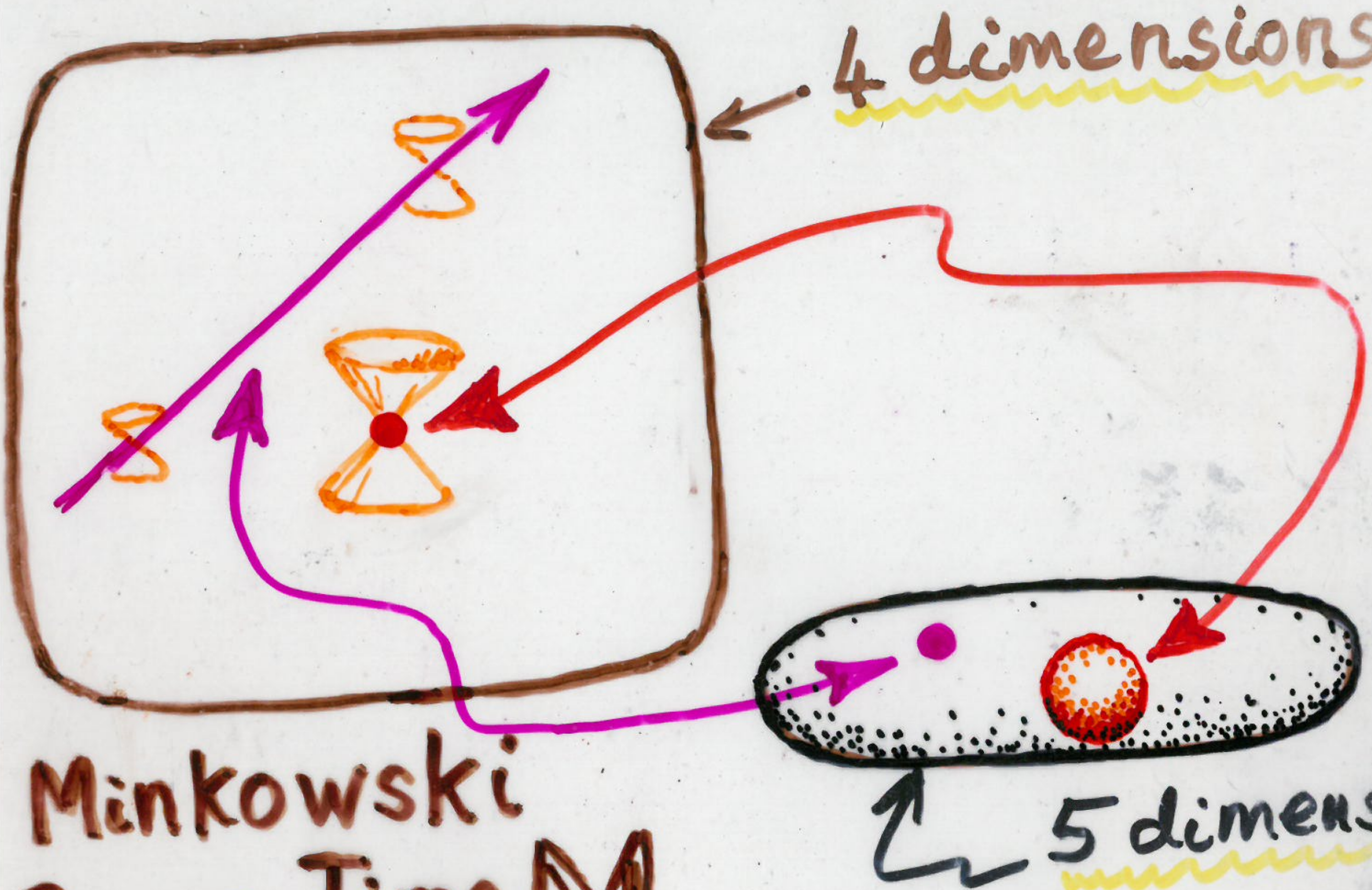


Complex

non-local

Spinors for O(2,4)

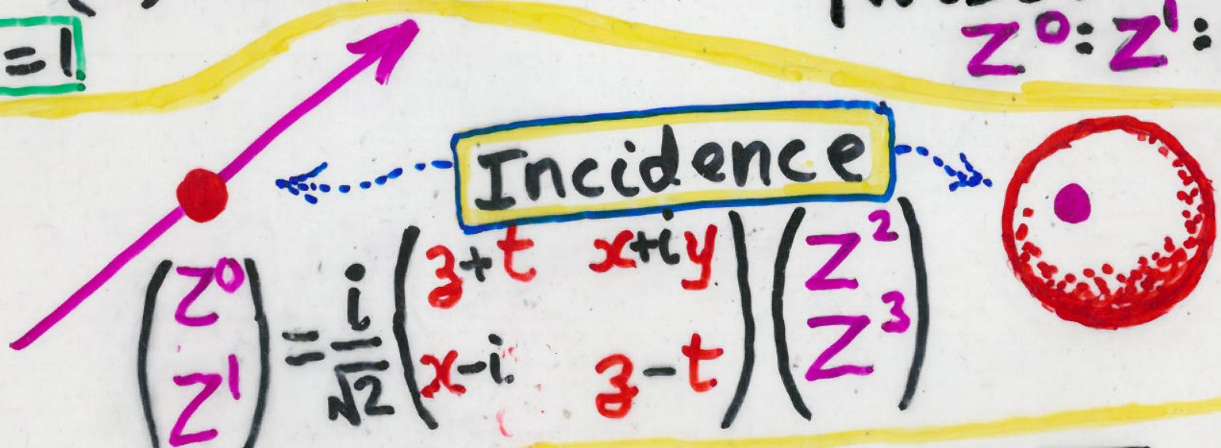
Twistor Theory



Minkowski Space-Time $\mathbb{M}$
(t, x, y, z)

$\boxed{c=1}$

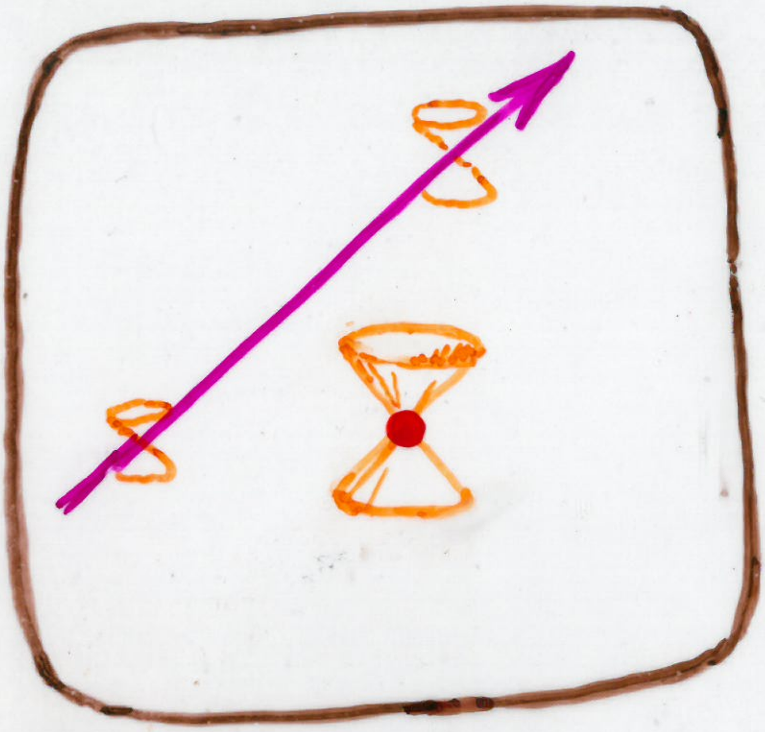
Twistor space
$Z^0 : Z^1 : Z^2 : Z^3$



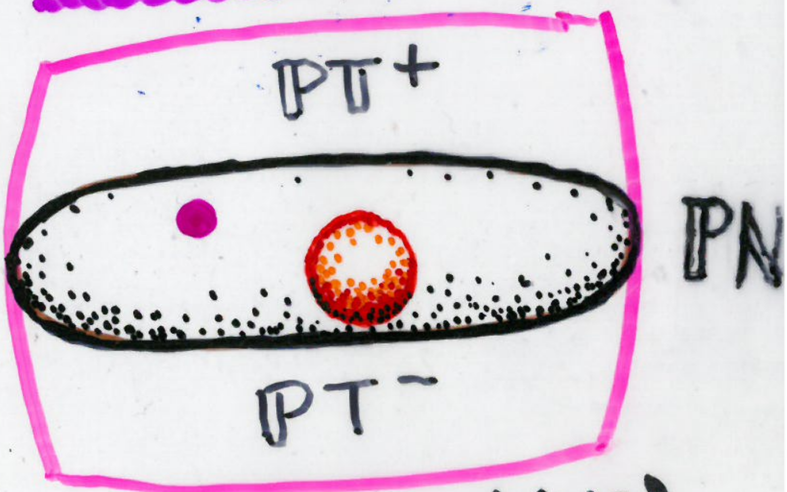
$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} z+t & x+iy \\ x-iy & z-t \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

Eqn. of $\mathbb{PN}$: $Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} = 0$

