All-regime Lagrangian-Remap numerical schemes for the gas dynamics equations. Applications to the large friction and low Mach coefficients

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Outline

Outline

Introduction

Motivation : numerical study of two-phase flows in nuclear reactors

We consider the following model

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ &\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0 \\ &\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = 0 \end{aligned}$$

where ρ, $\mathbf{u} = (u, v)^t$, E denote respectively the density, the velocity vector and the total energy of the fluid.

Let $e = E - \frac{|\mathbf{u}|^2}{2}$ be the specific and $\tau = 1/\rho$ the covolume

Introduction

We are especially interested in the design of numerical schemes when the model depends on a parameter $\epsilon > 0$.

The following three flow regimes are of interest

Classical regime : $\epsilon = O(1)$
Low ϵ regime : $\epsilon << 1$
Limit regime : $\epsilon \to 0$

Outline

Large friction regime

We consider the following model with friction and gravity

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ &\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho(\mathbf{g} - \alpha \mathbf{u}) \\ &\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = \rho \mathbf{u}.(\mathbf{g} - \alpha \mathbf{u}) \end{aligned}$$

where $\mathbf{g}$, α denote the gravity field and the friction coefficient.

The large friction regime is obtained by replacing α with $\frac{\alpha}{\epsilon}$

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ &\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho(\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u}) \\ &\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = \rho \mathbf{u}.(\mathbf{g} - \frac{\alpha}{\epsilon} \mathbf{u}) \end{aligned}$$

with $\epsilon << 1$

Large friction regime

Remark 1. Setting $\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + O(\epsilon^2)$, the long time behaviour of the solutions is given by

$$
\begin{aligned}
&\mathbf{u}_0 = 0 \\
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}_1) = 0 \\
&\nabla p = \rho(\mathbf{g} - \alpha \mathbf{u}_1) \\
&\partial_t(\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}_1] = \rho \mathbf{u}_1.(\mathbf{g} - \alpha \mathbf{u}_1)
\end{aligned}
$$

see Hsiao-Liu, Nishihara, Junca-Rascle, Lin-Coulombel, Coulombel-Goudon, Marcati-Milani... for rigorous proofs

Remark 2. This system will not be considered in the design of the numerical strategy

Low Mach regime

Introducing the characteristic and non-dimensional quantities :

$$x = \frac{x}{L}, \quad t = \frac{t}{T}, \quad \rho = \frac{\rho}{\rho_0}, \quad u = \frac{u}{u_0},$$

$$v = \frac{v}{v_0}, \quad e = \frac{e}{e_0}, \quad p = \frac{p}{p_0}, \quad c = \frac{c}{c_0}$$

with $u_0 = v_0 = \frac{L}{T}$, $e_0 = p_0\rho_0$ and $p_0 = \rho_0 c_0^2$, the non-dimensional system is (no gravity, no friction)

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0 \\
&\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2}\Big(\partial_t(\rho \mathbf{u}.\mathbf{u}) + \nabla \cdot (\rho \mathbf{u}.\mathbf{u}\mathbf{u})\Big) = 0
\end{aligned}$$

where $M = \frac{u_0}{c_0}$ denotes the Mach number and plays the role of ϵ

Low Mach regime

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0$$
$$\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2}\Big(\partial_t (\rho \mathbf{u}.\mathbf{u}) + \nabla \cdot (\rho \mathbf{u}.\mathbf{u}\mathbf{u})\Big) = 0$$

Definition. The flow is said to be in the low Mach regime if $M << 1$ and $\nabla p = O(M^2)$

Remark 1. Using asymptotic expansions in powers of M in the governing equations of $\rho, \mathbf{u}, p$ and boundary conditions leads to

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0$$
$$\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \frac{1}{\rho_0} \nabla p_2 = 0$$
$$\nabla \cdot \mathbf{u}_0 = 0$$

Remark 2. This system will not be considered in the design of the numerical strategy

Numerical issue in the Low Mach regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(M)$
- a time step $\Delta t = O(hM)$

which is out of reach in practice

More details can be found in the large body of literature on this subject : A. Majda, E. Turkel, H. Guillard, C. Viozat, B. Thornber, S. Dellacherie, P. Omnes, P-A. Raviart, F. Rieper, Y. Penel, P. Degond, S. Jin, J.-G. Liu, P. Colella, K. Pao, E. Turkel, R. Klein, J-P Vila, M.G., B. Després, M. Ndjinga, J. Jung, M. Sun, ...

