Remarks on the nonlinear stability of Schwarzschild

Jérémie Szeftel

Laboratoire Jacques-Louis Lions,
Université Pierre et Marie Curie

(Joint work with Sergiu Klainerman)

Kerr stability conjecture

Let $(\mathcal{M}^{1+3}, \mathbf{g})$ Lorentzian, and $\mathbf{Ric}(\mathbf{g})$ its Ricci tensor

Einstein vacuum equations: $\mathbf{Ric}(\mathbf{g}) = 0$

The Einstein vacuum equations form an evolution problem

This talk focuses on the stability of the Kerr metrics which form a family of stationary solutions to the Einstein vacuum equations

Conjecture (first version) The Kerr family is stable as a solution of the Einstein vacuum equation under small perturbations of its initial data

Stability of Minkowski

The simplest member of the Kerr family is the Minkowski space-time

Minkowski: $(\mathbb{R}^{1+3}, \mathbf{m})$, $\mathbf{m} = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

The stability of the Minkowski space-time has been obtain by Christodoulou-Klainerman 93' (see also Klainerman-Nicolo 03', Lindblad-Rodnianski 05', Bieri 09')

So far, this is the only full stability result for a solution of the Einstein vacuum equations

The Kerr and the Schwarzschild solutions

Kerr metric given in Boyer-Lindquist (t, r, θ, φ) coordinates by

$$\mathbf{g}_{a,m} = -\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2 (\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2 (d\theta)^2$$

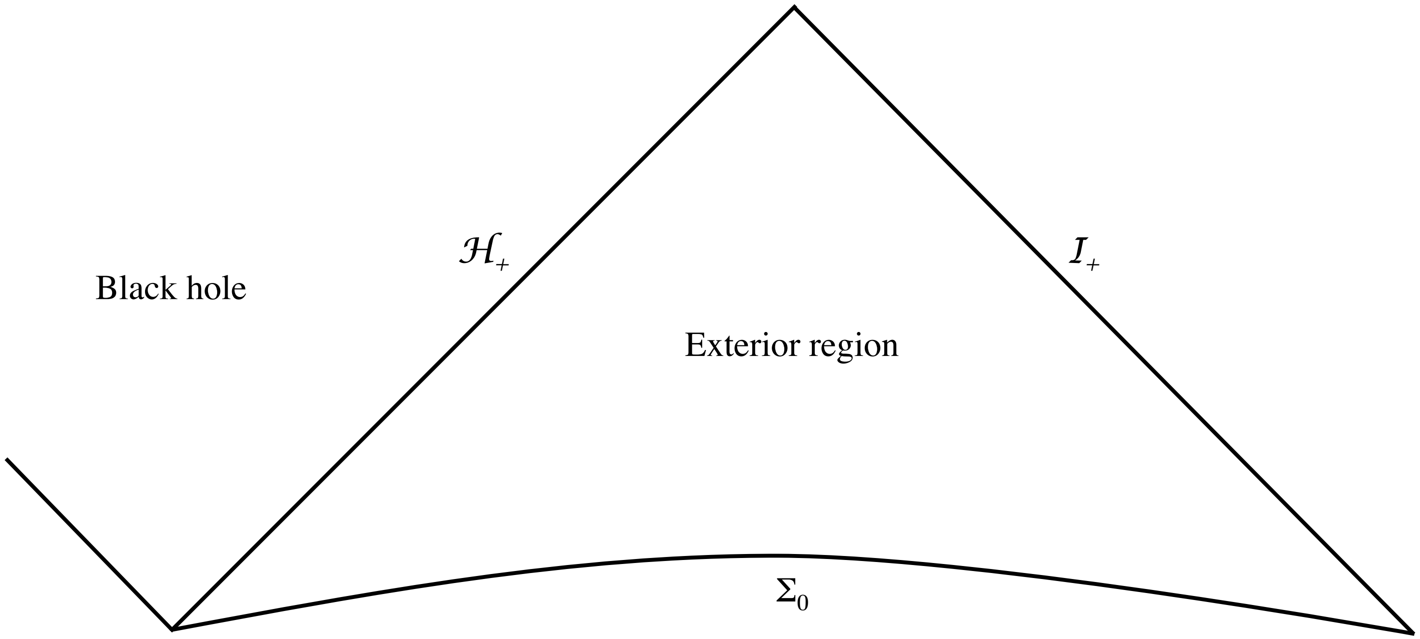
$$\Delta = r^2 + a^2 - 2mr,\ \rho^2 = r^2 + a^2(\cos\theta)^2,\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta$$