Twistor Theory



An unusual complex-geometric approach to relativistic physics



Minkowski Space-Time $\mathbb{M}$ is a flat real Lorentzian 4-space

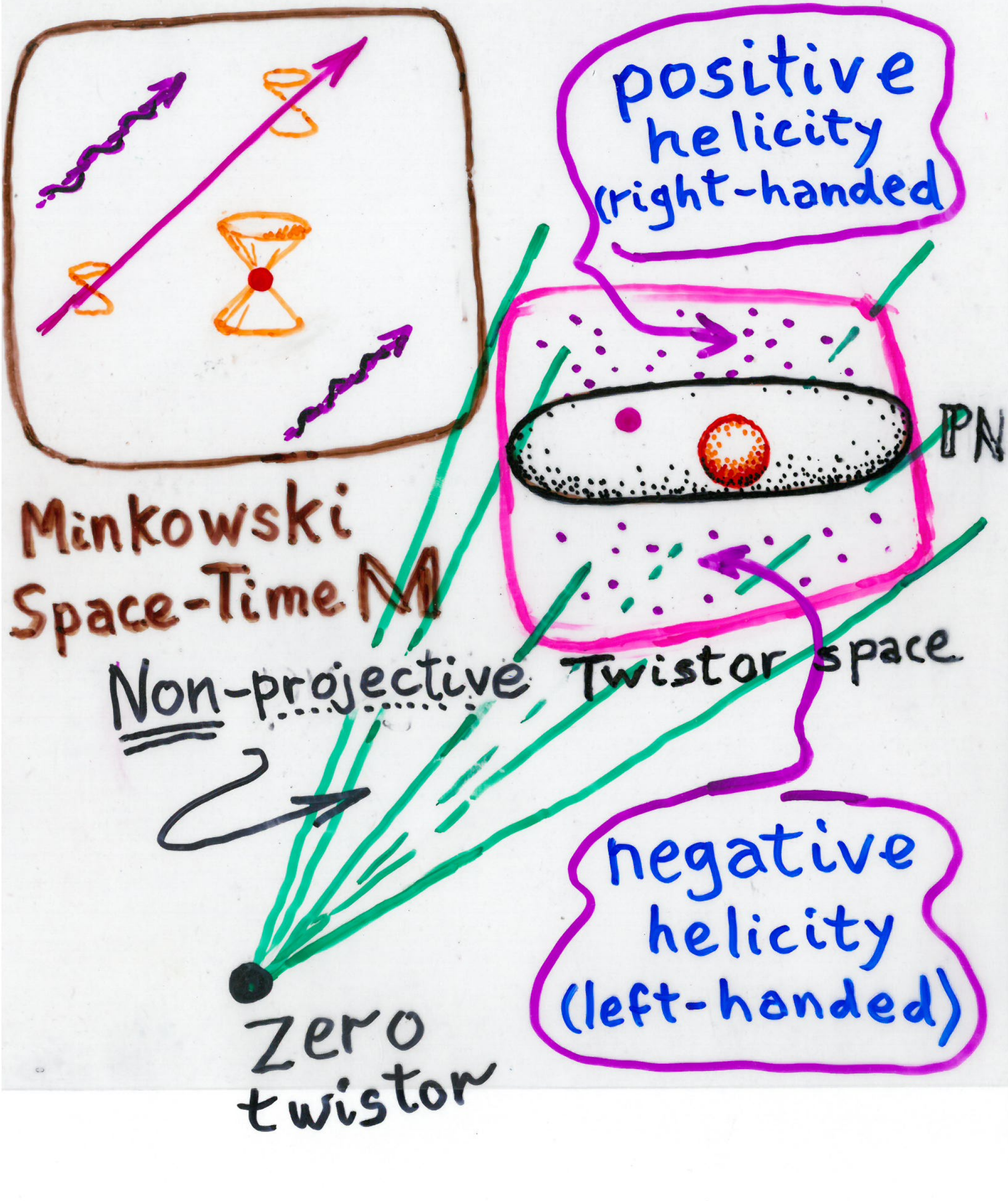
(projective) Twistor space is a complex projective 3-space

twistor correspondence

Brings together key features of relativity & quantum theory

2 geometrical roles for Riemann sphere ① Rel.: celestial spere ② QM spin directns.

Twistor Theory



New ideas:

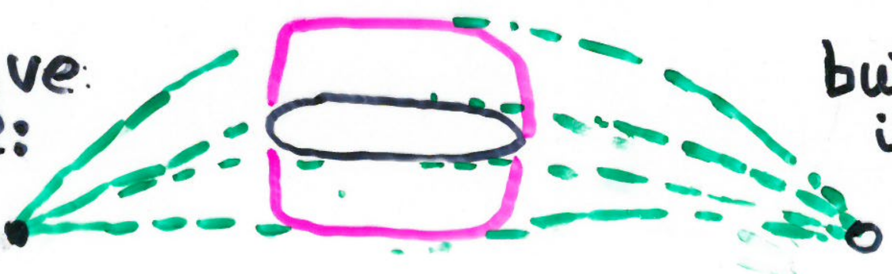
PALATIAL Twistor theory

In order to describe a general Lorentzian, globally hyperbolic 4-dimensional space-time in twistor terms, we appear to be driven to a holomorphic

Non-commutative

twistor geometry — even for a classical space-time!

still have picture:



but the geom is non-commutative

Twistors

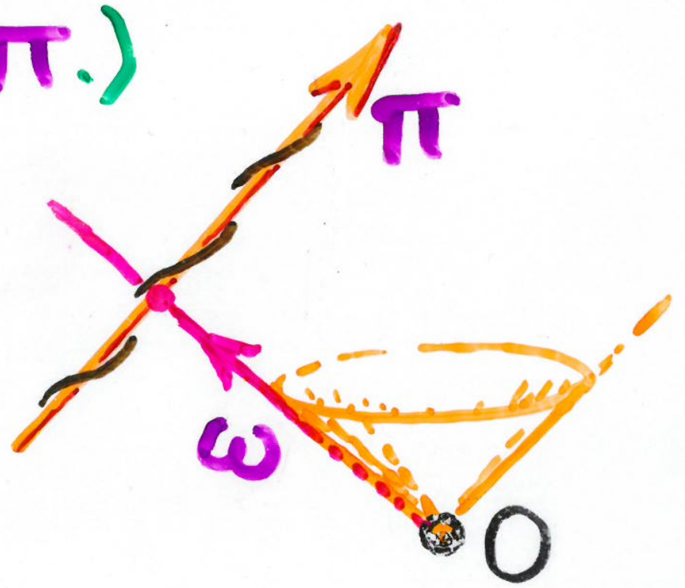
A twistor Z can be represented as a pair of 2-spinors ω, π, where π describes the 4-momentum of a massless entity (e.g. photon) with spin

energy + 3mom.

(strictly $p = \bar{\pi}\pi$, where $\bar{\pi}$ is the complex conjugate of π), and where ω tells us its angular momentum about a chosen origin O. (More accurately, the 6-angular mom. M is expressed linearly in terms of $\omega\bar{\pi}$ and $\bar{\omega}\pi$.)

Picture of a null twistor

$Z = (\omega, \pi)$



2-Spinor notation

B.L. van der Waerden
L. Infeld & "

$$V^a \simeq V^{AA'} : \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0+V^3 & V^1+iV^2 \\ V^1-iV^2 & V^0-V^3 \end{pmatrix}$$

abstract correspondence

components in local frame

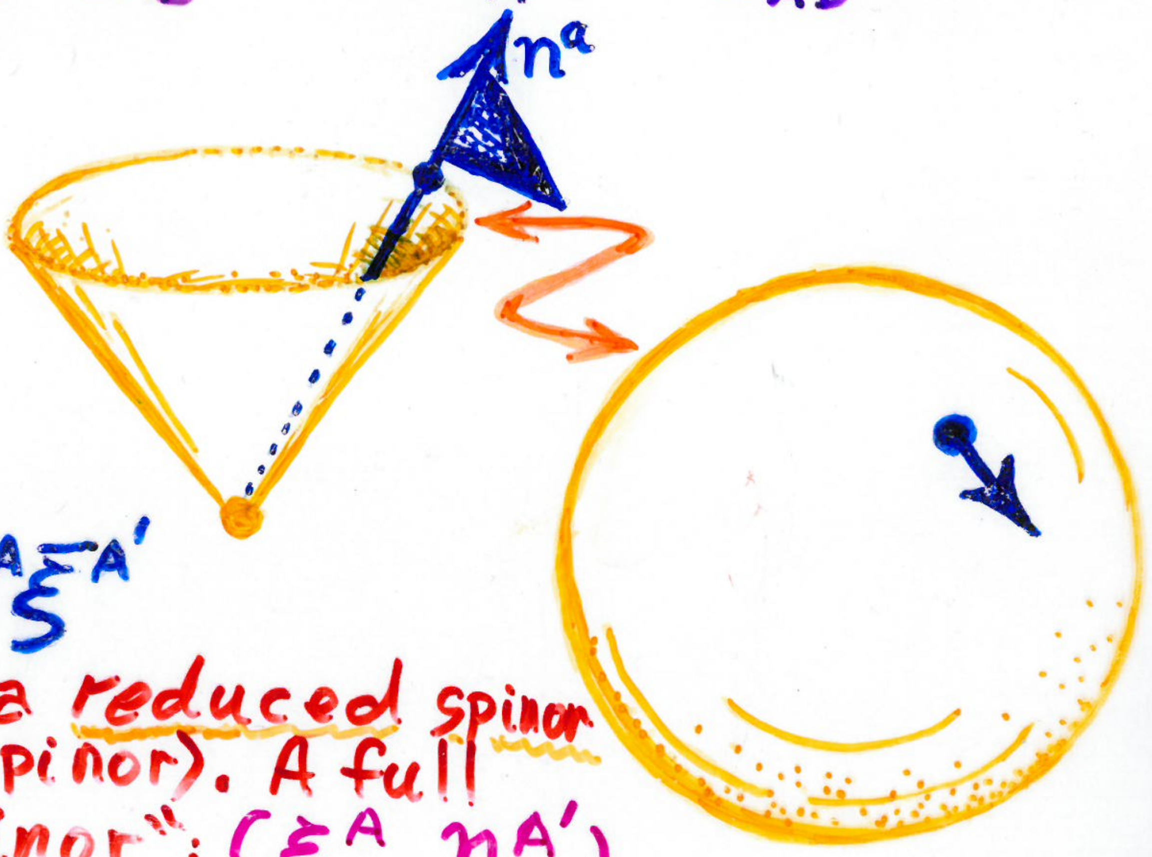
$$\det V^{AA'} = \tfrac{1}{2}\left[(V^0)^2-(V^1)^2-(V^2)^2-(V^3)^2\right] = \tfrac{1}{2} g_{ab} V^a V^b$$

$$= \tfrac{1}{2} \varepsilon_{AB}\, \varepsilon_{A'B'}\, V^{AA'} V^{BB'}$$

$$g_{ab} \simeq \varepsilon_{AB}\, \varepsilon_{A'B'} \qquad \begin{pmatrix} a \simeq AA' \\ b \simeq BB' \\ \text{etc.} \end{pmatrix}$$

$$\xi^A = \varepsilon^{AB} \xi_B \qquad \xi_A = -\varepsilon_{AB}\, \xi^B$$

ξ^A :



$$n^a \simeq \xi^A \bar{\xi}^{A'}$$

N.B. ξ is a reduced spinor (or half-spinor). A full "Dirac spinor": $(\xi^A, \eta^{A'})$

