Nonconservative products and Shallow Water models: an overview.

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Outline

EDANYA Team: main goals

- Simulation of geophysical flows: coastal and turbidity currents, sediment transport, prediction of emergency situations such as river floods, oil spills, tsunamis, debris slides, . . .
- Development of high order finite volume numerical schemes for nonconservative hyperbolic systems.
- Design of new models: turbidity current, sediment transport, multilayer shallow-water type approach, . . .
- High Performance Computing: from standard domain decomposition techniques using MPI to multi-GPU implementations, nested meshes, AMR,...
- Provide numerical tools for natural hazard early warning systems, Civil protection agencies, . . .

An example: the Lituya bay mega-Tsunami

- At 10:16pm (local time) on July 10, 1958. M_w 8.3 earthquake.
- Southwest sides and bottoms of Gilbert and Crillon inlets moved northwestward and relative to the northeast shore at he head of the bay, on the opposite side of the Fairweather fault.
- Shaking lasted about 4 minutes. Estimated total movements of 6.4m horizontally and 1 m vertically.
- About 2 minutes after the beginning of the earthquake the landslide was triggered.
- Slide volume estimated by Miller, 1960 in $30.6 \times 10^6\ m^3$.

Picture from Weiss et. al., 2009

An example: the Lituya bay mega-Tsunami

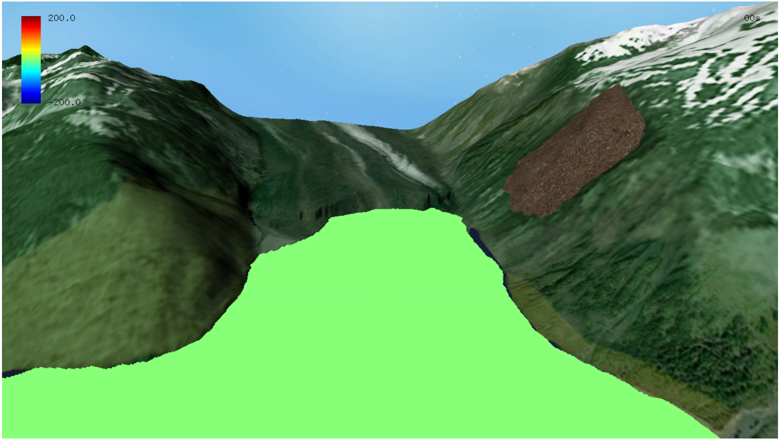


Figure : Lituya Bay event: Time $t = 0$ sec.

An example: the Lituya bay mega-Tsunami

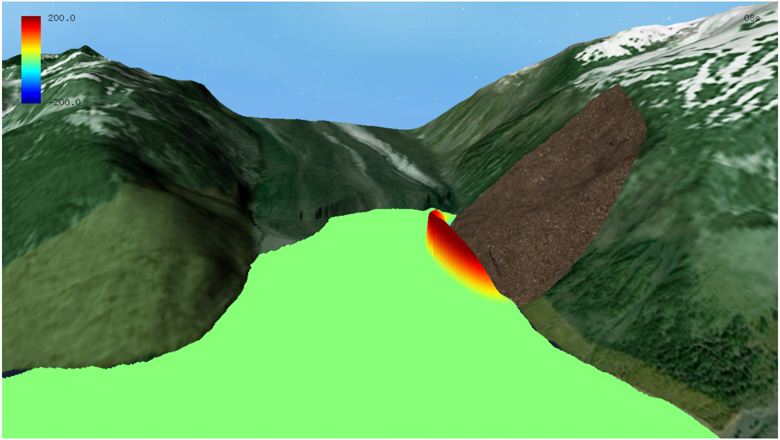


Figure : Lituya Bay event: Time $t = 8$ sec.

An example: the Lituya bay mega-Tsunami

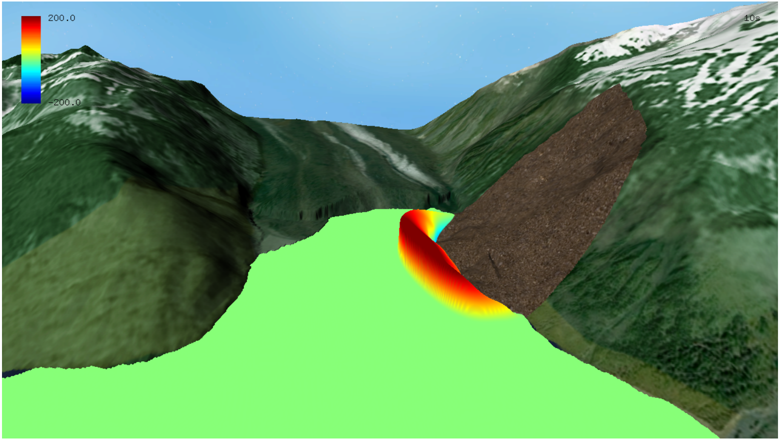


Figure : Lituya Bay event: Time $t = 10$ sec.