Kerr is stationary (∂_t is Killing) and axisymmetric (∂_φ is Killing)

The Schwarzschild metric is static and spherically symmetric and corresponds to the particular case $a = 0, m > 0$

$$\mathbf{g}_m = -\left(1 - \frac{2m}{r}\right)(dt)^2 + \left(1 - \frac{2m}{r}\right)^{-1}(dr)^2 + r^2\Big((d\theta)^2 + (\sin\theta)^2 (d\varphi)^2\Big)$$

Kerr space-time for $|a| \leq m$



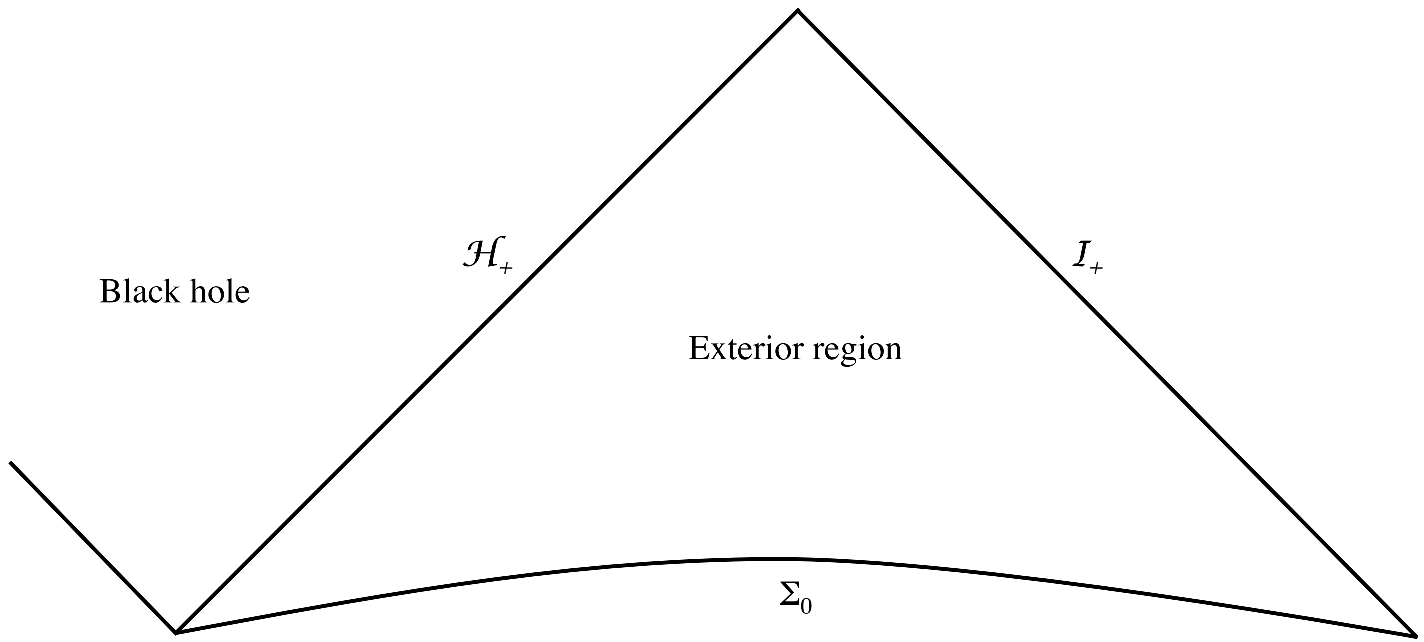
Exterior region $\quad r > m + \sqrt{m^2 - a^2}$ ($r > 2m$ in Schwarzschild)

Event horizon $\quad r = m + \sqrt{m^2 - a^2}$ ($r = 2m$ in Schwarzschild)

Black hole $\quad r < m + \sqrt{m^2 - a^2}$ ($r < 2m$ in Schwarzschild)

Stability conjecture for the Kerr family

Conjecture (Stability of the exterior region of Kerr). Small perturbations of given initial conditions of an exterior Kerr $\mathbf{g}_{a,m}$ with $|a| < m$ have maximal future developments converging to another exterior Kerr solution $\mathbf{g}_{a_f,m_f}$ with $|a_f| < m_f$