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{bmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

$$\omega^A = i \, r^{AA'} \, \pi_{A'}$$

$$Z^\alpha = (\omega^A, \pi_{A'})$$

$$\boxed{\text{incidence:} \quad \omega = i\, r \cdot \pi}$$

Shift of origin

$$\overset{Q}{\omega}{}^A = \overset{O}{\omega}{}^A - i\, q^{AA'} \overset{O}{\pi}_{A'}$$

$$\overset{Q}{\pi}_{A'} = \overset{O}{\pi}_{A'}$$

complex conjugate twistor

$$\boxed{\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})}$$

Null twistor Z

$$Z^\alpha \bar{Z}_\alpha = 0$$

← equation of $\mathbb{PN}$

Momentum-angular mom. for massless ptcle.

p_a = 4-momentum $\quad \left. \begin{matrix} p_a p^a = 0 \\ p_0 > 0 \end{matrix} \right\}$

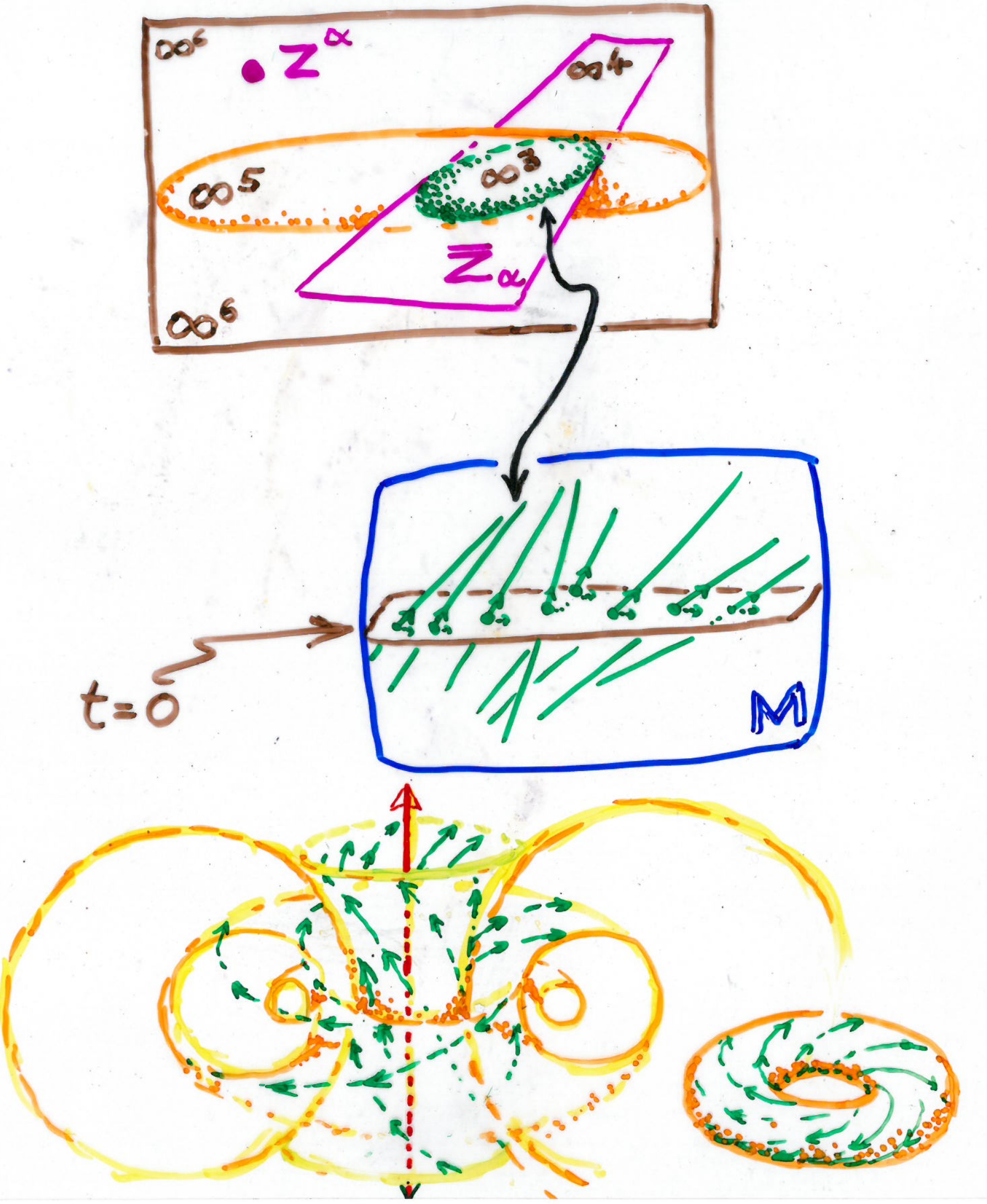
M^{ab} = 6-angular momentum

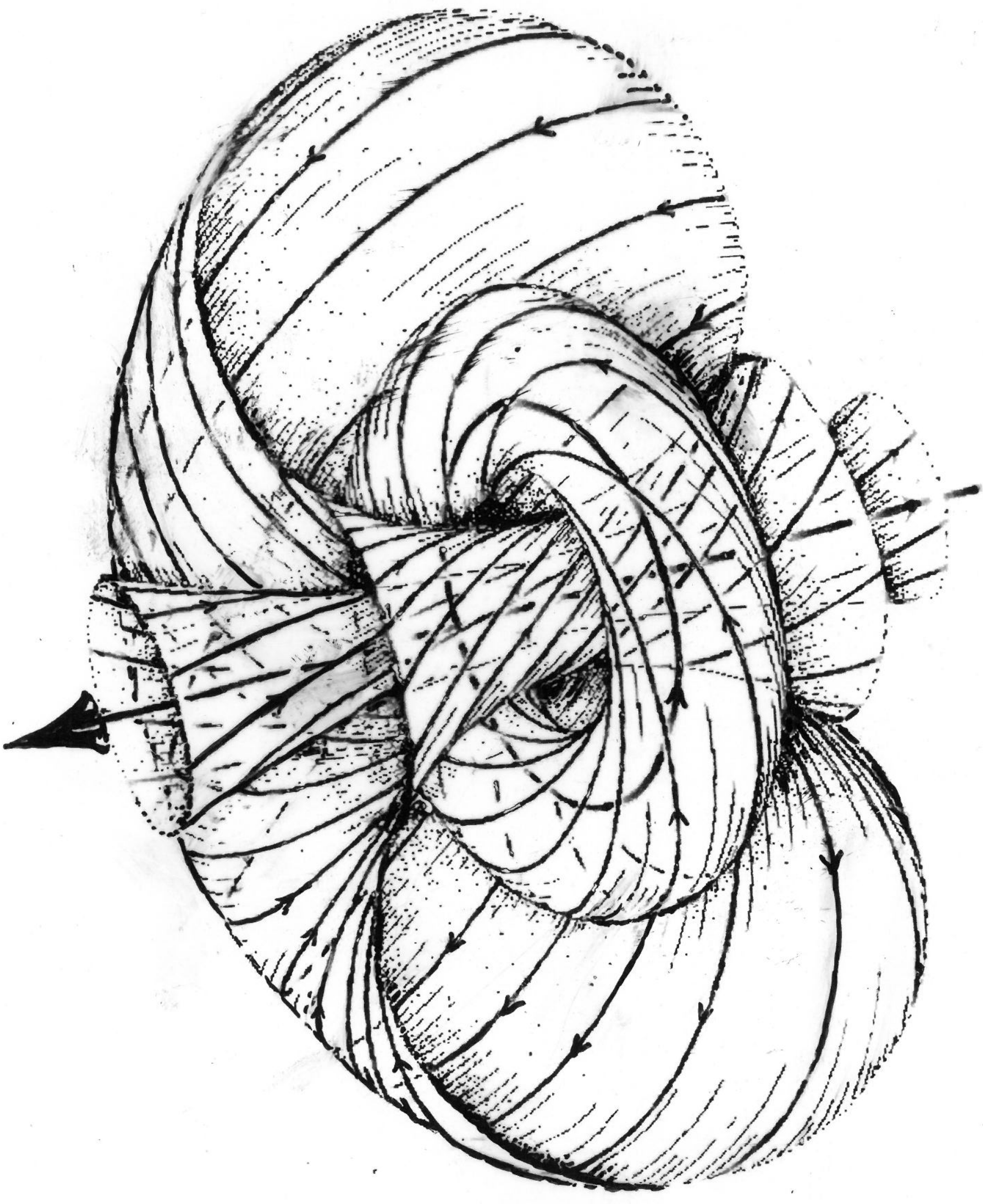
Pauli-Lubanski spin vector $\quad S_a = \frac{1}{2} e_{abcd} \underbrace{p^b M^{cd}}_{*(p \wedge M)} = \underbrace{s}_{\text{helicity}} p_a$