General cure : change the treatment of acoustic waves in the low Mach regime by centering the pressure gradient

Numerical issue in the large friction regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(\epsilon)$
- a time step $\Delta t = O(\epsilon)$

which is out of reach in practice

More details can be found in the large body of literature on this subject : L. Hsiao, T.-P. Liu, S. Jin, L. Pareschi, L. Gosse, G. Toscani, F. Bouchut, H. Ounaissa, B. Perthame, C. C., F. Coquel, E. Godlewski, P.-A. Raviart, N. Seguin, C. Berthon, P.-G. LeFloch, R. Turpault, F. Filbet, A. Rambaud, M. Girardin, S. Kokh, C. Cancès, H. Mathis, N. Seguin, S. Cordier, B. Després, E. Franck, C. Buet, ...

General cure : upwinding of the source terms at interfaces (USI)

A couple of definitions

Uniform stability

A scheme is said to be stable in the uniform sense if the CFL condition is uniform with respect to ϵ

This avoids stringent CFL restrictions $\Delta t = O(hM)$ or $\Delta t = O(\epsilon)$

Uniform consistency

A scheme is said to be consistent in the uniform sense if the truncation error is uniform with respect to ϵ

This avoids large numerical diffusion and mesh size restrictions $h = o(M)$ or $h = O(\epsilon)$

All-regime scheme

A scheme is said to be all-regime if it is able to compute accurate solutions with a mesh size h and a time step Δt independent of ϵ

Objectives

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

How to do that...

Outline

How to reach these objectives

How to get the uniform stability ?

- implicit treatment of the fast phenomenon
- explicit treatment of the slow phenomenon (sake of accuracy)

→ Lagrange-Projection strategy Coquel-Nguyen-Postel-Tran

How to get the uniform consistency ?

- modify the numerical fluxes to reduce the numerical diffusion

→ Truncation errors in equivalent equations

How to deal with any (possibly strongly nonlinear) pressure law p ?

- overcome the non linearities, "linearization"

→ Relaxation strategy Suliciu, Jin-Xin, Bouchut, C.-Coquel, C.-Coulombel

How to deal with unstructured meshes in multi-D ?

- work on a fixed mesh (no need to deform unstructured meshes)

→ Operator splitting strategy and rotational invariance

Lagrange-Projection strategy

Let us first focus on the 1D system

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \varrho u + \partial_x (\varrho u^2 + p) = 0 \\ \partial_t (\varrho E) + \partial_x (\varrho E u + p u) = 0 \end{cases}$$

Using chain rule arguments, we also have

$$\begin{cases} \partial_t \varrho + u \partial_x \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + u \partial_x \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + u \partial_x \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases}$$

so that splitting the transport part leads to

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases} \qquad \begin{cases} \partial_t \varrho + u \partial_x \varrho = 0 \\ \partial_t \varrho u + u \partial_x \varrho u = 0 \\ \partial_t \varrho E + u \partial_x \varrho E = 0 \end{cases}$$

Lagrangian-step — Transport-step

Lagrange-Projection strategy

The Lagrangian-step

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases} \quad \text{also writes} \quad \begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m p u = 0 \end{cases}$$

with $\tau = 1/\varrho$ and $\tau\partial_x = \partial_m$.