Main difficulties

- Linearized gravity system, as discussed by Teukolski and al., does not posses a conserved energy. No exponentially growing modes (Whiting 89'), but one cannot a priori establish, even formally, the boundedness
- The experience from the stability of the Minkowski space shows that establishing decay is crucial to prove nonlinear stability
- Until recently even the simplest toy model of a linear wave equation on a fixed Kerr background was not understood Dafermos-Rodnianski-Schlapentokh Rothman 14'
- The linearized gravity system must in fact have instabilities corresponding to mass and angular momentum, as well as infinitesimal motions of the center of mass of the Kerr solution

Polarized axial symmetry

Assume that $(\mathcal{M}^{1+3}, \mathbf{g})$ possess an axial Killing vectorfield $\mathbf{Z}$ which is hypersurface orthogonal

Let $X = \mathbf{g}(\mathbf{Z}, \mathbf{Z})$ and let g the 2+1 metric such that

$$\mathbf{g} = g + X(d\varphi)^2$$

Then, the Einstein vacuum equations reduce to

$$\begin{cases} \square_g X &= \dfrac{1}{X} D^a X D_a X \\ \mathrm{Ric}(g)_{ab} &= \dfrac{1}{2X} D_a D_b X - \dfrac{1}{4X^2} D_a X D_b X \end{cases}$$

Stability conjecture in the axial polarized case

In the axial polarized case, the final state has to be Schwarzschild!

Conjecture (Stability of the exterior region of Schwarzschild under axial polarized perturbations). Small axial polarized perturbations of given initial conditions of an exterior Schwarzschild $\mathbf{g}_m$ with $m > 0$ have maximal future developments converging to another exterior Schwarzschild solution $\mathbf{g}_{m_f}$ with $m_f > 0$

Recent progress on the axial polarized case

In this talk, we report on the following progress

- Identify a main quantity satisfying a suitable energy estimate from which one can in principle reconstruct the solution

  See also Dafermos-Holzegel-Rodnianski (DHR, announced 13')

- How to treat the (two) instabilities

Note that once these instabilities have been treated, another crucial step is to derive decay estimates for linearized gravity, see DHR

Connection and curvature coefficients

In the reduced 2+1 picture, we consider a null frame (e_4, e_3, e_θ)

$$\chi = g(D_\theta e_4, e_\theta),\ \underline{\chi} = g(D_\theta e_3, e_\theta),\ \omega = \frac{1}{4} g(D_4 e_4, e_3),\ \underline{\omega} = \frac{1}{4} g(D_3 e_3, e_4), \cdots$$

$$R_{33} = \underline{\alpha}, \quad R_{44} = \alpha, \quad R_{3\theta} = \underline{\beta}, \quad R_{4\theta} = \beta, \quad R_{\theta\theta} = \rho, \quad R_{34} = \rho$$

$$\kappa = \chi + \frac{e_4(X)}{2X},\ \underline{\kappa} = \underline{\chi} + \frac{e_3(X)}{2X},\ \vartheta = \chi - \frac{e_4(X)}{2X},\ \underline{\vartheta} = \underline{\chi} - \frac{e_3(X)}{2X}$$

In Schwarzschild, we have $X = r^2(\sin\theta)^2$ and

$$\chi = \frac{\kappa}{2} = -\underline{\chi} = -\frac{\underline{\kappa}}{2} = \frac{\sqrt{1-\frac{2m}{r}}}{r},\ \omega = -\underline{\omega} = -\frac{m}{2r^2\sqrt{1-\frac{2m}{r}}},\ \rho = -\frac{2m}{r^3}$$

$$\vartheta = \underline{\vartheta} = \zeta = \eta = \underline{\eta} = \xi = \underline{\xi} = \beta = \underline{\beta} = \alpha = \underline{\alpha} = 0$$

Null structure and Bianchi equations

We decompose Γ (connection) and R in the null frame (e_4, e_3, e_θ):

$$\Gamma = \Gamma_S + \check{\Gamma}, \quad \check{\Gamma} = O(\epsilon), \quad R = R_S + \check{R}, \quad \check{R} = O(\epsilon)$$

The null structure and projected Bianchi equations take the form

$$\begin{aligned} \partial\check{\Gamma} + \Gamma_S \cdot \check{\Gamma} &= \check{R} + O(\epsilon^2) \\ \partial\check{R} + \Gamma_S \cdot \check{R} + R_S \cdot \check{\Gamma} &= O(\epsilon^2) \end{aligned}$$

In Minkowski, there is a decoupling as $\check{\Gamma}$ does not appear in the linearized Bianchi equations!