An example: the Lituya bay mega-Tsunami

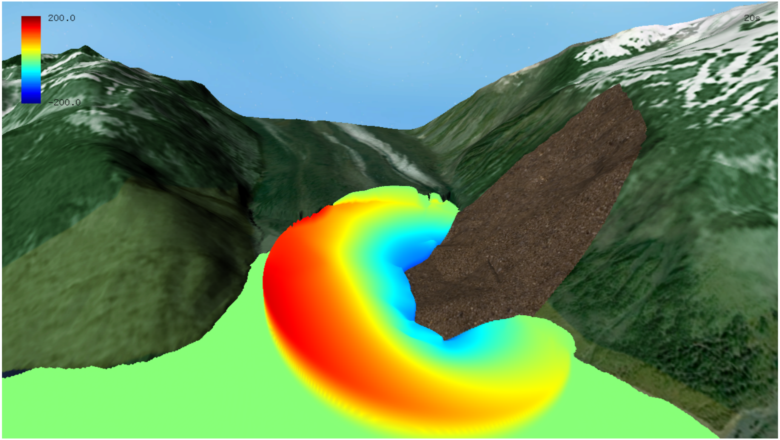


Figure : Lituya Bay event: Time $t = 20$ sec.

An example: the Lituya bay mega-Tsunami

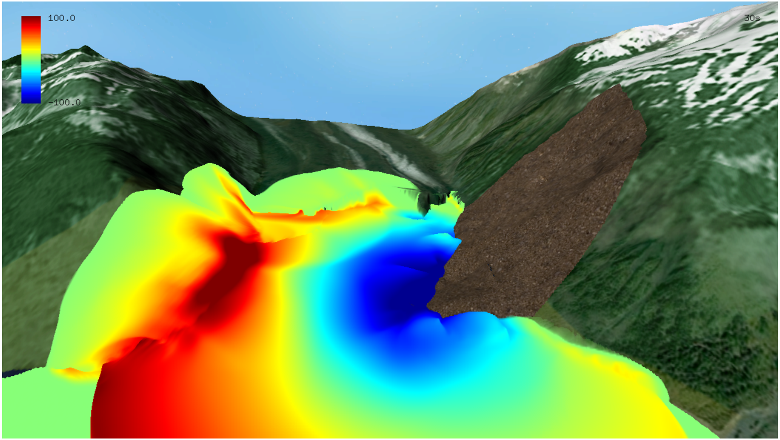


Figure : Lituya Bay event: Time $t = 30$ sec.

An example: the Lituya bay mega-Tsunami

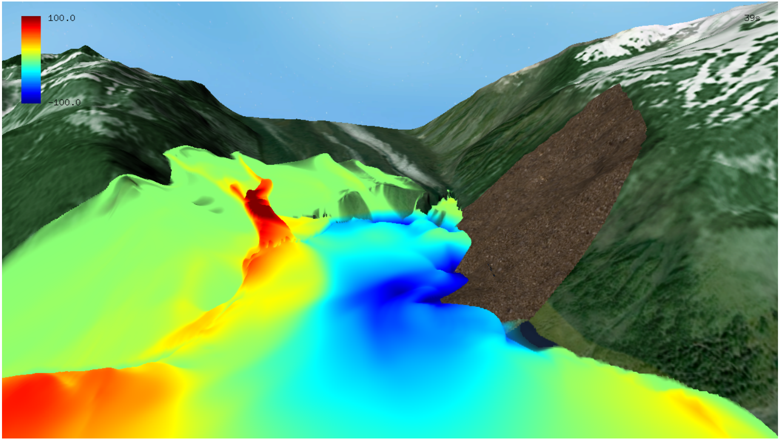
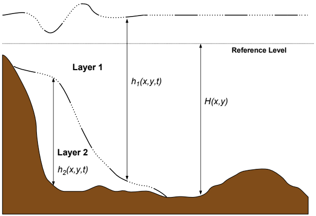


Figure : Lituya Bay event: Time $t = 39$ sec (max runup).