- Eigenvalues are given by $-\rho c,\ 0,\ \rho c$
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{h} \max(\rho c) \leq \frac{1}{2}$$

The idea is to propose a time-implicit scheme to avoid this time-step restriction ($\Delta t = O(hM)$ in the low Mach regime)

Lagrange-Projection strategy

The Transport-step is

$$\begin{cases} \partial_t \varrho + u\partial_x \varrho = 0 \\ \partial_t \varrho u + u\partial_x \varrho u = 0 \\ \partial_t \varrho E + u\partial_x \varrho E = 0 \end{cases} \quad \text{also writes} \quad \begin{cases} \partial_t \varrho + \partial_x \varrho u - \varrho\partial_x u = 0 \\ \partial_t \varrho u + \partial_x \varrho u^2 - \varrho u\partial_x u = 0 \\ \partial_t \varrho E + \partial_x \varrho E u - \varrho E\partial_x u = 0 \end{cases}$$

- Eigenvalues are given by u
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{h} \max(|u|) \leq \frac{1}{2}$$

The idea is then to propose a standard time-explicit scheme to keep accuracy on the slow phenomenon ($\Delta t = O(h)$ in all regime)

Operator splitting strategy

We will consider the following three-step numerical scheme :

First step $(t^n \rightarrow t^{Lag})$: solve implicitly the acoustic system with the solution at time t^n as initial solution

Second step $(t^{Lag} \rightarrow t^{n+1-})$ solve implicitly the source terms when present with the solution at time t^{Lag} as initial solution

Third step $(t^{n+1-} \rightarrow t^{n+1})$ solve explicitly the transport system with the solution at time t^{n+1-} as initial solution

Solving implicitly the source terms avoid the time-step restriction $\Delta t = O(\epsilon)$ when $\epsilon << 1$ ($\Delta t = O(h)$ in all regime)

A few words about the relaxation approach

The gas dynamics equations in Lagrangian coordinates :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

with $p = p(\tau, e)$ and

$$e = E - \frac{1}{2}u^2$$

Due to the nonlinearities of p, the Riemann problem is difficult to solve. The relaxation strategy :

- Idea : to deal with a larger but simpler system
- Design principle : to understand $p(\tau, e)$ as a new unknown that we denote Π

A few words about the relaxation approach

The proposed relaxation system is

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda(p - \Pi) \end{cases}$$

At least formally, observe that

$$\lim_{\lambda \to +\infty} \Pi = p \quad (\text{if} \quad a > \rho c(\tau, e))$$

(see e.g. Chalons-Coulombel for a rigorous proof)

Why is it interesting ? The characteristic fields are linearly degenerate whatever the pressure law is !

A few words about the relaxation approach

The time-explicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

$$\begin{cases} \tau_j^{Lag} = \tau_j^n + \dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ u_j^{Lag} = u_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\ \Pi_j^{Lag} = \Pi_j^n - a^2\dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ E_j^{Lag} = E_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2}u^*_{j+1/2} - p^*_{j-1/2}u^*_{j-1/2}) \end{cases}$$

with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u^*_{j+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(\Pi_{j+1}^n - \Pi_j^n)$$

$$p^*_{j+1/2} = \frac{1}{2}(\Pi_j^n + \Pi_{j+1}^n) - \frac{a}{2}(u_{j+1}^n - u_j^n)$$

A few words about the relaxation approach

The time-implicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

$$\begin{cases} \tau_j^{Lag} = \tau_j^n + \dfrac{\Delta t}{\Delta m}(u_{j+1/2}^* - u_{j-1/2}^*) \\ u_j^{Lag} = u_j^n - \dfrac{\Delta t}{\Delta m}(p_{j+1/2}^* - p_{j-1/2}^*) \\ \Pi_j^{Lag} = \Pi_j^n - a^2 \dfrac{\Delta t}{\Delta m}(u_{j+1/2}^* - u_{j-1/2}^*) \\ E_j^{Lag} = E_j^n - \dfrac{\Delta t}{\Delta m}(p_{j+1/2}^* u_{j+1/2}^* - p_{j-1/2}^* u_{j-1/2}^*) \end{cases}$$

with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u_{j+1/2}^* = \frac{1}{2}(u_j^{Lag} + u_{j+1}^{Lag}) - \frac{1}{2a}(\Pi_{j+1}^{Lag} - \Pi_j^{Lag})$$

$$p_{j+1/2}^* = \frac{1}{2}(\Pi_j^{Lag} + \Pi_{j+1}^{Lag}) - \frac{a}{2}(u_{j+1}^{Lag} - u_j^{Lag})$$