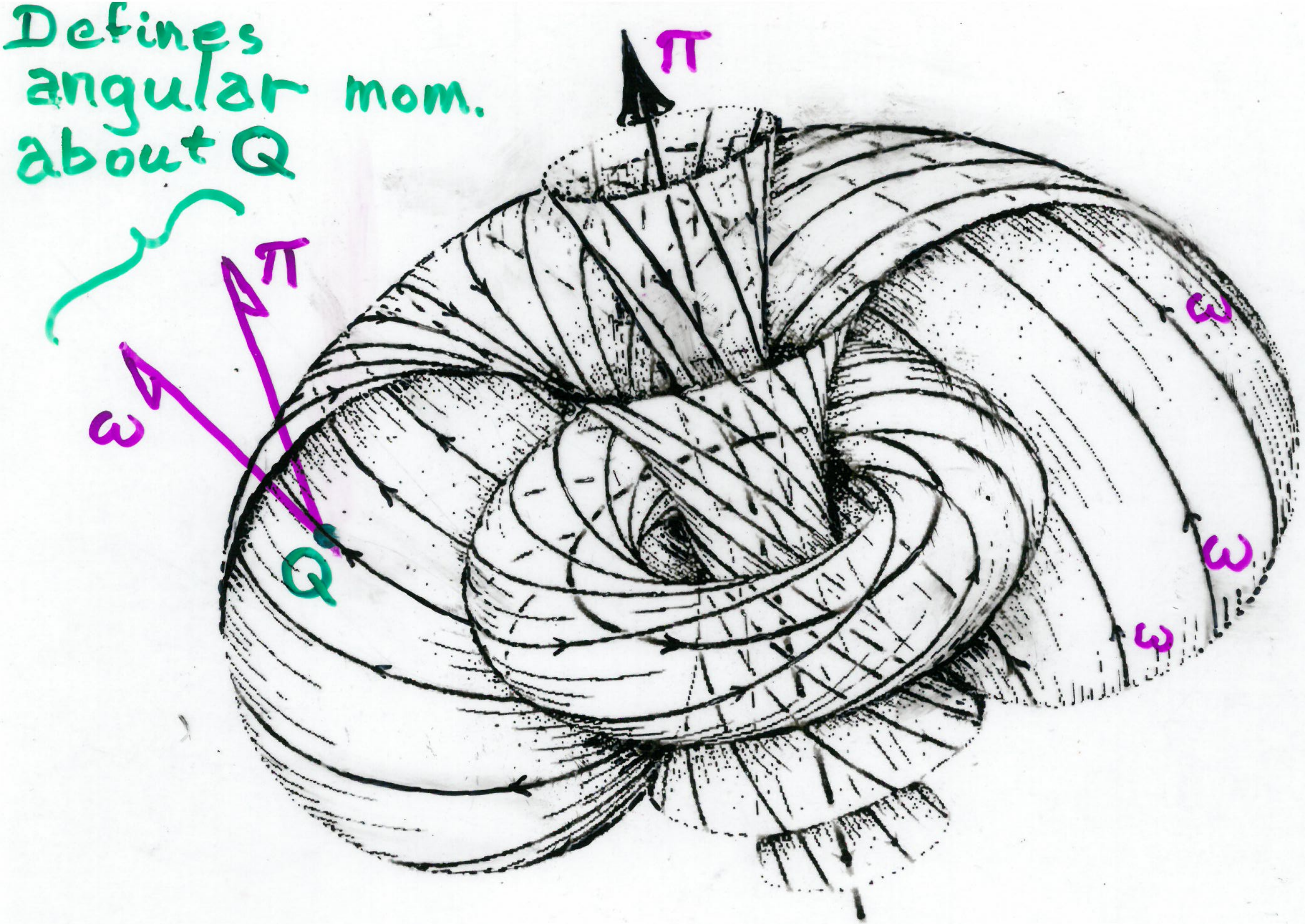
$$p_a = \pi_{A'} \bar{\pi}_A, \quad M^{ab} = i\, \omega^{(A} \bar{\pi}^{B)} \varepsilon^{A'B'} - i\, \varepsilon^{AB} \bar{\omega}^{(A'} \pi^{B')}, \quad 2s = Z^\alpha \bar{Z}_\alpha$$

Note: (...) denotes symmetrization

Spatial "TWISTOR" interpretation of Z^α (via $\bar{Z}_\alpha$)






Defines angular mom. about Q
π
π
ω
ω
ω
ω
ω
Q

Quantum Twistor Theory

Z^α and $\bar{Z}_\alpha$ become non-commuting

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$

$$\bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar\, \delta^\alpha_\beta$$

So Z^α and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugates).

Choose $\hbar = 1$, for convenience. We find

$$p_a = \pi_A \bar{\pi}_{A'}, \qquad M^{ab} = i\omega^{(A}\pi^{B)}\varepsilon^{A'B'} - i\,\varepsilon^{AB}\bar{\omega}^{(A'}\bar{\pi}^{B')},$$

undisturbed by factor ordering, but

$$S = \frac{1}{4}\left(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha\right)$$

The standard commutators for p_a and M^{ab} follow

$p_a p_b - p_b p_a = 0$, $\quad p_a M^{bc} - M^{bc} p_a = i\,(g_a{}^b p^c - g_a{}^c p^b)$,

$M^{ab}M^{cd} - M^{cd}M^{ab} = i(g^{bc}M^{ad} - g^{bd}M^{ac} + g^{ad}M^{bc} - g^{ac}M^{bd})$.

Lie algebra generators for the Poincaré group

⑦ # Twistor Wavefunctions

Since Z^α and $(i)\bar{Z}_\alpha$ are conjugate variables, a twistor wavefunction f ought to depend on *either* Z^α or $\bar{Z}_\alpha$, but not both. But what does it mean to say that $f(Z^\alpha)$ does not depend on $\bar{Z}_\alpha$? The condition is $\frac{\partial f}{\partial \bar{Z}_\alpha} = 0$, the Cauchy-Riemann eqns, asserting that f is *holomorphic* in Z^α.

Helicity eigenstates

These are eigenstates of the helicity operator $s = \frac{\hbar}{2}\left(-2 - Z^\alpha \frac{\partial}{\partial Z^\alpha}\right)$.

But $Z^\alpha \frac{\partial}{\partial Z^\alpha}$ is the Euler homogeneity operator, whose eigenstates are homogeneous functions, with eigenvalue = degree of homogeneity.

Take the integer $n = 2s/\hbar$ to represent the helicity; then f is homogeneous (holomorphic) in Z^α of $\boxed{\text{degree } -2-n}$. Alternatively use $\tilde{f}(W_\alpha)$, with $W_\alpha = \bar{Z}_\alpha$. Then $\tilde{f}$ is hol. hom. $\boxed{\text{deg. } -2+n}$

Massless Particles of Arbitrary Spin

Wave equations:

$s = 0$: $$\Box \phi = 0$$

$s > 0$: $$\phi_{\underbrace{A'B'\ldots L'}_{2s}} = \phi_{(A'B'\ldots L')}$$

$$\nabla^{AA'} \phi_{A'B'\ldots L'} = 0$$

$s < 0$: $$\phi_{\underbrace{AB\ldots L}_{2s}} = \phi_{(AB\ldots L)}$$

$$\nabla^{AA'} \phi_{AB\ldots L} = 0$$

	2×helicity	Twistor hom.	Dual Twistor hom.
graviton	2	-6	2
photon	1	-4	0
massles anti-neutrino	1/2	-3	-1
Scalar	0	-2	-2
massles neutr.	-1/2	-1	-3
photon	-1	0	-4
graviton	-2	2	-6

Spinor form, and Conformal Invariance of Maxwell & gravitational wave eqns.

$$F_{ab} = \underbrace{\varphi_{AB}\varepsilon_{A'B'}}_{\text{anti-self-dual part (−helicity)}} + \underbrace{\varepsilon_{AB}\tilde{\varphi}_{A'B'}}_{\text{self-dual part (+helicity)}}$$

complex conjugates for a <u>real</u> field

Gravitational case: use Weyl conformal tensor to describe the free gravitational field

$$C_{abcd} = \underbrace{\Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}}_{\text{anti-self-dual part (−helicity)}}$$

$$+ \underbrace{\varepsilon_{AB}\varepsilon_{CD}\tilde{\Psi}_{A'B'C'D'}}_{\text{self-dual part (+helicity)}}$$

complex conj., for a <u>real</u> field

Free-field eqns.: $\nabla^{AA'}\varphi_{AB} = 0$, $\nabla^{AA'}\psi_{ABCD} = 0$

(& ~ versions)

conformal invariance:

$$\varphi_{AB} \rightsquigarrow \Omega^{-1}\varphi_{AB} \qquad \psi_{ABCD} \rightsquigarrow \Omega^{-1}\psi_{ABCD}$$

BUT NOTE conf. curv.: $\Psi_{ABCD} \rightsquigarrow \Psi_{ABCD}$!

Massless Field Contour Integral

Whittaker 1903, Bateman 1904, 1944, RP 1968, 1969, 1975
Hughston 1973, 1979

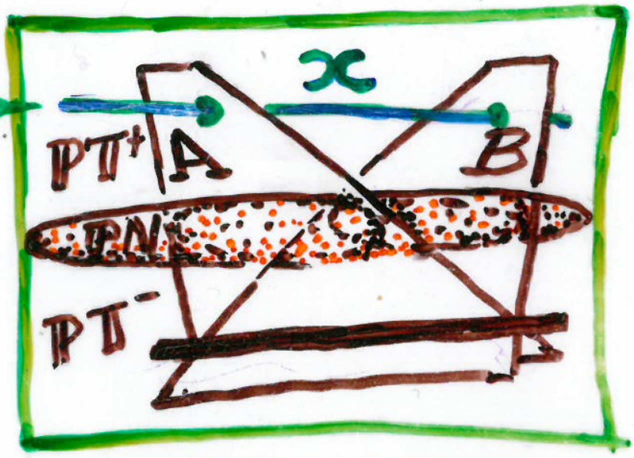
$S=0$: Wave eqn. $\Box\phi=0$ $\quad f(Z)$ hom. deg. $=-2$

$$\boxed{\phi(x)=\oint_{\omega=ix\pi} f(Z)\,\delta Z}$$

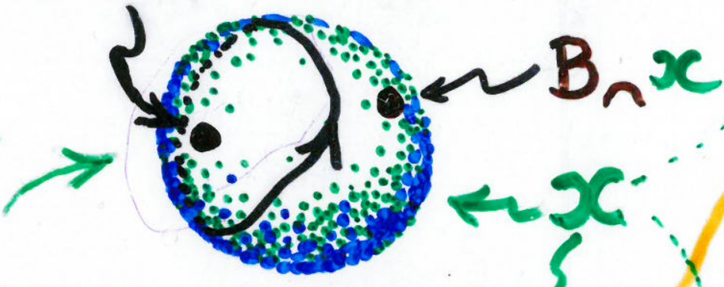
$$\delta Z=\varepsilon^{A'B'}\pi_{A'}d\pi_{B'}$$

