L^2-ESTIMATES FOR GEOMETRIC TRANSPORT EQUATIONS

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1. Motivation

Nonlinear stability of expanding spacetimes in the class of solutions to the Einstein-Vlasov system

$$(\mathrm{EV}) \begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}(\hat{f}) \\ \mathscr{X} \hat{f} = 0 \end{cases}$$

$\mathscr{X}$: Geodesic flow field on $T(\mathbb{R} \times M)$

$$T(\mathbb{R} \times M) \supset \mathscr{P} = \{(x,p) \in TM \,:\, -m^2 = |p|^2_{g_{\overline{M}}}, p^0 > 0\} = \mathsf{dom}(\hat{f})$$

$$T_{\mu\nu}(x) = \int_{\mathscr{P}_x} \hat{f} p_\mu p_\nu \mu_{\mathscr{P}_x}$$

Aim: Develop an **energy estimates approach** to control the energy-momentum tensor of Vlasov matter on spacetimes with compact spatial hypersurfaces without symmetries and with different future asymptotics.

(EV) models ensembles of self-gravitating collisionless matter.

Transport equation is regular on a fixed background.

Decay mechanism:

- Dispersion in the asymptotically flat setting.
  Cf. talk by *J. Smulevici* in November on *Vector field method for transport equations* (j. w. J. Joudioux & F.)
- In expanding spacetimes matter dilutes.
  Volume increases while the number of particles is conserved.

Results

- Static and Stationary solutions: Rein-Rendall, Rein, Andréasson, Andréasson-Kunze-Rein, Wolansky, Rein-Ramming, Andréasson-F.-Thaller
- Cauchy problem in AF case: Rein-Rendall, Dafermos, Hadžić-Rein, Taylor, Andréasson, Andréasson-Rein, Andréasson-Kunze-Rein
- Stability, Global existence, Asymptotics in the cosmological case: Andréasson, Andréasson-Ringström, Andréasson-Rendall-Weaver, Dafermos-Rendall, Lee, Weaver, Rein, Nungesser, Smulevici, Tchapnda, Noutchegueme, Tegankong
- [Ringström, '13] $\Lambda > 0$, no symmetry assumptions, massive particles, localized stability (+ local existence with L^2-norms)

Expanding spacetimes: $\bar{M} = I \times M$, M compact, $\partial M = \emptyset$

Lorentzian metric on $\bar{M}$, CMC foliated

$$g_{\bar{M}} = -N^2 dt \otimes dt + g_{ab}(dx^a + X^a dt) \otimes (dx^b + X^b dt)$$

with behavior

$$g \approx T(t)\mathbf{g}, \quad N \approx n, \quad X \approx 0,$$

where $\mathbf{g}$ is some fixed Riemannian metric on M and $T(t)$ is strictly monotonically increasing.

$$k = -\frac{1}{2N}\left[\partial_t g - \mathcal{L}_X g\right] \equiv \Sigma + \frac{\tau}{n} g_{ab},$$

where τ is the mean curvature. Here we assume $\Sigma \approx 0$.

This is compatible with

- Generalized de Sitter ($\Lambda > 0$): $T(t) = \sinh(t)^2$,
- Power law inflation (scalar field): $T(t) = t^{2p}$ for $p > 1$,
- Milne universe: $T(t) = t^2$.

Christoffel symbols:

$$
\begin{aligned}
{}^{(n+1)}\Gamma^a_{00} &= N\partial_0(X^a/N) - 2N/n\tau X^a + \mathsf{Q}^a \equiv \mathbf{Q}^a \\
2\,{}^{(n+1)}\Gamma^a_{(b0)} &= -2\frac{N}{n}\tau\delta^a_b + 2\mathbf{P}^a_b \\
{}^{(n+1)}\Gamma^a_{bc} &= \Gamma^a_{bc} + \mathbf{W}^a_{bc},
\end{aligned}
$$

where Q^a, $\mathbf{P}^a_b$, $\mathbf{W}^a_{bc}$ tensors on M, depending on Σ, N, X, g, perturbation terms.

Consider distribution functions

$$f : I \times TM \to \mathbb{R}_+$$

($TM \setminus \{0\}$ for massless particles) solutions to

$$
\begin{aligned}
\partial_t f = &-p^e/p^0\mathbf{A}_e f - 2\tau\frac{N}{n}p^e\mathbf{B}_e f + p^0(N\partial_0(X^e/N) - 2N\tau X^e)\mathbf{B}_e f \\
&+ p^0\mathbf{Q}^e\mathbf{B}_e f + p^u 2\mathbf{P}^e_u\mathbf{B}_e f + p^a p^b/p^0\mathbf{W}^e_{ab}\mathbf{B}_e f,
\end{aligned}
$$

with $\mathbf{A}_e = \partial_a - p^i\Gamma^b_{ie}\mathbf{B}_b$ and $\mathbf{B}_e = \partial_{p^e}$, where

$$p^0 = N^{-1}(1-|\hat{X}|^2_g)^{-1}\left(\hat{X}_j p^j + \sqrt{(\hat{X}_j p^j)^2 + (1-|\hat{X}|^2_g)(m^2 + |p|^2_g)}\right).$$

2. Sasaki metric

Construction of invariant Sobolev-type norms for functions on TM requires a Riemannian metric on TM.