An example: two layer Savage-Hutter model

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial q_1}{\partial x} = 0 \\ \dfrac{\partial q_1}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q_1^2}{h_1} + \dfrac{g}{2}h_1^2\right) = -gh_1\dfrac{\partial h_2}{\partial x} + gh_1\dfrac{dH}{dx} + S_{i_1} \\ \dfrac{\partial h_2}{\partial t} + \dfrac{\partial q_2}{\partial x} = 0 \\ \dfrac{\partial q_2}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q_2^2}{h_2} + \dfrac{g}{2}h_2^2\right) = -g\dfrac{\rho_1}{\rho_2}h_2\dfrac{\partial h_1}{\partial x} + gh_2\dfrac{dH}{dx} + S_{i_2} + \tau, \end{cases}$$



- Here h_2 is the sediment (granular) layer depth.
- ρ_1 is the density of the fluid, ρ_s is the density of the granular material and $\rho_2 = (1-\psi_0)\rho_s + \psi_0\rho_1$;

An example: two layer Savage-Hutter model

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- τ is a parametrization of the Coulomb friction term.

$$\tau = \begin{cases} gh_2(1 - \frac{\rho_2}{\rho_1})\frac{u_2}{|u_2|}\tan(\delta_0) & \text{if } \tau > \sigma_c \\ u_2 = 0 & \text{otherwise} \end{cases}$$

with

$$\sigma_c = gh_2(1 - \frac{\rho_1}{\rho_2})\tan(\delta_0).$$

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E. Fernández Nieto, F. Bouchut, D. Bresch, M.J. Castro, A. Mangeney. *A new Savage-Hutter type model for submarine avalanches and generated tsunami.* J. Comp. Phys., 227: 7720-7754, 2008.

E.B. Pitman, I. Le, *A two-fluid model for avalanche and debris flows*, Phil. Trans. Roy. Soc. A 363:1573–1601, 2005.

1D Model problem

Let us consider the system

$$w_t + F(w)_x + B(w) \cdot w_x = S(w)H_x + S_f(w), \tag{1}$$

- $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$,
- F is a regular function from $\mathcal{O}$ to $\mathbb{R}^N$,
- B is a regular matrix function from $\mathcal{O}$ to $\mathcal{M}_{N\times N}(\mathbb{R})$,
- S is a function from $\mathcal{O}$ to $\mathbb{R}^N$,
- S_f is a function from $\mathcal{O}$ to $\mathbb{R}^N$, and
- H is a function from $\mathbb{R}$ to $\mathbb{R}$.

The system (1) can be rewritten under the form

$$W_t + \mathcal{A}(W) \cdot W_x = \widehat{S}_f(W), \tag{2}$$

W is the augmented vector

$$W = \begin{bmatrix} w \\ H \end{bmatrix} \in \Omega = \mathcal{O} \times \mathbb{R} \subset \mathbb{R}^{N+1}$$

1D Model problem

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The system (1) can be rewritten under the form

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$\mathcal{A}(W)$ is the matrix whose block structure is given by:

$$\mathcal{A}(W) = \left[\begin{array}{c|c} A(w) & -S(w) \\ \hline 0 & 0 \end{array}\right],$$

$$A(w) = J(w) + B(w), \quad \text{being } J(w) = \frac{\partial F}{\partial w}(w).$$

Nonconservative systems: weak solutions

We consider a strictly hyperbolic nonconservative numerical system:

$$W_t + \mathcal{A}(W) W_x = 0,$$

- The main difficulty of nonconservative system is that there is not a unique definition of weak solution.

- Following the theory developed by Dal Maso-LeFloch-Murat 1995, given a family of paths φ, a piecewise smooth function is a weak solution is:
  - It is a classical solution where it is regular.
  - Across a discontinuity the following jump condition is satisfied:

    $$\int_0^1 \mathcal{A}(\varphi(s; W^-, W^+)) \frac{\partial \varphi}{\partial s}(s; W^-, W^+)\, ds = \sigma (W^+ - W^-),$$

    where σ is the speed of propagation and $W^{\pm}$ the limits to the left and to the right of the discontinuity.
  - A concept of entropy is also necessary to select the meaningful physical solutions as it occurs in systems of conservation laws.