Typical case:

$$f(Z)=\frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$



Riemann sphere x



Generally:

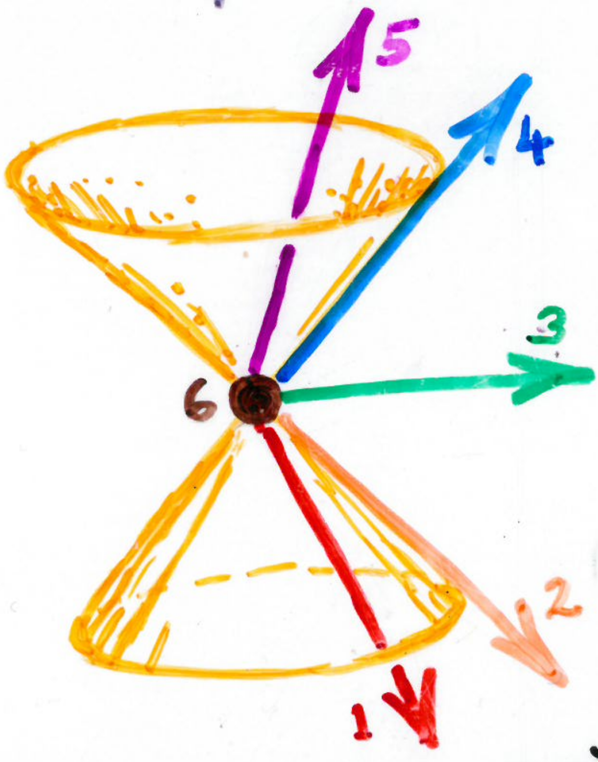
$$\nabla^{AA'}\phi_{A'B'C'\dots L'}=0$$

$S>0$ $\quad \boxed{\phi_{A'B'\dots L'}(x)=\oint_{\omega=ix\pi}\pi_{A'}\pi_{B'}\dots\pi_{L'}\,f(Z)\,\delta Z}$

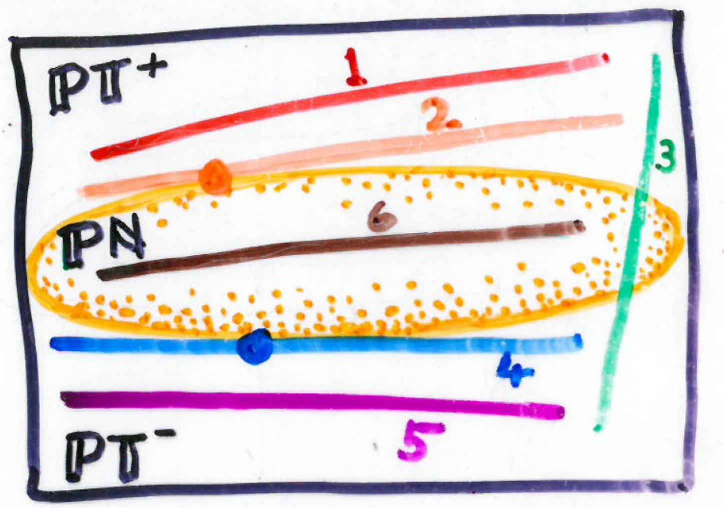
$$\nabla^{AA'}\phi_{ABC\dots L}=0$$

$S<0$ $\quad \boxed{\phi_{AB\dots L}(x)=\oint_{\omega=ix\pi}\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\dots\frac{\partial}{\partial\omega^L}f(Z)\,\delta Z}$

Complex Minkowski Points

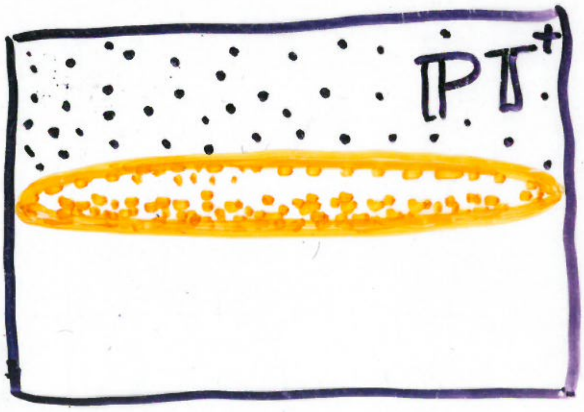


Imaginary part of complex position vector



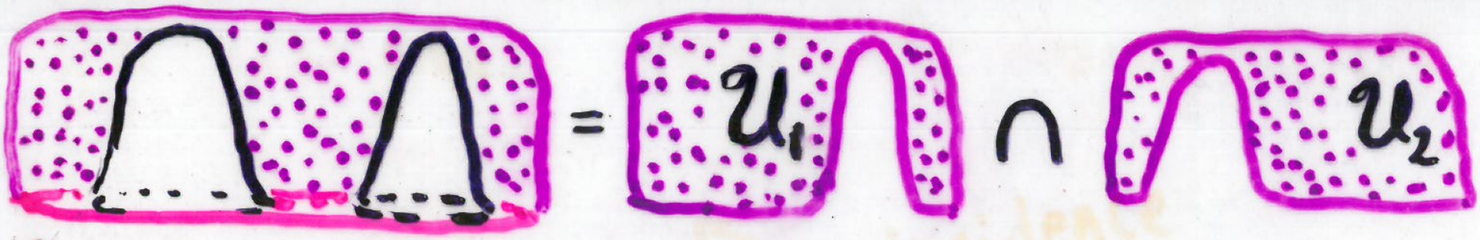
Projective twistor space $\mathbb{PT}$

Corresponds to forward tube of complex Minkowski space: past-timelike imaginary part



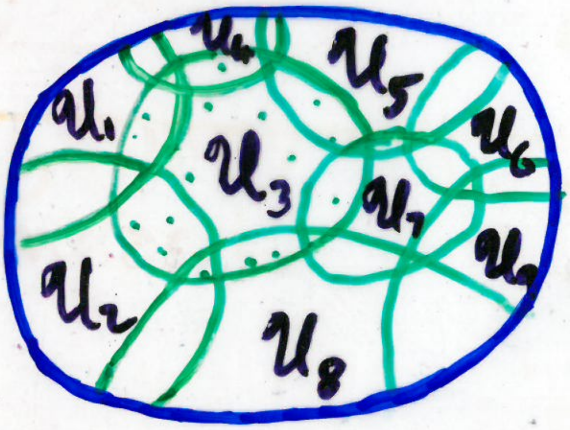
Positive-frequency fields: extend holomorphically to the forward tube

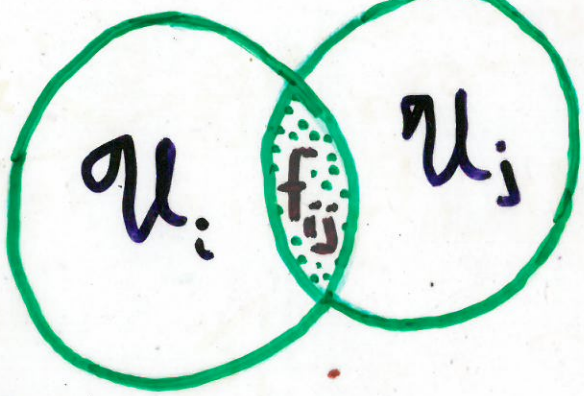
Composed of $e^{P_a x^a / i\hbar}$ with  : tails off in forward tube



↗ domain of def of f, whereas:

More generally, need



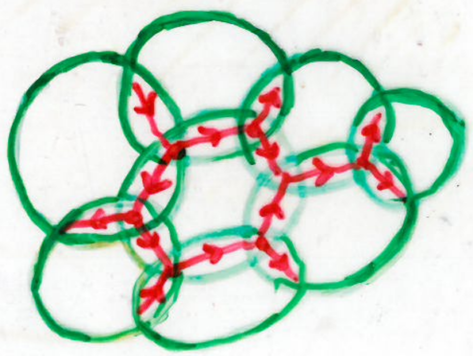


$f_{ij} = -f_{ji}$ on overlaps

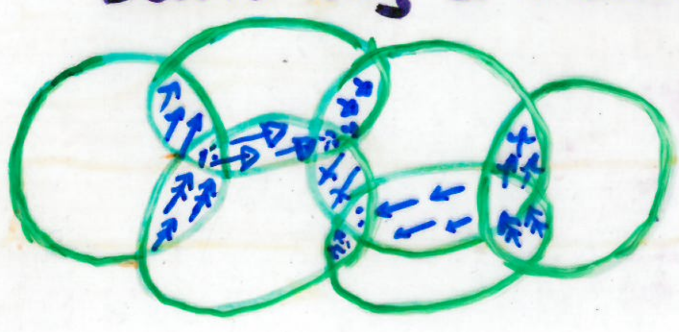
$f_{ij} + f_{jk} = f_{ik}$ on triple overlaps

$f_{ij} \equiv f_{ij} + h_i - h_j$ $\quad h_i$ def'd on $\mathcal{U}_i$ etc.