The following construction bases on given metric on the base manifold M. We choose the rescaled metric g.

Consider the decomposition in horizontal and vertical parts

$$T_{(x,p)}(TM) = \mathbf{H}(T_{(x,p)}(TM)) \oplus \mathbf{V}(T_{(x,p)}(TM)).$$
$$\mathbf{H}(T_{(x,p)}(TM)) = span\{D_{x^a} := \partial/\partial x^a - p^k \Gamma^b_{ak} \partial/\partial p^b\}$$
$$\mathbf{V}(T_{(x,p)}(TM)) = span\{\partial/\partial p^a\}$$

Sasaki metric g on TM is defined by demanding orthogonality of horizontal and vertical vectors in a metrical sense.

$$\mathbf{g}(D_{x^a}, D_{x^b}) = \mathsf{g}_{ab}$$
$$\mathbf{g}(D_{x^a}, \partial/\partial p^b) = 0$$
$$\mathbf{g}(\partial/\partial p^a, \partial/\partial p^b) = \mathsf{g}_{ab}.$$

In connection coframe: $\mathbf{g} = \mathsf{g}_{ij} dx^i \otimes dx^j + \mathsf{g}_{ij} Dp^i \otimes Dp^j$. $Dp^i = dp^i + \Gamma^i_{jk} p^j dx^k$.

Volume form: $\mu_{TM} = |g| dx^1 \wedge \ldots \wedge dx^n \wedge dp^1 \wedge \ldots \wedge dp^n$; Covariant derivative: $\widehat{\nabla}$

3. L^2-energies

Weighted Sasaki metric:

$$\underline{\mathbf{g}} = \mathrm{g}_{ij} dx^i \otimes dx^j + (T(t)^{-2} + |p|^2_{\mathrm{g}})^{-1} \mathrm{g}_{ij} Dp^i \otimes Dp^j$$

Sasaki-Vlasov norm for distribution functions $f : TM \to [0, \infty)$.

$$\|f\|_{\mathsf{Vl},s} \equiv \sqrt{\sum_{k=0}^{s} \int_{TM} |\widehat{\nabla}^k f|^2_{\underline{\mathbf{g}}} \mu_{\mathbf{g}}}$$

Decay of the momentum-support in combination with uniform boundedness of these energies yields decay of Sobolev norms of the components of the energy-momentum tensor.

Next: Energy estimates for $\|f\|_{\mathsf{Vl},s}$.

Lemma. [Standard energy estimate]
Let $(\mathscr{M} = I \times M, g_{\mathscr{M}})$, where $I \subset \mathbb{R}$ is a connected open interval, be a smooth globally hyperbolic manifold such that $1 - |\hat{X}|_g^2 > 0$, $N > 0$ (...) holds, CMC foliated. Let f be a smooth solution to the transport equation on $[T_0, T_1) \times M$ with initial data f_0 at T_0 with $f_0 \in \mathsf{H}^s_{\mathrm{Vl}}(TM)$ for $n/2 + 1 < s \in \mathbb{Z}_+$. Then

$$\frac{d}{dt}\|f\|_{\mathsf{Vl},s} \leq \kappa(t)\|f\|_{\mathsf{Vl},s},$$

for $T_0 \leq t \in I$ with

$$\begin{aligned}\kappa(t) \leq C\Big[&|\tau|(\|\nabla N\|_{H^s} + \|N/n - 1\|_{H^{s+1}}) + T^{-1}(t)\|\Sigma\|_{H^{s-1}} \\ &+ \|X\|_{H^s} + T^{-1}(t)\|\mathcal{L}_X g\|_s + \frac{\mathsf{P}_{\infty,T}}{\mathsf{P}^0_{\infty,T}} + \mathsf{P}_{\infty,T} \cdot \|\mathbf{Rm}\|_{H^{s-1}} \\ &+ \mathfrak{p}_1\left(\frac{\mathsf{P}_{\infty,T}}{\mathsf{P}^0_{\infty,T}}\right) \cdot \mathfrak{q}_1\left(\|N\|_{H^{s+1}}, \|\hat{X}\|_{H^{s+1}}, \mathsf{P}_{\infty,T}, \mathsf{P}_{\infty,T} \cdot \|\mathbf{Rm}\|_{H^{s-1}}\right)(1 + \|\mathbf{Q}, \mathbf{W}\|_{H^s}) \\ &+ \mathfrak{p}_2(\mathsf{P}_{\infty,T}) \cdot \mathfrak{q}_2(\|\mathbf{Rm}\|_{H^{s-1}})\Big(\|\mathbf{P}\|_{H^s} + T(t)\|\mathbf{Q}\|_{H^s}\Big) \\ &+ \mathfrak{p}_3(\mathsf{P}_{\infty,T}) \cdot \mathfrak{q}_3(\|\mathbf{Rm}\|_{H^{s-1}})\|\mathbf{W}\|_{H^s}\Big],\end{aligned}$$

where C denotes a numerical constant, $\mathfrak{p}_i$ and $\mathfrak{q}_i$ denote polynomials, where $\mathfrak{p}_1$ and $\mathfrak{p}_3$ are at least first order in their arguments.