Path-conservative methods: Definition

- We consider semidiscrete path-conservative numerical schemes introduced in CP 2006:

$$\frac{d}{dt}W_i + \frac{1}{\Delta x}(\mathcal{D}^-_{i+1/2}(W_i, W_{i+1}) + \mathcal{D}^+_{i-1/2}(W_{i-1}, W_i)) = 0.$$

- $\mathcal{D}^\pm_{i+1/2}(W_i, W_{i+1})$ satisfy:

$$W_i = W_{i+1} \qquad \Rightarrow \qquad \mathcal{D}^\pm_{i+1/2}(W_i, W_{i+1}) = 0$$

$$\mathcal{D}^-_{i+1/2}(W_i, W_{i+1}) + \mathcal{D}^+_{i+1/2}(W_i, W_{i+1}) = \int_0^1 \mathcal{A}(\varphi(s; W_i, W_{i+1}))\partial_s\varphi(s; W_i, W_{i+1})ds.$$

- It is a generalization of the notion of **conservative methods**: if the system is conservative, every path-conservative method is equivalent to a conservative one.

Roe's method

$$\mathcal{D}^{\pm}(W_l, W_r) = \mathcal{A}^{\pm}_{\varphi}(W_l, W_r) \cdot (W_r - W_l),$$

where, $\mathcal{A}_{\varphi}(W_l, W_r)$ is a consistent linearization satisfying the generalized Roe property:

$$\mathcal{A}_{\varphi}(W_l, W_r) \cdot (W_r - W_l) = \int_0^1 \mathcal{A}(\varphi(s; W_l, W_r)) \frac{\partial \varphi}{\partial s}(s; W_l, W_r)\, ds.$$

(LeFloch 1990, Toumi 1992, CP-Castro 2004).

Path-conservative methods: Pros

- Easy extension to nonconservative systems of families of conservatives methods:
  - Godunov (CP-Muñoz 2008).
  - Simple approximate Riemann solvers: HLL, HLLC, etc (CP 2006, Castro-Fernández-Morales-Narbona-CP 2013).
  - Entropy preserving and entropy stable schemes (Fjordholm-Mishra-Tadmor 2008, Fjordholm-Mishra 2010 and Castro-Fjordholm-Mishra-CP 2013).
  - Osher type schemes (Dumbser-Toro 2011, Castro-Gallardo-Marquina 2015).
  - Central schemes on staggered grids (Castro-CP-Puppo-Russo 2012).
  - Central Upwind scheme (Castro-Kurganov-Morales 2013 and Castro-Cheng-Chertock-Kurganov 2013).
  - PVM methods (Castro-Fernández 2012).
- Design of new families of schemes that automatically produce new conservative methods for the particular case of systems of conservation laws:
  - RVM methods Castro-Gallardo-Marquina-2013.

Path-conservative methods: Pros

- Conservative for the conservation laws of the system (mass equation, for instance).
- Easy extension to high order:
  - Reconstruction operators (Castro-Gallardo-CP 2006, ...)
  - ADER (Dumbser-Castro-CP-Toro 2009, A. Hidalgo-Dumbser et al. 2010, ...).
  - WAF type schemes (Castro-Fernández-Narbona-Asunción 2013).
  - Discontinous Galerkin methods for nonconservative systems (Rhebergen-Bokhove-van der Vegt 2008...).
- Helpful for the design of well-balanced methods: the paths have to be related to the integral curves of the null eigenvalue.
  - Water at rest equilibria.
  - Generalized Hydrostatic Reconstruction (Castro-Pardo-CP 2007)
  - High order numerical methods that preserve every stationary solution of the 1d shallow water model (Castro-López-CP 2013) or a 1d model of the flood flow in vessels (Müller-CP-Toro 2013).
  - Euler equations with gravity: under study in collaboration with A.Marquina.

Path-conservative methods: Cons

- How to choose the 'good' family of paths?
  - When the hyperbolic system is the vanishing diffusion and dispersion limit of a family of problems, say

    $$W_t + A(W)W_x = \varepsilon B W_{xx} + \delta \varepsilon^2 C W_{xxx}, \tag{3}$$

    the adequate family of paths should be consistent with the *traveling waves* of the regularized system (LeFloch 1989).
  - A traveling wave

    $$U_\varepsilon(x,t) = V\left(\frac{x-\sigma t}{\varepsilon}\right), \tag{4}$$

    is a solution of (3) satisfying

    $$\lim_{\xi\to-\infty} V(\xi) = U^-, \quad \lim_{\xi\to+\infty} V(\xi) = U^+, \quad \lim_{\xi\to\pm\infty} V'(\xi) = 0, \quad \lim_{\xi\to\pm\infty} V''(\xi) = 0.$$

  - If there exists a traveling wave of speed σ linking the states U^-, U^+, the limit when ε tends to 0 of U_ε is:

    $$U(x,t) = \begin{cases} U^- & \text{if } x < \sigma t; \\ U^+ & \text{if } x > \sigma t; \end{cases}$$

    should be an admissible shock.