Branched contour $\oint$

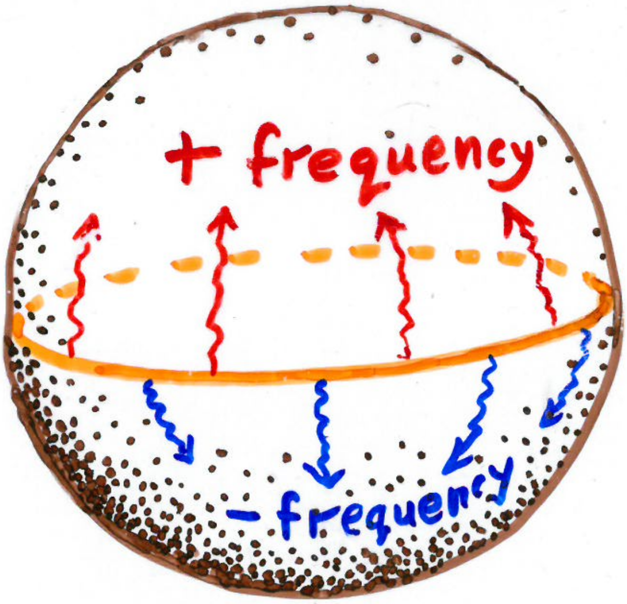


Non-linear (e.g. building a manifold,



Positive/Negative Frequency Splitting

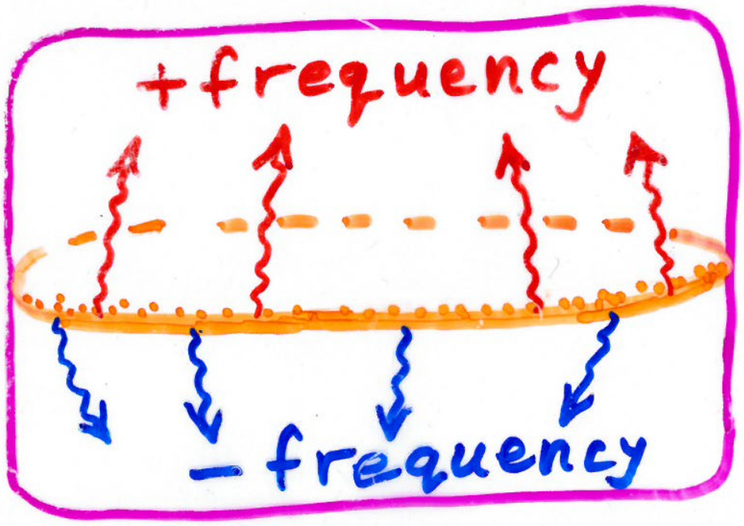
Riemann sphere $\mathbb{CP}^1$



Splitting of H^0s (ordinary functions) by holomorphic extension

real axis

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$



splitting of H^1s, by holomorphic extension

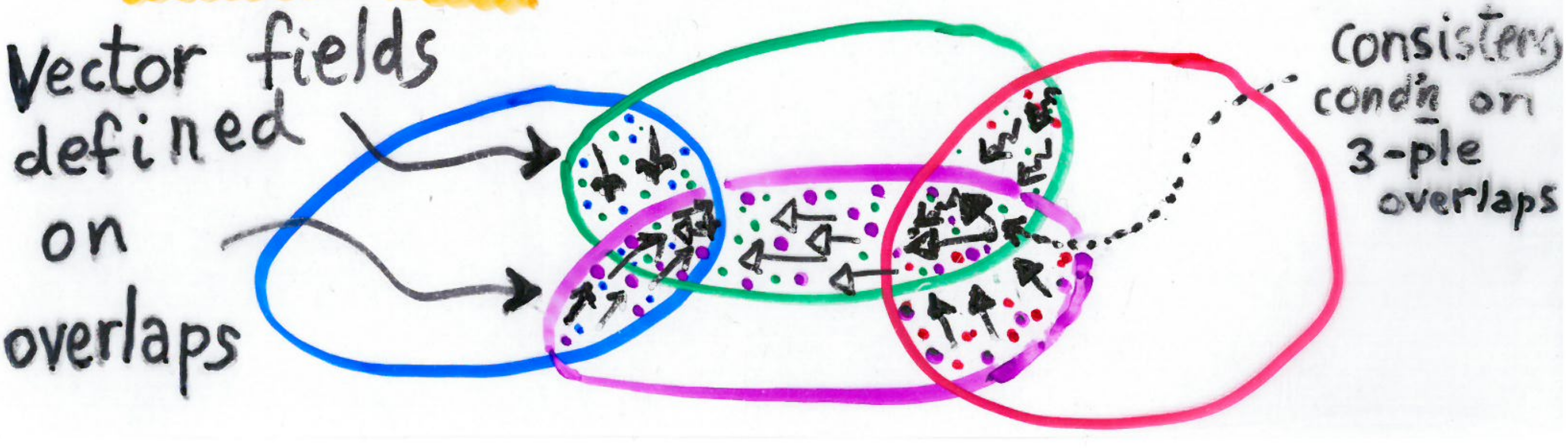
$\mathbb{PN}$ (null twistors)

This realized (finally!) one of the very early motivations behind twistor theory

"Non-linearization" of 1st (sheaf) Cohomology

• Construction of a bundle (Ward constr'n) (1977)

• Construction of a curved manifold (R.P. 1976)

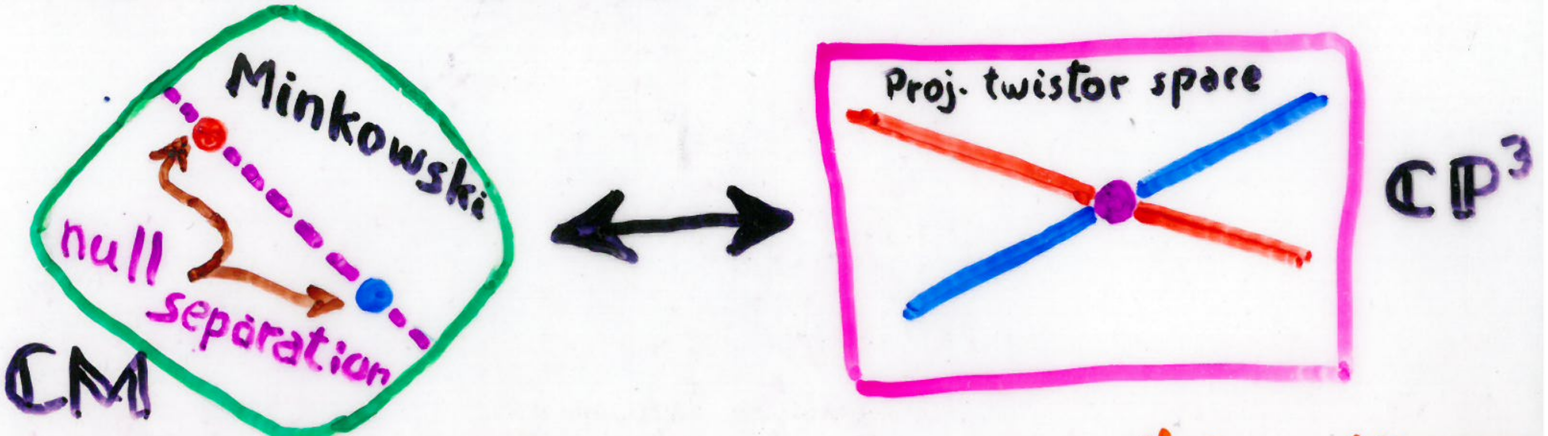


General Relativity

Numerous special applications
(e.g. Woodhouse-Mason: stationary axi-symm.)

As part of general programme:
"non-linear graviton construction" [R.P. '76]

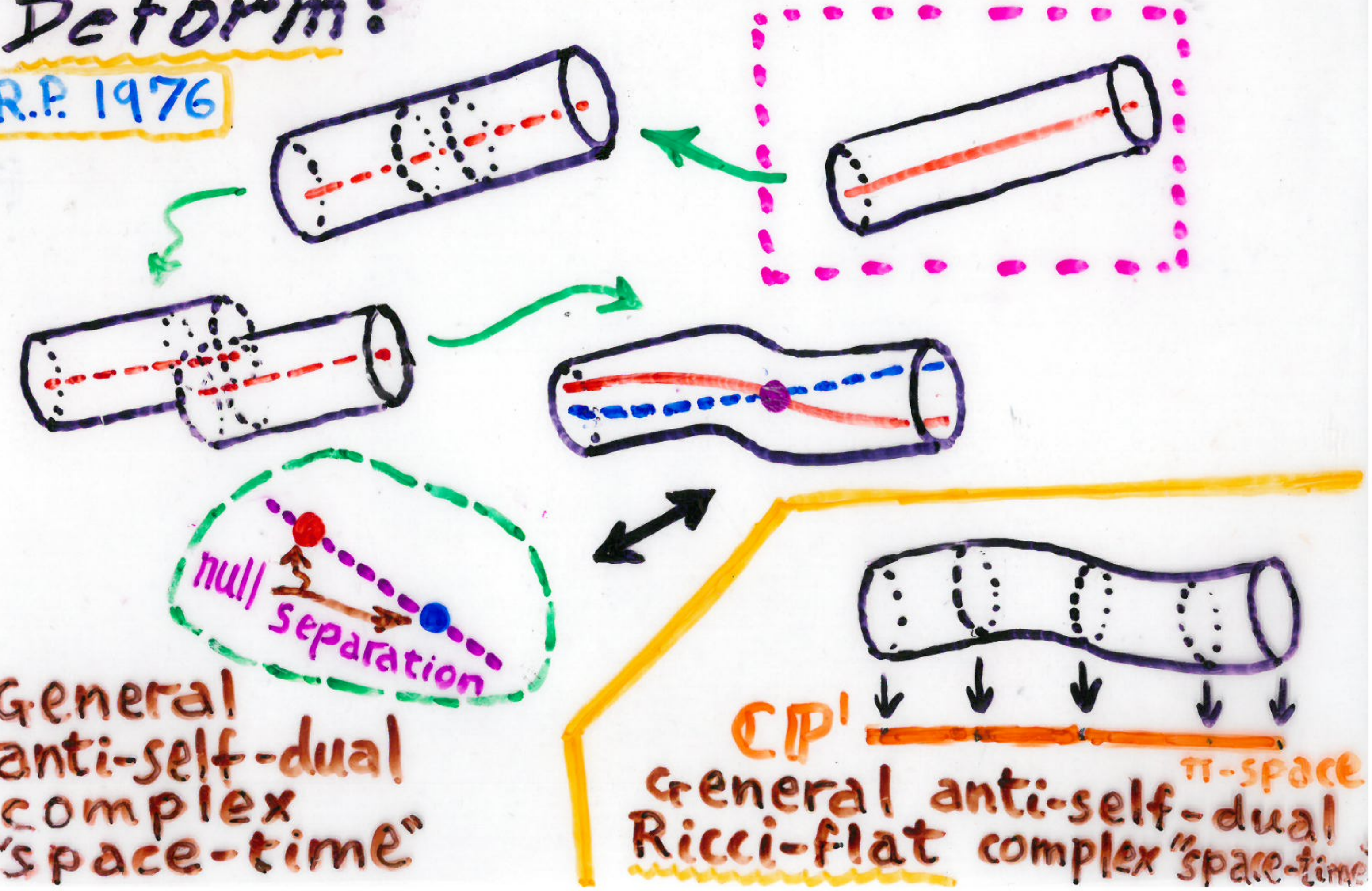
N.B. for flat space:



null separation ⟷ meeting lines

Deform:

R.P. 1976



Infinity twistors

Provide the space-time **metric** for Minkowski space $\mathbb{M}$ or (anti) de Sitter $\mathbb{D}$

$$I_{\alpha\beta} = \begin{pmatrix} \frac{\Lambda}{6}\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}, \quad I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \frac{\Lambda}{6}\varepsilon_{A'B'} \end{pmatrix}$$

Λ = cosmological constant > 0

$$I_{\alpha\beta} = \overline{I^{\alpha\beta}}, \quad I_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta} I^{\gamma\delta}$$ Skew

$$I^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta}, \quad I_{\alpha\beta} I^{\beta\gamma} = -\frac{\Lambda}{6}\delta^{\gamma}_{\alpha}$$

inverse if $\Lambda \neq 0$

Complex symplectic structure

$$\mathbb{I} = I_{\alpha\beta}\, dZ^{\alpha} \wedge dZ^{\beta}$$

Sympl. potential: $I_{\alpha\beta} Z^{\alpha} dZ^{\beta}$

degenerate if $\Lambda = 0$

Deformation to obtain a general left-handed (–helicity) ("leg break") non-linear graviton — from a homogeneity +2 fn.

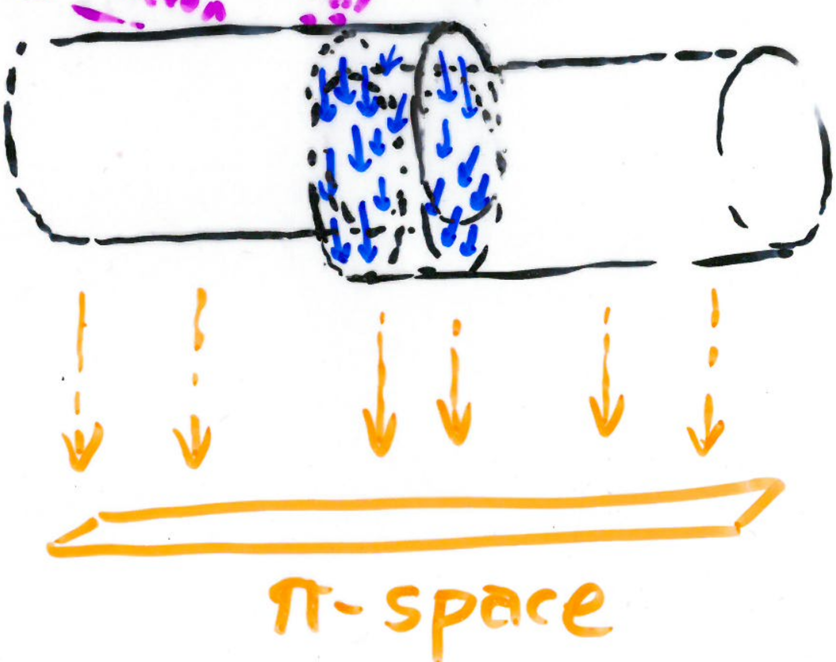
Vector field:

twistor function (1st cohomology)

This is for $\Lambda = 0$; but expression OK if $\Lambda \neq 0$

$$I^{\alpha\beta}\frac{\partial f_{+2}}{\partial Z^{\alpha}}\frac{\partial}{\partial Z^{\beta}} = \varepsilon^{AB}\frac{\partial f_{+2}(\omega,\pi)}{\partial \omega^{A}}\frac{\partial}{\partial \omega^{B}}$$

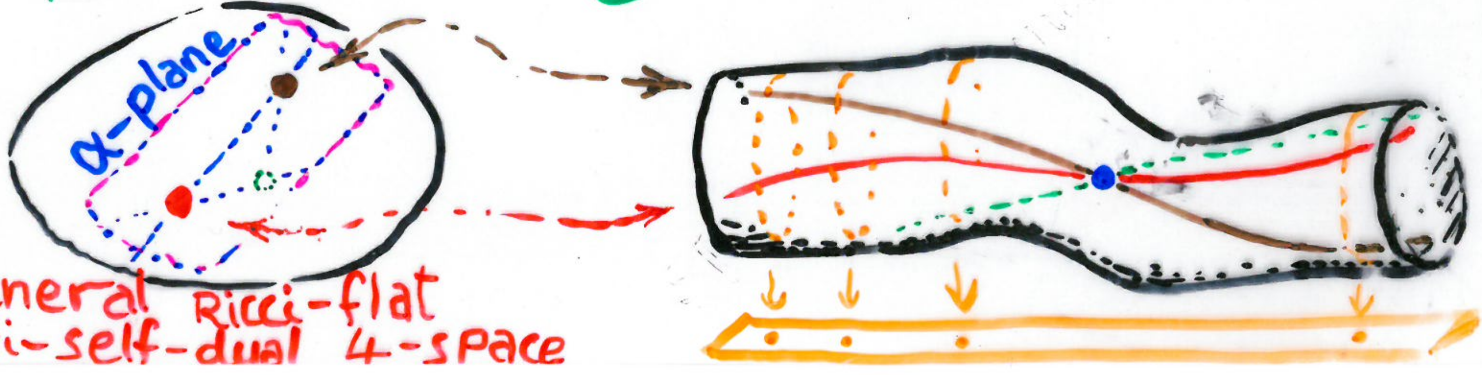
Vector field gives infinitesimal deformation



deformation preserves $\pi_{A'}$

∴ projection to π-space is preserved

"Exponentiate" to give finite deformation



General Ricci-flat anti-self-dual 4-space

Googly (graviton)

A cricket ball bowled so as to look as though it spins left-handed whereas it actually spins right-handed

Left-handed gravitons are described by anti-self-dual Weyl curvature, whereas right-handed gravitons are described by self-dual Weyl curvature.

Compare: photons, linear spin 2 massless quanta

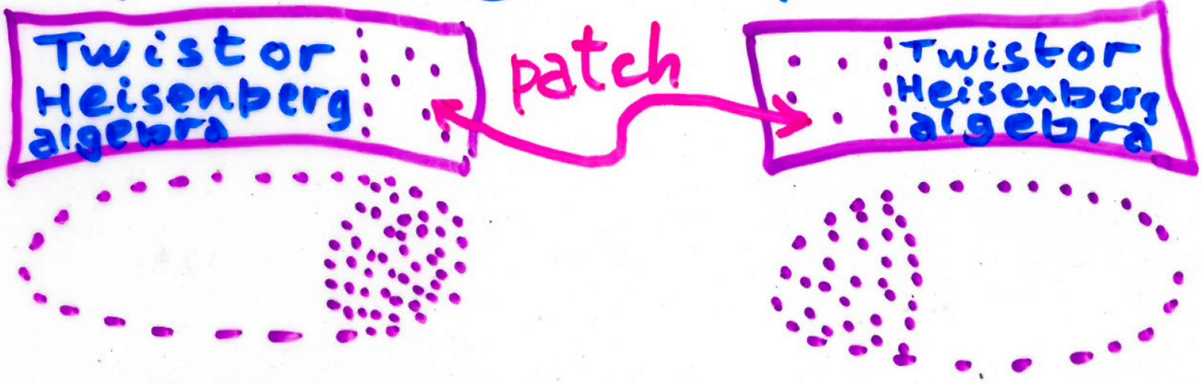
[positive-frequency wave function]

Twistors, with usual conventions, had seemed to describe anti-self-dual interactions so far!

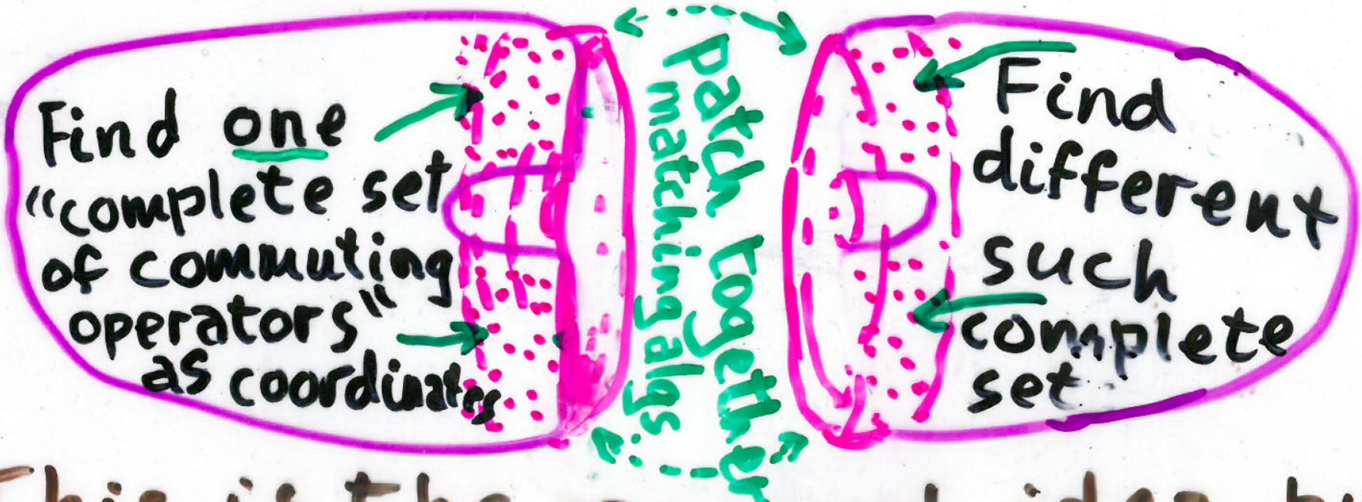
In twistor theory, holomorphicity plays a crucial role, and this is indeed retained through this twistor quantization procedure

$$\boxed{\bar{Z}_\alpha = -\partial/\partial Z^\alpha \,, \quad \partial/\partial \bar{Z}_\alpha = Z^\alpha} \quad \hbar = 1$$