A few words about the relaxation approach

The time-implicit scheme

- deals with (possibly strongly nonlinear) pressure laws
- is free of CFL restriction !
- is not expensive in the sense that only a linear problem w.r.t. u and Π has to be solved. Thanks to the relaxation strategy !

Formulation on unstructured meshes

On unstructured meshes, the time-explicit ($\sharp = n$) and time-implicit ($\sharp = Lag$) schemes write

$$\begin{aligned}
\mathbf{u}_j^{Lag} &= \mathbf{u}_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} \Pi_{jk}^* \mathbf{n}_{jk} \\
\Pi_j^{Lag} &= \Pi_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} (a_{jk})^2 u_{jk}^* \\
\tau_j^{Lag} &= \tau_j^n + \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \\
E_j^{Lag} &= E_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} p_{jk}^* u_{jk}^*
\end{aligned}$$

$$u_{jk}^* = \frac{1}{2}\mathbf{n}_{jk}^T(\mathbf{u}_j^\sharp + \mathbf{u}_k^\sharp) - \frac{1}{2a_{jk}}(\Pi_k^\sharp - \Pi_j^\sharp), \quad p_{jk}^* = \frac{1}{2}(\Pi_j^\sharp + \Pi_k^\sharp) - \frac{a_{jk}}{2}\mathbf{n}_{jk}^T(\mathbf{u}_k^\sharp - \mathbf{u}_j^\sharp)$$

Source terms

The time-implicit point-wise scheme for the gravity terms and external forces writes

$$
\begin{aligned}
\tau_j^{n+1-} &= \tau_j^{Lag} \\
\mathbf{u}_j^{n+1-} &= \mathbf{u}_j^{Lag} + \Delta t(\mathbf{g} - \alpha \mathbf{u}_j^{n+1-}) \\
E_j^{n+1-} &= E_j^{Lag} + \Delta t\, \mathbf{u}_j^{n+1-}.(\mathbf{g} - \alpha \mathbf{u}_j^{n+1-})
\end{aligned}
$$

It is free of CFL restriction

Transport step

In order to approximate the solutions of the transport step

$$\begin{aligned} \partial_t \rho + (\mathbf{u}\cdot\nabla)\rho &= 0 \\ \partial_t(\rho\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u} &= 0 \\ \partial_t(\rho E) + (\mathbf{u}\cdot\nabla)\rho E &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) - \rho\nabla\cdot\mathbf{u} &= 0 \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \rho\mathbf{u}\nabla\cdot\mathbf{u} &= 0 \\ \partial_t \rho E + \nabla\cdot(\rho E\mathbf{u}) - \rho E\nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

we simply use the time-explicit upwind finite-volume scheme

$$\varphi_j^{n+1} = \varphi_j^{n+1-} - \Delta t \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \varphi_{jk}^{n+1-} + \Delta t \varphi_j^{n+1-} \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^*$$

where $\varphi = \rho, \rho\mathbf{u}, \rho E$ and $\varphi_{jk}^{n+1-} = \begin{cases} \varphi_j^{n+1-} & \text{if } u_{jk}^* > 0 \\ \varphi_k^{n+1-} & \text{if } u_{jk}^* \leq 0 \end{cases}$

This scheme is stable under a material CFL condition $(\Delta t = O(h))$

Objectives

Our objective was to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

What about the uniform consistency ?