The Bianchi and null structure equations form a coupled system of 20 equations which is a priori hard to analyze

Exhibiting a main quantity

Goal: find a quantity satisfying a good equation and from which one can reconstruct all the other quantities

Since the main problems come from the fact that ρ is not small, one would like to obtain a quantity which is a good linearization of ρ

A good candidate is frame invariant, has signature 0, and vanishes in Schwarzschild

The following candidate satisfies all the above criteria

$$\mathfrak{p} \quad := \quad \frac{1}{4X^2} D_s X D^s X - \rho - \frac{1}{X}$$

Exhibiting a main quantity

$$\begin{aligned}
\square_{\mathbf{g}}(r^2\mathfrak{p}) - \left(\frac{2}{r^2} + \frac{4\cos(\theta)^2}{r^2\sin(\theta)^2}\right) r^2\mathfrak{p} &= r^2(\underline{\chi}^2\alpha + \chi^2\underline{\alpha}) + O(\epsilon^2)\\
-e_3(\mathfrak{p}) - 2\underline{\chi}\mathfrak{p} &= e_4(\underline{\alpha}) + 2(\chi - 2\omega)\underline{\alpha} + O(\epsilon^2)\\
-e_4(\mathfrak{p}) - 2\chi\mathfrak{p} &= e_3(\alpha) + 2(\underline{\chi} - 2\underline{\omega})\alpha + O(\epsilon^2)
\end{aligned}$$

Suggests to introduce $\mathfrak{q} := r^2\Omega^{-1}e_3(\Omega e_4(r^2\mathfrak{p}))$ which satisfies

$$\square_{\mathbf{g}}(\mathfrak{q}) + \left(\frac{8m}{r^3} - \frac{4}{r^2\sin(\theta)^2}\right)\mathfrak{q} = O(\epsilon^2)$$

This wave equation satisfies a good energy estimate!

Reconstruction and instabilities

In view of definition of $\mathfrak{q}$ and relation between α and $\mathfrak{p}$, we have

$$\Omega^{-1} e_3(\Omega^{-1} e_3(r^2\alpha)) = -\frac{\mathfrak{q}}{r^2} + O(\epsilon^2)$$

To control β and ρ, consider in particular the following Bianchi identities

$$\begin{aligned} e_\theta(\beta) - \frac{e_\theta(X)}{2X}\beta &= e_3(\alpha) + (\underline{\chi} - 4\underline{\omega})\alpha + \frac{3}{2}\vartheta\,\rho + O(\epsilon^2) \\ e_\theta(\rho) &= e_3(\beta) + 2(\underline{\chi} - \underline{\omega})\beta - 3\eta\rho + O(\epsilon^2) \end{aligned}$$

The 2 instabilities correspond to the kernel of the operators e_θ and $e_\theta - \frac{e_\theta(X)}{2X}$, i.e. the constant and $\sin\theta$ modes in the Fourier series decomposition

The first instability

We control $e_\theta(\rho)$ and hence $\rho - \overline{\rho}$, but not the average $\overline{\rho}$ of ρ. This instability is related to the definition of the mass

Ideally, we look for a definition of the mass such that

$$\overline{\rho} = -\frac{2m}{r^3} + O(\epsilon^2), \quad e_\theta(m) = 0, \quad e_4(m) = O(\epsilon^2), \quad e_3(m) = O(\epsilon^2)$$

These properties turn out to be satisfied by the Hawking mass

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi} \int_S \kappa \underline{\kappa}$$

In particular, m_f should be given by the limit of m_H along $\mathcal{I}_+$

The second instability

$e_\theta(\beta) - \frac{e_\theta(X)}{2X}\beta$ provides no control the $\sin\theta$ mode of β

Consider a fixed axis of symmetry, say the z-axis, and a Schwarzschild metric $\mathbf{g}_m$. Then, $\mathbf{g}_m(t, x, y, z - z_0), z_0 \in \mathbb{R}$ forms a 1-parameter family of solutions which are symmetric with respect to this axis

The kernel to the linearized equations generated by this 1-parameter family can be explicitly parametrized by a function $f(u, \underline{u})$

To control this instability requires to take advantage of this kernel

Concluding remarks

Recall that the main difficulties where

- Identifying a main quantity satisfying a suitable energy estimate
- Removing obstructions linked with instabilities
- Obtain decay for linearized gravity

In view of the above discussion as well as the work of DHR, we expect no additional conceptual obstructions in proving the stability of Schwarzschild in the axially symmetric polarized case

However, many technical difficulties still need to be addressed: suitable gauge choices, avoiding potential loss of derivatives (see Holzegel 10'), quadratic terms and null condition,...