Path-conservative methods: Cons

- An easy computation shows that V has to solve the equation

$$-\sigma V' + A(V)\, V' = BV'' + \delta C V'''. \tag{5}$$

- By integrating (5) from $-\infty$ to ∞ and taking into account the boundary conditions, we obtain the jump condition

$$\int_{-\infty}^{\infty} A(V(\xi))\, V'(\xi)\, d\xi = \sigma(U^+ - U^-).$$

- If this jump condition is compared with the generalized Rankine-Hugoniot condition:

$$\int_0^1 \mathcal{A}(\varphi(s; U^-, U^+)) \frac{\partial \varphi}{\partial s}(s; U^-, U^+)\, ds = \sigma(U^+ - U^-),$$

  it is clear that the good choice for the path connecting the states U^- and U^+ would be, after a reparameterization, the viscous profile.
- The computation of viscous profiles may be a very difficult task. In that case, the family of straight segments is a sensible choice, as their corresponding jump conditions give a third order approximation of the physically correct ones (Castro-Fernández-Morales-Narbona-CP 2013).

Path-conservative methods: Cons

Are the limits of the numerical solutions weak solutions according to the chosen family of paths?

- The following result has be shown (Castro-LeFloch-Muñoz-CP 2008) for first order schemes: if the numerical solutions have bounded total variations and if they converge to a function W *uniformly in the sense of graphs*, then W is a weak solution of the system according to the chosen family of paths and thus its discontinuities satisfy the generalized Rankine-Hugoniot condition.

- If the sequence of numerical solutions is assumed to converge only almost everywhere to a function W, this function solves a perturbed problem

$$W_t + \mathcal{A}(W)\, W_x = \nu_W,$$

where ν_W is a bounded measure supported on the discontinuities of W, called the *convergence error measure*.

Path-conservative methods: Cons

- The appearance of this convergence error has been first studied by Hou-LeFloch, 1994 when a nonconservative numerical scheme is applied to a conservative system. This is also the case if you consider a nonconservative reformulation of a conservative system and a numerical scheme based of the correct family of paths (Karni-Abgrall 2010).

- This convergence error is due to the fact that the numerical solutions converge to a function which is a weak solution in a different sense: the satisfied Rankine-Hugoniot conditions are related to the viscous profile of a regularized problem whose viscous term is given by the numerical viscosity of the scheme and not by the physical one.

- This is a common difficulty for every numerical scheme having a viscosity term which is different from the physical one.

Path-conservative methods: Cons

- In order to overcome this difficulty, several strategies are possible: viscosity-free methods as Glimm of front-tracking methods, methods based in the equivalent equation (Karni 1992), nonlinear projection methods (Berthon, Coquel and collaborators), kinetic relations (Berthon-Coquel-LeFloch 2012), modified cells (Chalons, Coquel, LeFloch and collaborators) etc. . .
- Among these strategies:
  - Entropy stable path-conservative methods (Castro-Fjordholm-Mishra-CP 2013).
  - Methods with well-controlled diffusion (WCD): Ernst-LeFloch-Mishra 2015, Beljadid-LeFloch-Mishra-CP 2016.
- In some cases, this convergence error identically vanishes: this is the case for system of balance laws with regular enough source terms (Muñoz-CP 2011) or for particular cases in which the nonconservative products only act across contact discontinuities.
- According to Alouges-Merlet 2012 when the numerical viscosity of the method commutes with $\mathcal{A}(W)$, as it is the case for many schemes (Roe, Rusanov, LF, FORCE, HLL, PVM, WAF, . . .), the numerical solutions provided by different paths for small /medium shocks do not differ too much, as the numerical shock curves associated to each scheme (or path) coincide up to third order in the size of the discontinuity.

Further developments

- Improvements of the HySEA platform.
- Dispersive shallow water models.
- Development of new models: sediment transport, avalanches, etc.
- Numerical methods with Welll Controlled Dispersion for nonconservative systems.
- Entropy-stable methods for degenerate parabolic systems.
- Uncertainty quantification: Multilevel Monte Carlo methods and alternative approaches.
- Numerical methods for MHD and relativistic MHD: WB methdos, free Jacobian methods, etc.
- Data assimilation using gradient free methods.
- Model reduction methods.
- Asymptotic-preserving methods.

Thanks

Webpage:

http://edanya.uma.es

Youtube Channel:

http://youtube.com/grupoedanya