Uniform consistency in the low Mach regime

Let us first recall that the Lagrangian-step in 1D writes

$$\begin{aligned}
\tau_j^{n+1-} &= \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\
u_j^{n+1-} &= u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\
E_j^{n+1-} &= E_j^n - \frac{\Delta t}{\Delta m}\big((pu)^*_{j+1/2} - (pu)^*_{j-1/2}\big)
\end{aligned}$$

with

$$\begin{aligned}
u^*_{j+1/2} &= \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j) \\
p^*_{j+1/2} &= \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)
\end{aligned}$$

Uniform consistency in the low Mach regime

In dimensionless form we get

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}\left((pu)^*_{j+1/2} - (pu)^*_{j-1/2}\right)$$

with, since $p_{j+1} - p_j = \mathcal{O}(\Delta m M^2)$,

$$u^*_{j+1/2} = \frac{u_j + u_{j+1}}{2} - \frac{M}{2a}\frac{(p_{j+1} - p_j)}{M^2} = \frac{u_j + u_{j+1}}{2} + \mathcal{O}(M\Delta m)$$ ☺

$$p^*_{j+1/2} = \frac{p_j + p_{j+1}}{2M^2} - \frac{a}{2M}(u_{j+1} - u_j) = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O}\left(\frac{\Delta m}{M}\right)$$ ☹

Uniform consistency in the low Mach regime

The problem comes from the numerical diffusion in $p^*_{j+1/2}$

To obtain the uniform consistency w.r.t. M we introduce the parameter $\theta_{j+1/2}$ and simply consider the new definition of $p^*_{j+1/2}$

$$p^*_{j+1/2} = \frac{1}{2}(p^n_j + p^n_{j+1}) - \theta_{j+1/2}\frac{a}{2}(u^n_{j+1} - u^n_j)$$

Then we get $p^*_{j+1/2} = \dfrac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\dfrac{\theta_{j+1/2}\Delta m}{M})$

Which gives the uniform consistency provided that $\theta_{j+1/2} = \mathcal{O}(M)$

The modification is extremely simple and applies directly in multi-D

Uniform consistency in the large friction regime

Let us first recall that the first two steps write

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) + \Delta t(g - \frac{\alpha}{\epsilon} u_j^{n+1-})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2}) + \Delta t\, u_j^{n+1-}.(g - \frac{\alpha}{\epsilon} u_j^{n+1-})$$

with

$$u^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j)$$
$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)$$

Uniform consistency in the large friction regime

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$

$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) + \Delta t(g - \frac{\alpha}{\epsilon} u_j^{n+1-})$$

$$u^*_{j+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(p_{j+1}^n - p_j^n)$$

$$p^*_{j+1/2} = \frac{1}{2}(p_j^n + p_{j+1}^n) - \frac{a}{2}(u_{j+1}^n - u_j^n)$$

Numerical asymptotic analysis. $u_j = u_j^{(0)} + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$

- Multiply the second equation by ϵ and let $\epsilon \to 0$: $u_j^{(0)} = 0$
- Let $\epsilon \to 0$ in the second equation : $\dfrac{p_{j+1} - p_{j-1}}{2\Delta m} = (g - \alpha u_j^{(1)})$
- It remains to insert $u_j = 0 + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$ in the first equation

Uniform consistency in the large friction regime

Let us insert $u_j = 0 + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$ in the first equation, we immediately get

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}\epsilon(u_{j+1/2}^{(1)} - u_{j-1/2}^{(1)}) + \mathcal{O}(\epsilon^2)$$

with

$$u_{j+1/2}^{(1)} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} - \frac{1}{\epsilon}\frac{p_{j+1} - p_j}{2a} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} + \mathcal{O}(\frac{\Delta m}{\epsilon}) \quad ☹$$

which is clearly not consistent with $\partial_t \tau - \epsilon \partial_m u_1 = O(\epsilon^2)$

Uniform consistency in the large friction regime

The problem comes from the numerical diffusion in $u^*_{j+1/2}$

To obtain an uniform consistency w.r.t. ϵ we introduce the parameter $\theta_{j+1/2}$ and simply consider the following definition of $u^*_{j+1/2}$

$$u^*_{j+1/2} = \frac{1}{2}(u^n_j + u^n_{j+1}) - \frac{\theta_{j+1/2}}{2a}(p^n_{j+1} - p^n_j)$$

Then we get $u^{(1)}_{j+1/2} = \dfrac{u^{(1)}_j + u^{(1)}_{j+1}}{2} + \mathcal{O}(\dfrac{\theta_{j+1/2}\Delta m}{\epsilon})$

Which gives the uniform consistency provided that $\theta_{j+1/2} = \mathcal{O}(\epsilon)$

The modification is extremely simple and applies directly in multi-D

Remarks

All the objectives are reached !