A special case:
In particular, this concerns the class of metrics of the form

$$g_{\bar{M}} = -dt \otimes dt + T(t)\mathsf{g}.$$

with g time-independent of constant, positive sectional curvature $K_{\mathsf{g}} = const. > 0$. We choose the weighted Sasaki metric adapted to this geometry by

$$\underline{\mathbf{g}} = \mathsf{g}_{ij} dx^i \otimes dx^j + \frac{\ell^{-1}}{|p|^2_{\mathsf{g}}} \mathsf{g}_{ij} Dp^i \otimes Dp^j,$$

on $TM \setminus \{0\}$, where

$$\ell = \frac{R(\mathsf{g})}{2n(n-1)}.$$

Let f be the solution to the massless transport equation. Then

$$\|f\|_{\mathsf{VI},1} \equiv \sqrt{\int f^2 + |\widehat{\nabla} f|^2_{\underline{\mathbf{g}}} \mu_{\mathbf{g}}}$$

is conserved.

4. Corrected energies

Model problem: transport equation in $1+1+1$-dimensions

distribution function: $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, sufficiently regular.

Model equation:

$$\partial_t f = a \partial_p f + \partial_t b \cdot \partial_p f,$$

where $a = a(t,x)$ and $b = b(t,x)$ sufficiently regular;

Decay properties:

$$\|\nabla b\|_\infty < \epsilon,$$

$$\int_1^\infty \|\nabla a\|_\infty \, dt < \infty,$$

$$\int_1^T \|\nabla \partial_t b\|_\infty \, dt \overset{T\to\infty}{\longrightarrow} \infty.$$

Aim: Show **uniform boundedness** of an energy of the form

$$E(f) = \iint |\nabla f|^2 + |\partial_p f|^2 dp dx.$$

Taking the time derivative, substituting by the equation, integration by parts yields

$$\partial_t E = 2 \iint \nabla f (\nabla a) \partial_p f dp dx + 2 \iint \nabla f \nabla (\partial_t b) \partial_p f dp dx.$$

In turn

$$|\partial_t E| \leq C(\|\nabla a\|_\infty + \|\nabla \partial_t b\|_\infty) E.$$

Model solution: Introduce correction terms.

Consider the correction term

$$C = 2 \iint \nabla f \nabla b \partial_p f dp dx.$$

Then as before

$$\partial_t C = \underbrace{2 \iint \nabla f \nabla (\partial_t b) \partial_p f dp dx}_{(*)} + 2 \iint (\nabla a) \partial_p f \nabla b \partial_p f dp dx$$
$$+ \underbrace{2 \iint \nabla (\partial_t b) \partial_p f \nabla b \partial_p f dp dx}_{(**)}.$$

$(*)$ neutralizes the first bad term in $\partial_t E$. $(**)$ is a **new bad term**.

Require a second correction term

$$D = \iint |\nabla b|^2 |\partial_p f|^2 dp dx.$$

Require a second correction term

$$D = \iint |\nabla b|^2 |\partial_p f|^2 dp dx,$$

which has the time derivative

$$\partial_t D = 2 \iint \nabla b \nabla \partial_t b |\partial_p f|^2 dp dx.$$

We then define the corrected energy by

$$E_{\mathsf{cor}} \equiv E - C + D.$$

Due to the smallness of $\|\nabla b\|_\infty$ we have the equivalency of energies

$$\frac{1}{C} E \leq E_{\mathsf{cor}} \leq C E$$

for some constant C and consequently the desired energy estimate

$$|\partial_t E_{\mathsf{cor}}| \lesssim C \|\nabla a\|_\infty E_{\mathsf{cor}}.$$

In the nonlinear setting:

The idea of corrections can be applied to **avoid the appearance of certain shift vector terms** in the energy estimates.

$$\partial_t f = -p^e/p^0 \mathbf{A}_e f - 2\tau \frac{N}{n} p^e \mathbf{B}_e f + \underbrace{p^0 (N\partial_0 (X^e/N) - 2N\tau X^e)\mathbf{B}_e f}_{(*)}$$
$$+ p^0 \mathbf{Q}^e \mathbf{B}_e f + p^u 2\mathbf{P}^e_u \mathbf{B}_e f + p^a p^b/p^0 \mathbf{W}^e_{ab} \mathbf{B}_e f,$$

These terms are problematic in the $2+1$ – dimensional case for $\Lambda = 0$ and **massive particles**.

The first corrected energy takes the form

$$\mathbf{E}^{(1)}_{\mathrm{Vl},\mathbf{c}}(f) \equiv \|f\|^2_{\mathsf{Vl},1} - \int \mathbf{A}^a f \nabla_a X^e \mathbf{B}_e f \mu_{TM} + \frac{1}{4} \int \nabla^a X^i \mathbf{B}_i f \nabla_a X^e \mathbf{B}_e f \mu_{TM}$$
$$+ \int (\langle p, X\rangle_\sigma + \tfrac{1}{4}|X|^2_\sigma)|\mathbf{B} f|^2_g \mu_{TM} \equiv \|f\|^2_{\mathsf{Vl},1} + \mathbf{C}^{(1)}_{\mathrm{Vl}}(f)$$