Now, we try to patch:



The patching could proceed somewhat like this:



This is the general idea, but the patching could involve sublties

Real Symplectic & contact structures on twistor & null-geodesic 7-space $\mathcal{N}$

Choose local twistor coords. (or use local twistors). We can define a contact structure from the 1-form

$$\chi = i Z^\alpha d\bar{Z}_\alpha$$

and a symplectic structure from

$$\Sigma = d\chi = i\, dZ^\alpha \wedge d\bar{Z}_\alpha$$

In terms of ordinary x^a, p_a with

$$\omega^A = i x^{AA'} \pi_{A'}$$

and

$$Z^\alpha = (\omega^A, \pi_{A'})$$

we find

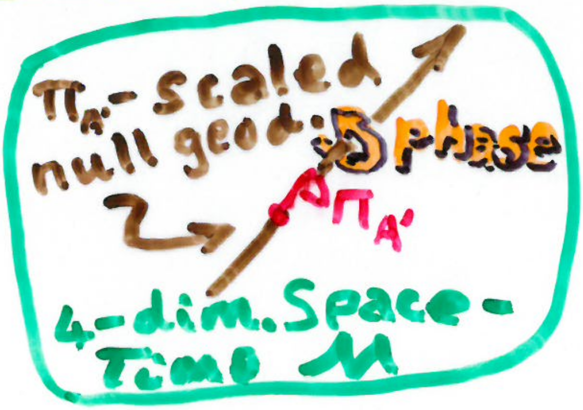
$$\chi = p_a dx^a$$

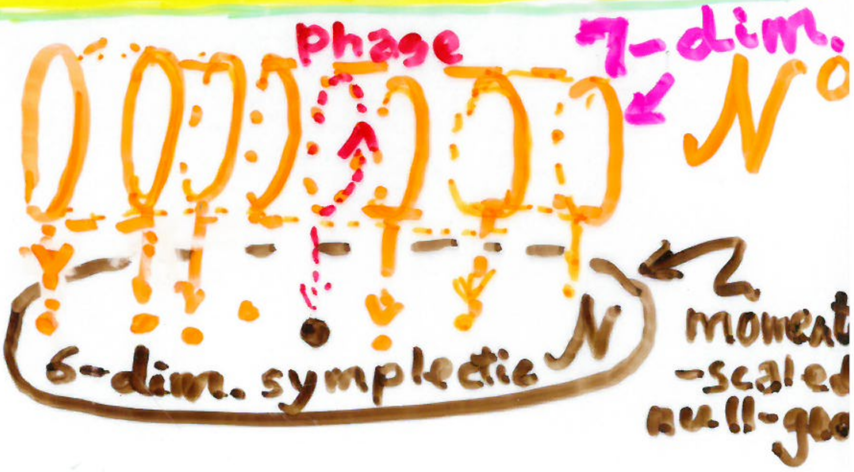
and

$$\Sigma = dp_a \wedge dx^a$$

standard forms o.k. for curved $\mathcal{N}$

Geometric Quantization of Light-Ray Space $\mathcal{N}$, for Curved Space Time $\mathcal{M}$



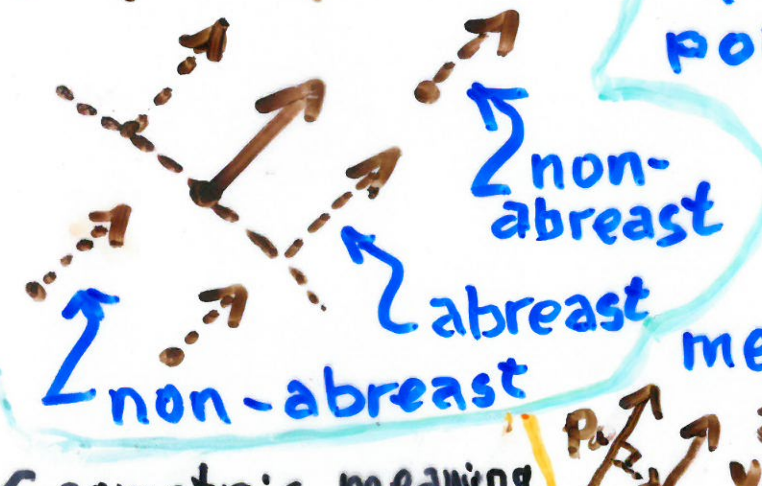


For geom. quant'n of $\mathcal{N}$, we need a circle-bundle $\mathcal{N}^{\circ}$ over $\mathcal{N}$. Here, the circles are already given, by phase (flag-plane) of $\pi_{A'}$

Geom. quant. supplies a connection $\tilde{\nabla}_a$ on $\mathcal{N}^{\circ}$, whose curvature is $(2\pi)^{-1}\times$ the natural Symplectic form:

$$\Sigma = dp_a \wedge dx^a = i\, dZ^{\alpha} \wedge d\bar{Z}_{\alpha} \quad \text{on } \mathcal{N}$$

3-space picture



The symplectic potential, $\Sigma = d\Phi$

$$\Phi = p_a dx^a = i Z^{\alpha} d\bar{Z}_{\alpha}$$
$$= -i \bar{Z}_{\alpha} dZ^{\alpha} \quad (Z^{\alpha}\bar{Z}_{\alpha}=0)$$

measures "abreastness"

abreast if connecting vector $\perp$ to p_a.

Geometric meaning of $\tilde{\nabla}$:

$$\tilde{\nabla}_{AA'}\xi^C = \nabla_{AA'}\xi^C + \Theta_{AA'B}{}^C \xi^B \quad \text{where } \Theta_{AA'B}{}^C = i\Pi_{AA'}\varepsilon_B{}^C + \Upsilon_{AA'B}\varepsilon_A{}^C$$

for conformal rescaling

Take $\Theta_{AA'B}{}^A = -\frac{1}{\hbar}\pi_{A'}\bar{\pi}_B$ canonical & conf. invar.

Local twistor transport

Local twistor: $(\omega^A, \pi_{A'})$

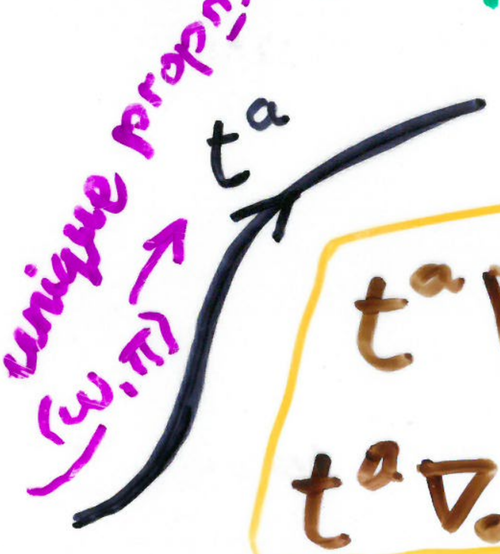
under conformal rescaling

$$g_{ab} \rightsquigarrow \Omega^2 g_{ab}, \quad \varepsilon_{AB} \rightsquigarrow \Omega\,\varepsilon_{AB}, \quad \varepsilon^{AB} \rightsquigarrow \Omega^{-1}\varepsilon^{AB}$$

$$\boxed{\omega^A \rightsquigarrow \omega^A, \quad \pi_{A'} \rightsquigarrow \pi_{A'} + i\Upsilon_{AA'}\omega^A}$$

where $\Upsilon_{AA'} = \Omega^{-1}\nabla_{AA'}\Omega$

$= \nabla_{AB'}\log\Omega$



$$\boxed{\begin{aligned} t^a\nabla_a\omega^B &= -i\,t^{BA'}\pi_{A'} \\ t^a\nabla_a\pi_{B'} &= -i\,t^{AA'}P_{AA'BB'}\,\omega^B \end{aligned}}$$

where $P_{ab} = \frac{1}{12}R\,g_{ab} - \frac{1}{2}R_{ab}$

Note $I_{\alpha\beta}$ is loc. tw. CONSTANT

iff Einstein's vacuum eqns.

$(R_{ab} = \Lambda g_{ab})$ hold [Λ in defn. $I_{\alpha\beta}$]

As things stand, palatial twistor theory could not be a proposal for a quantum gravity theory, if only for the reason that the Planck length/time does not appear in the formalism.

Crazy new proposal:

retain

$$Z^{\alpha}\bar{Z}_{\beta} - \bar{Z}_{\beta}Z^{\alpha} = \hbar\,\delta^{\alpha}_{\beta}$$

but now take

$$Z^{\alpha}Z^{\beta} - Z^{\beta}Z^{\alpha} = \varepsilon\, I^{\alpha\beta}$$

$$\bar{Z}_{\beta}\bar{Z}_{\alpha} - \bar{Z}_{\alpha}\bar{Z}_{\beta} = \bar{\varepsilon}\, I_{\alpha\beta}$$

where $\varepsilon\bar{\varepsilon}$ is extremely small (related to the Planck length), in comparison with Λ.

Cosmol. const.

Note: $\boxed{I_{\alpha\beta}I^{\beta\gamma} = -\frac{\Lambda}{6}\delta^{\gamma}_{\alpha}}$.