How does the modifications affect the stability properties ?

- conservative (with no source terms and external forces)
- positive
- unconditionally entropy satisfying for all $\theta \geq 0$ in the linear case
- conditionally entropy satisfying in the non linear case. $\theta = 0$ is also possible in practice ! (numerical diffusion in the transport step)

Interestingly, operator-splitting techniques are compatible with the all-regime property. USI approach not mandatory

High-order extension under progress using DG methods, as well as shallow-water equations and diffusion terms

Outline

Numerical results

We want to assess the following properties of the numerical scheme :

- Accuracy of the numerical scheme in the large friction regime if $\tilde{\theta} = O(\epsilon)$
- Accuracy of the numerical scheme in the low Mach regime if $\theta = O(M)$
- Robustness of the numerical scheme with respect to the choice of θ (resp. $\tilde{\theta}$) in and outside the low Mach regime (resp. large friction regime)
- Performance in terms of CPU time of the mixed implicit-explicit numerical scheme

Large friction modification

Large friction modification

test case : sensitivity w.r.t. the space step

The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

We consider the domain $\Omega = (0, 1)$.
The initial condition is given by

$$\begin{cases} (\rho, u, p) & = (1.0, 0, 10000.0), \quad \text{if } x \in [0, 0.35] \cap [0.65, 1], \\ (\rho, u, p) & = (2.0, 0, 26390.2), \quad \text{if } x \in [0.35, 0.65]. \end{cases}$$

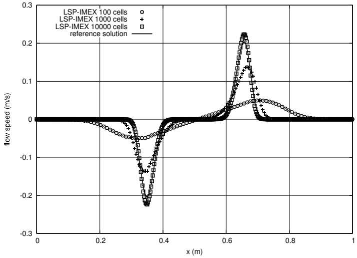
We impose periodic boundary conditions.

The friction parameter is given by $\alpha = 10^6 s^{-1}$, so that we are in the large fraction regime.

test case : sensitivity w.r.t. the space step

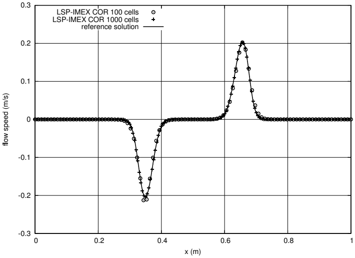
We compute approximate solutions with a 100-cell, 1000-cell and a 10 000-cell grid, with $\beta = n$

$\tilde{\theta} = 1$



flow speed

$\tilde{\theta} = \min\left(\frac{2a}{\alpha \Delta x}, 1\right)$

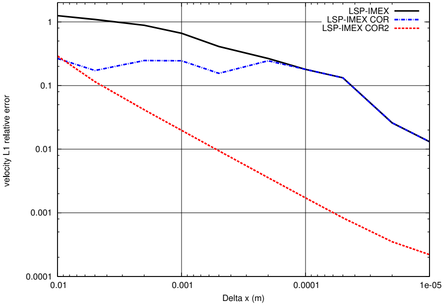


flow speed

test case : sensitivity w.r.t. the space step

We plot convergence curves in L^1 norm for

$$\tilde{\theta} = 1 \text{ (black)}, \quad \tilde{\theta} = \min\left(\frac{2a}{\alpha\Delta x}, 1\right) \text{ (blue)}, \quad \tilde{\theta} = \frac{1}{\alpha} \text{ (red)}$$