$$
\begin{aligned}
\mathbf{C}^{(2)}_{\mathrm{V1}}(f) \equiv & -\int_{TM} \widehat{\nabla}^a\widehat{\nabla}^b f[2\nabla_{(a}X^e\widehat{\nabla}_E\widehat{\nabla}_{b)}f + \nabla_a\nabla_b X^e\widehat{\nabla}_E f + \tfrac{1}{2}\mathbf{Rm}^e_{kab}X^k\widehat{\nabla}_E f]\mu_{TM} \\
& + \int_{TM}(\nabla^a\nabla^b X^e + \tfrac{1}{2}\mathbf{Rm}^{e\ \ ab}_{\ k}X^k)\widehat{\nabla}_E f[\nabla_{(b}X^i\widehat{\nabla}_I\widehat{\nabla}_{a)}f]\mu_{TM} \\
& + \tfrac{1}{2}\int_{TM}\widehat{\nabla}^a\widehat{\nabla}^b f[\nabla_a X^e\nabla_b X^i\widehat{\nabla}_I\widehat{\nabla}_E f]\mu_{TM} + \int_{TM}(\nabla^{(a}X^i\widehat{\nabla}_I\widehat{\nabla}^{b)}f)(\nabla_{(a}X^e\widehat{\nabla}_E\widehat{\nabla}_{b)}f)\mu_{TM} \\
& + \tfrac{1}{4}\int_{TM}(\nabla^a\nabla^b X^e + \tfrac{1}{2}\mathbf{Rm}^{e\ \ ab}_{\ k}X^k)\widehat{\nabla}_E f[\nabla_a\nabla_b X^i + \tfrac{1}{2}\mathbf{Rm}^i_{lab}X^l]\widehat{\nabla}_I f\mu_{TM} \\
& - \tfrac{1}{2}\int_{TM}(\nabla^{(a}X^i\widehat{\nabla}_I\widehat{\nabla}^{b)}f)(\nabla_{(a}X^e\nabla_{b)}X^l\widehat{\nabla}_L\widehat{\nabla}_E f)\mu_{TM} \\
& - \tfrac{1}{4}\int_{TM}(\nabla^a\nabla^b X^k + \tfrac{1}{2}\mathbf{Rm}^{k\ \ ab}_{\ l}X^l)\widehat{\nabla}_K f(\nabla_a X^e\nabla_b X^i\widehat{\nabla}_I\widehat{\nabla}_E f)\mu_{TM} \\
& + \tfrac{1}{16}\int_{TM}(\nabla^a X^i\nabla^b X^k\widehat{\nabla}_K\widehat{\nabla}_I f)(\nabla_a X^e\nabla_b X^l\widehat{\nabla}_L\widehat{\nabla}_E f)\mu_{TM} \\
& + \int_{TM}(\langle p, X\rangle + \tfrac{1}{4}|X|^2_\sigma)g^{au}g^{vb}\widehat{\nabla}_U\widehat{\nabla}_v f\widehat{\nabla}_A\widehat{\nabla}_b f\mu_{TM} \\
& - \int_{TM}(\omega^2 t^{-4} + |p|^2_\sigma + \langle X, p\rangle + \tfrac{1}{4}|X|^2_\sigma)\widehat{\nabla}^B\widehat{\nabla}^a f\nabla_a X^i\widehat{\nabla}_I\widehat{\nabla}_B f\mu_{TM} \\
& + \tfrac{1}{4}\int_{TM}(\omega^2 t^{-4} + |p|^2_\sigma + \langle X, p\rangle + \tfrac{1}{4}|X|^2_\sigma)\nabla_a X^e\nabla^a X^k\widehat{\nabla}_E\widehat{\nabla}_B f\widehat{\nabla}_K\widehat{\nabla}^B f\mu_{TM} \\
& + \int_{TM}(\langle X, p\rangle + \tfrac{1}{4}|X|^2_\sigma)g^{au}g^{vb}\widehat{\nabla}_u\widehat{\nabla}_V f\widehat{\nabla}_a\widehat{\nabla}_B f\mu_{TM} \\
& - \int_{TM}(\omega^2 t^{-4} + |p|^2_\sigma + \langle X, p\rangle + \tfrac{1}{4}|X|^2_\sigma)\widehat{\nabla}^a\widehat{\nabla}^B f(\nabla_a X^e)\nabla_E\nabla_B f\mu_{TM} \\
& + \tfrac{1}{4}\int_{TM}(\omega^2 t^{-4} + |p|^2_\sigma + \langle X, p\rangle + \tfrac{1}{4}|X|^2_\sigma)\nabla_a X^e\nabla^a X^k\widehat{\nabla}_E\widehat{\nabla}_B f\widehat{\nabla}_K\widehat{\nabla}^B f\mu_{TM} \\
& + \int_{TM}(|p + \tfrac{1}{2}X|^4_\sigma - |p|^4_\sigma)g^{au}g^{bv}\left({}^{\mathbf{g}}\nabla_U{}^{\mathbf{g}}\nabla_V f\right)\left({}^{\mathbf{g}}\nabla_A{}^{\mathbf{g}}\nabla_B f\right)\mu_{TM}.
\end{aligned}
$$

Lemma

Let the shift vector field be such that

$$\|X\|_{H^s} \lesssim \varepsilon t^{-2}.$$

Then for every $n/2 < s \in \mathbb{Z}_+$ there exists a correction term

$$\mathbf{C}_{\mathsf{VI},s} = \mathbf{C}_{\mathsf{VI},s}(X, g, f),$$

such that

$$\|f\|^2_{\mathsf{VI},s} \lesssim \|f\|^2_{\mathsf{VI},s} + \mathbf{C}_{\mathsf{VI},s} \lesssim \|f\|^2_{\mathsf{VI},s}$$

and an estimate of the form

$$\left|\partial_t \mathbf{E}^{(s)}_{\mathsf{VI,c}}\right| \lesssim \kappa(t) \cdot \mathbf{E}^{(s)}_{\mathsf{VI,c}}$$

holds without the corresponding shift vector terms in the first order terms of κ.

5. Nonlinear Stability I, $\Lambda > 0$

(j. w. K. Kröncke, L. Andersson)

Nonlinear Stability for the massless Einstein-Vlasov system with positive Λ. [complementary to the massive case of [Ringström,13]]

Background geometries: $\Lambda = \frac{n(n-1)}{2}$

Class A: negative curvature

(M, γ) is an n-dimensional compact Riemannian Einstein manifold with $Ric(\gamma) = -n(n-1)\gamma$.

$$g = -dt^2 + \sinh(t)^2 \gamma$$

on $(0, \infty) \times M$.

Class B: positive curvature

(M, γ) is an n-dimensional compact Riemannian Einstein manifold with $Ric(\gamma) = n(n-1)\gamma$.

$$g = -dt^2 + \cosh(t)^2 \gamma$$

on $\mathbb{R} \times M$.

Results for the vacuum Einstein flow, $\Lambda > 0$:

Theorem 1 (F.-Kröncke, 2015)
Let M be a closed n-dim. mnfld ($n \geq 2$) and γ an Einstein metric satisfying $\mathrm{Ric}(\gamma) = -(n-1)\gamma$. Then for $s > n/2 + 2$, $s' > n/2 + s$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ s.t. for initial data (g_0, k_0) satisfying

$$\|g_0 - \gamma\|_{H^{s'}} + \left\|k_0 + \sqrt{2}\gamma\right\|_{H^{s'-1}} < \delta$$

its maximal globally hyperbolic development under the Einstein flow admits CMC-foliations $\{M_t\}$, such that the induced metrics g_t satisfy

$$\left\|\sinh^{-2}(t) g_t - \gamma\right\|_{H^s} < \varepsilon.$$

All corresponding homogeneous solutions are **orbitally stable** and future developments of small perturbations are future geodesically complete.

Theorem 2 (F.-Kröncke, 2015)
Let M be a closed n-dim. mfld. ($n \geq 2$) and γ an Einstein metric with $\mathrm{Ric}(\gamma) = (n-1)\gamma$ which does not admit Killing vector fields and such that $-2(n-1)$ is not an eigenvalue of the Laplacian. Then for $s > n/2+2$, $s' > n/2+s$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ s.t. for initial data (g_0, k_0) satisfying

$$\|g_0 - \gamma\|_{H^{s'}} + \|k_0\|_{H^{s'-1}} < \delta$$

its maximal globally hyperbolic development under the Einstein flow admits a CMC-foliation $\{M_t\}_{t\in\mathbb{R}}$ such that the induced metrics g_t satisfy

$$\left\|\cosh^{-2}(t) g_t - \gamma\right\|_{H^s} < \varepsilon.$$

All corresponding homogeneous solutions are **orbitally stable** and the developments of small perturbations are complete.

Remark. The stability can be deduced from "localized stability result" in [Ringström, 08] (also in higher dimensions).

New: Approach via CMC-foliations (inspired by [Andersson-Moncrief,'11]) provides a substantially simplified and concise proof.

Foliation by compact, spacelike hypersurfaces M_t, $I \subset \mathbb{R}$ compact interval.

$$\overline{M} = I \times M$$

ADM-Ansatz

$$^{(n+1)}g = -N^2 dt \otimes dt + g_{ab}(dx^a + X^a dt) \otimes (dx^b + X^b dt)$$

Lapse function N, shift vector field X, induced Riemannian metric g on Σ.

Second fundamental form k, Mean curvature $\tau = tr_g k$, $k = \Sigma + \frac{\tau}{n} g$.

Constant mean curvature - spatial harmonic - Gauge

$$\partial_a \tau = 0$$
$$V^k = g^{ij}(\Gamma^k_{ij} - \widehat{\Gamma}^k_{ij}) = 0$$

Elliptic - Hyperbolic system

$$\partial_t g_{ij} = -2N(\Sigma_{ij} + \tau/n g_{ij}) + \mathscr{L}_X g_{ij}$$

$$\partial_t \Sigma_{ij} = N(R_{ij} + \tau\Sigma_{ij} - 2\Sigma_{ik}\Sigma^k_j + (\tau^2/n - n)g_{ij})$$

$$+ \mathcal{L}_X \Sigma_{ij} - \frac{1}{n} g_{ij} - \frac{2N\tau}{n}\Sigma_{ij} - \nabla_i \nabla_j N$$

$$\Delta N = -1 + N\left[|\Sigma|^2_g + \frac{\tau^2}{n} - n\right]$$

$$\Delta X^i + R^i_m X^m - \mathcal{L}_X V^i = 2\nabla_j N k^{ji} - \nabla^i N tr_g k + 2N\nabla_j k^{ij}$$

$$-(2Nk^{mn} - (\mathcal{L}_X g)^{mn})(\Gamma^i_{mn} - \hat{\Gamma}^i_{mn})$$

Sketch of Proof.

1. Universality of the CMC-gauge:
Every development of a small perturbation of the background geometries contains a CMC surface.
Implicit function theorem argument.

2. Rescaling of the equations.

$$g = s(\tau)\tilde{g},\ N = s(\tau)\tilde{N},$$
$$\Sigma = s(\tau)^{1/2}\tilde{\Sigma},\ X = s(\tau)^{1/2}\tilde{X}$$

for $s(\tau) = (\frac{\tau}{n})^2 - 1$ and define a new time function T by

$$\tau = -n\frac{\cosh(T)}{\sinh(T)}.$$

3. Uniform energy estimate.

Rescaled evolution equations:

$$
\begin{aligned}
\partial_T g_{ij} &= -\frac{n}{\sinh(T)} 2N\Sigma_{ij} + \frac{\cosh(T)}{\sinh(T)} A + \frac{n}{\sinh(T)} B, \\
\partial_T \Sigma_{ij} &= -\frac{\cosh(T)}{\sinh(T)} (n-1)\Sigma_{ij} \\
&\quad + \frac{n}{\sinh(T)} N\left(-\frac{1}{2}\Delta_{g,\gamma}(g-\gamma) - \mathring{R}_{\gamma}(g-\gamma)\right) \\
&\quad + \frac{1}{\sinh(T)} (\mathcal{L}_X \Sigma_{ij}) + \frac{n}{\sinh(T)} C + \frac{\cosh(T)}{\sinh(T)} D.
\end{aligned}
$$

Constraining the energy estimate:

- Not all terms have an exponentially decaying coefficient.
- Decay inducing term only for Σ.
- Laplacian does not allow for an isolated estimate of Σ due to regularity problems.

Resolution of the energy estimate:

- Give up on decay in the first step.
- The negative Σ-term
$$-(n-1)\sum_{k\leq s}\int|\nabla^k\Sigma|^2dx$$
can be used to absorb terms which are higher order in Σ.
- Main observation: **All** terms without an explicitly decaying time factor e^{-T} **are higher order in** Σ.

Conclusion: An estimate where "bad" terms disappear holds for small data.

Energy:

$$\mathbf{E}_s(g,\Sigma) \equiv \|g-\gamma\|^2_{L^2(\gamma)} + \sum_{k=0}^{s-1} (-1)^k (\Sigma, \Delta^k_{g,\gamma}\Sigma)_{L^2(g,\gamma)} + \frac{1}{2}\sum_{k=1}^{s} (-1)^k (g-\gamma, \Delta^k_{g,\gamma}(g-\gamma))_{L^2(g,\gamma)}$$

$$\text{Generalized Laplacian:} \quad \Delta_{g,\gamma} h_{ij} = \frac{1}{\mu_g}\nabla[\gamma]_m (g^{mn}\mu_g \nabla[\gamma]_n h_{ij}),$$

$$L^2\text{-norm:} \quad \langle u, v\rangle \equiv u_{ij}v_{kl}\gamma^{ik}\gamma^{jl}; \quad (u,v)_{L^2(g,\gamma)} \equiv \int_M \langle u, v\rangle \mu_g.$$

Lemma

Let $s > n/2+1$ and $(g,\Sigma) \in H^s \times H^{s-1}$ be a solution to the evolution equations. Then there exists an $\varepsilon > 0$ such that for

$$(g,\Sigma) \in \mathcal{B}^s_\varepsilon(\gamma) \times \mathcal{B}^{s-1}_\varepsilon(0),$$

the estimate

$$\left|\partial_{\mathrm{T}} \mathbf{E}_s(g,\Sigma)\right| \leq \frac{C(\varepsilon)}{\sinh(\mathrm{T})} \mathbf{E}_s(g,\Sigma)$$

holds in the case of negative curvature of γ. The analogous estimate with $\sinh(\mathrm{T})$ replaced by $\cosh(\mathrm{T})$ holds in the case of positive curvature of γ.

Energy estimate yields global existence.

Problem:
The established boundedness does not imply completeness of the resulting solution.

- Boundedness of the metric in highest regularity is given by previous estimate.
- May apply estimate for Σ individually, in one order regularity lower.
- This yields decay for Σ.
- Elliptic equations improve the regularity in which gradient of the lapse and shift are decaying.

4. Lower order estimate for energy of Σ, ($\mathbf{H}_{s-2}$) yields decay for the perturbations. If $n > 2$,

$$\sqrt{\mathbf{H}_{s-2}(\Sigma)} \leq \frac{C\sqrt{\varepsilon}}{\sinh(T)}.$$

If $n = 2$, the estimate is

$$\sqrt{\mathbf{H}_{s-2}(\Sigma)} \leq \frac{C\sqrt{\varepsilon}}{\sinh^{1/2}(T)}.$$

5. Decay is sufficient to deduce completeness. $\square$

Consider now the **massless Einstein-Vlasov system**.

Theorem 3 (Andersson-F.-Kröncke, 2015)
Spacetimes in the class A are future nonlinearly stable under the massless Einstein-Vlasov flow. The future developments of small perturbations are future geodesically complete and admit CMC-foliations.

Theorem 4 (Andersson-F.-Kröncke, 2015)
Spacetimes in the class B (excluding certain symmetries) are future- and past non-linearly stable under the massless Einstein-Vlasov flow. The future developments and past developments of small perturbations are future- and past geodesically complete, respectively and admit CMC-foliations.

Elliptic - Hyperbolic system (non-vacuum case, geometry rescaled)

$$
\begin{aligned}
\Delta_g N &= -1 + N(|\Sigma|_g^2 + n) + \tilde{N}\tilde{\eta} \\
\Delta_g X^i + R^i_m X^m &= 2\nabla_j N \Sigma^{ij} + 2\sinh(T)\tilde{N}\tilde{\jmath}^i \\
&\quad - \cosh(T)(2-n)\nabla^i N - (2N\Sigma^{mn} - (\mathcal{L}_X g)^{mn})(\Gamma^i_{mn} - \hat{\Gamma}^i_{mn}) \\
\partial_T g_{ij} &= -2\frac{\cosh(T)}{\sinh(T)}(1-nN)g_{ij} - n\sinh(T)^{-1}(2N\Sigma_{ij} - \mathcal{L}_X g_{ij}) \\
\partial_T \Sigma_{ij} &= -n^2\frac{\cosh(T)}{\sinh(T)}(\frac{1}{n^2} + N - \frac{2N}{n})\Sigma_{ab} \\
&\quad + n\sinh(T)^{-1}N(R_{ij} + ng_{ab} - 2\Sigma_{ik}\Sigma^k_j) \\
&\quad + n\sinh(T)^{-1}(\mathcal{L}_X\Sigma_{ij} - \frac{1}{n}g_{ij} - \nabla_i\nabla_j N) + 2n\sinh(T)^{-3}\tilde{N}\tilde{T}_{ij},
\end{aligned}
$$

$$
\begin{aligned}
p^0 &= N^{-1}(1-|\hat{X}|_g^2)^{-1}\left[\hat{X}_j p^j + \sqrt{(\hat{X}_j p^j)^2 + (1-|\hat{X}|_g^2)|p|_g^2}\right], \\
\hat{p} &= \left[\hat{X}_j p^j + \sqrt{(\hat{X}_j p^j)^2 + (1-|\hat{X}|_g^2)|p|_g^2}\right] \\
\rho &= 2N^2\int f\frac{(p^0)^2}{\hat{p}}\sqrt{g}dp, \quad \eta = (2-\tfrac{n}{n-1})\rho + \tfrac{1}{n-1}\int f\frac{|p+p^0X|_g^2}{\hat{p}}\sqrt{g}dp \\
\jmath_a(f) &= N\int f\frac{p_a p^0}{\hat{p}}\sqrt{g}dp, \quad \hat{T}_{ab} = \sqrt{g}\int f p_a p_b \hat{p}^{-1}dp
\end{aligned}
$$

Rescaling the matter via

$$f = \sinh(T)^n \tilde{f}, \quad p^a = \sinh(T)^2 \tilde{p}^a$$

yields

$$\partial_T f = -Nn\sinh(T)^{-1}\frac{p^e}{\bar{p}}\mathbf{A}_e f + \frac{\bar{p}}{N}\Big[\frac{1}{n}\big(\frac{\cosh(T)}{\sinh(T)}X^a + \partial_T X^a\big)\Big]\mathbf{B}_a f$$
$$+ \frac{\bar{p}}{N}F_1^a\mathbf{B}_a f + p^u F_{2,u}^a\mathbf{B}_a f + p^a p^b/\bar{p}F_{3,ab}^c\mathbf{B}_c f + n\frac{\cosh(T)}{\sinh(T)}f - 2N\tau p^u\mathbf{B}_u f$$

and

$$\begin{aligned} \|\tilde{N}\tilde{\eta}\|_{H^{s-1}} &\leq C\mathbf{E}_{\mathsf{VI},s-1}(f)\cdot\sinh(T)^{-(n-1)}, \\ s(\tau)^{-1/2}\|2\tilde{N}\tilde{\jmath}\|_{H^{s-1}} &\leq C\mathbf{E}_{\mathsf{VI},s-1}(f)\cdot\sinh(T)^{-(n-1)}, \\ 2ns(\tau)^{3/2}\|\tilde{N}\tilde{T}_{ab}\|_{H^{s-1}} &\leq C\mathbf{E}_{\mathsf{VI},s-1}(f)\cdot\sinh(T)^{-n}, \end{aligned}$$

Observation:

- The coefficients on the RHS of the transport equations (their H_{s-1}-norms) are integrable in time, given the vacuum asymptotic of the geometry.
- The components of the energy-momentum tensor (in H_{s-1}) decay fast (compared to the leading terms in the vacuum problem.)

Proof of Theorems 3 & 4.

- Bootstrap the energy-momentum tensor by assuming

$$\mathbf{E}_{\mathsf{VI},s-1}(T) \leq C$$

same for the support.

- Under these assumptions the matter terms in the elliptic-hyperbolic system only appear as perturbation terms of fast decay.
- Same energy estimates as in the vacuum case can be achieved for $g - \gamma$ and Σ.
- Using the standard energy estimates (without correction) in combination with smallness of the initial data yield

$$\mathbf{E}_{\mathsf{VI},s-1}(T) \leq \varepsilon.$$

□

6. Nonlinear Stability II, $\Lambda = 0$

Let M be a closed surface of genus $\mathsf{gen}(M) \geq 2$, with σ^0 a fixed Riemannian metric of scalar curvature $R(\sigma^0) \equiv -1$.

We consider the vacuum solution

$$g_B = -4dt \otimes dt + 2t^2 \sigma^0_{ab} dx^a \otimes dx^b.$$

Vacuum stability (Andersson, Moncrief, Tromba '96)
Einstein+scalar field (from $U(1)$-symmetry) (Choquet-Bruhat, Moncrief '01)

Theorem 5 (F.'14)
g_B is future nonlinearly stable under the massive Einstein-Vlasov flow for perturbations with initially compact momentum support of the distribution function. The future developments of small perturbations are future geodesically complete.

The hardest problem for the proof are shift vector terms appearing in the Vlasov equation, which have insufficient decay.

These can be neutralized using the corrected energies.

Other topologies in $n = 2$

- Remaining closed surfaces are S^2 and T^2.
- In vacuum: no solutions with spatial S^2 topology ($\Lambda = 0$)
- For a scalar field initial data on S^2 exists (Choquet-Bruhat, Moncrief)
- On T^2 the conformal geometry degenerates asymptotically towards the future. Stability not understood.

Observation: The slow decay of the Vlasov energy-momentum tensor in 2+1 dimensions essentially changes the global behavior of the Einstein flow.

For "enough" matter any topology allows for future complete solutions.

Consider a vacuum solution (λ, Σ, N, X) to the Einstein equations on $[t_0, \infty) \times S^2$. The momentum constraint

$$D_b \Sigma_a^b = 0$$

implies that Σ is **transverse**, therefore a **TT-tensor**.

On S^2 this implies $\Sigma = 0$. The Hamiltonian constraint reads

$$2\Delta_\sigma \lambda = e^{2\lambda}\tau^2/2 + 1,$$

which by integration yields a contradiction. Therefore: **There is no solution of the vacuum constraint equations in CMC gauge on S^2.**

The Hamiltonian constraint is solvable with a non-vanishing energy density.

Consider the Hamiltonian constraint of the Einstein-Vlasov equations.

$$2\Delta_\sigma \lambda = e^{2\lambda}\tau^2/2 + 1 - e^{2\lambda}\rho - e^{-2\lambda}|\Sigma|_\sigma^2 \qquad \text{(HC)}$$

We define the *asymptotic mass* of the system by

$$\mathbf{m}_\infty \equiv \int_{TM} f \mu_{TM}.$$

The rescaled energy density for Vlasov matter is of the form

$$e^{2\lambda}\rho = \frac{\mathbf{m}_\infty}{\mathrm{vol}(S^2)} + \tilde{\rho},$$

where $\tilde{\rho}$ is a perturbation term.

This allows for solutions of (HC) if we set

$$\frac{\mathbf{m}_\infty}{\mathrm{vol}(S^2)} > 1.$$

Main Observation:

The *asymptotic mass* of Vlasov matter in 2+1 dimensions acts as a negative correction to the **scalar curvature** of the conformal metric - independent of the topology.

This presumably **avoids recollapse** of the corresponding space times.

We have the following explicit solutions using an ansatz $f = f(t, |p|_g)$.

Corresponding explicit solutions are of the form

$$g_{\mathrm{itr}} = -\left(\frac{2\tau^2}{\tau^2 + 2\eta}\right)^2 dt^2 + \frac{2\tau^2}{4\rho(f) - \tau^2} \cdot t^2 \sigma_{ab} dx^a \otimes dx^b.$$

Asymptotically,

$$\lim_{t \to \infty} \frac{2\tau^2}{\tau^2 + 2\eta} = 2$$

$$\lim_{t \to \infty} \frac{2\tau^2}{4\rho(f) - \tau^2} = 2\left[\frac{\mathbf{m}_\infty}{\mathrm{vol}(S^2)} - 1\right].$$

These are future complete solutions on $[T_0, \infty) \times S^2$ with σ the standard metric on S^2.

These models are counterexamples to a 2+1-dimensional version of the recollapse conjecture.

Theorem (F. '14/'15)

Every of the previous solutions to the Einstein-Vlasov system on $[T_0, \infty) \times S^2$ is future nonlinearly stable. Developments of the perturbed initial data are future complete.

The same holds for corresponding solutions on T^2.

Corollary

Let M be a closed surface. Then there exists initial data to the Einstein-Vlasov flow on M such that the future development of this initial data is geodesically complete and the space-time is asymptotically of the form

$$g_\infty = -4dt^2 + \mathbf{c} \cdot t^2 \sigma_{ab} dx^a dx^b,$$

where σ is a metric of constant scalar curvature on M and $\mathbf{c} = \mathbf{c}(\chi(M), \mathbf{m}_\infty(M))$ is a positive constant depending only on the genus of M and its total mass. These solutions are future stable.

Thank You.