Low Mach modification

Low Mach modification

Vortex in a box : test case

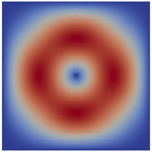
The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

We consider the domain $\Omega = (0,1)^2$.
The initial condition is given by

$$\begin{cases} \rho_0(x,y) = 1 - \frac{1}{2} tanh\left(y - \frac{1}{2}\right), & u_0(x,y) = 2sin^2(\pi x)sin(\pi y)cos(\pi y)), \\ p_0(x,y) = 1000, & v_0(x,y) = -2sin(\pi x)cos(\pi x)sin^2(\pi y). \end{cases}$$

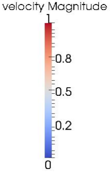
We impose a no-slip boundary condition.

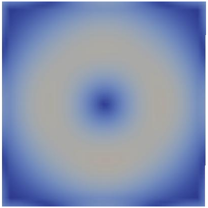


This configuration leads to a Mach number of order 0.026, so that we are in the low Mach regime.

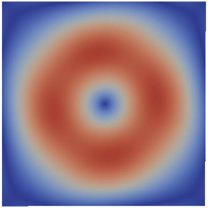
Vortex in a box $(M\#0.026)$: explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.

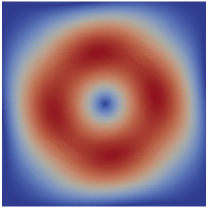




explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$



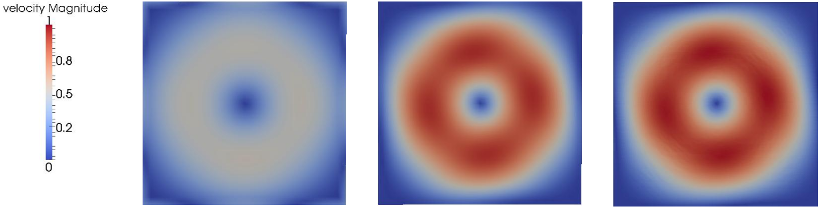
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$400 * 400 cells$



reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

Vortex in a box ($M\#0.026$) : modified explicit scheme

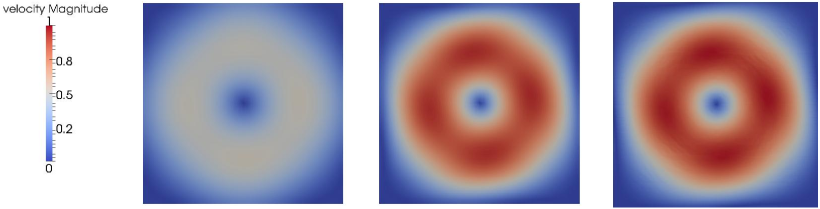
We plot the flow speed magnitude at time $T = 0.125s$.



explicit scheme	explicit scheme	reference solution
$(\theta = 1)$	$(\theta_{ij} = M_{ij}^n)$	explicit scheme
Cartesian Mesh	Cartesian Mesh	$(\theta = 1)$
$50 * 50 cells$	$50 * 50 cells$	Triangular Mesh

Vortex in a box ($M\#0.026$) : modified implicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.



implicit-explicit scheme ($\theta = 1$) Cartesian Mesh $50 * 50 cells$	implicit-explicit scheme ($\theta_{ij} = M_{ij}^n$) Cartesian Mesh $50 * 50 cells$	reference solution explicit scheme ($\theta = 1$) Triangular Mesh

Vortex in a box ($M\#0.026$) : CPU Time

EX : $\beta = n$, IMEX : $\beta = Lag$.

Numerical scheme	$EX(\theta = 1)$ (Mesh $400 * 400$)	$EX(\theta = 1)$ (Mesh $50 * 50$)	$EX(\theta_{ij} = M_{ij})$ (Mesh $50 * 50$)
Number of iterations	18 457	2 306	2 305
CPU time (s)	9 263.04 ($2h34min$)	17.14	19.3

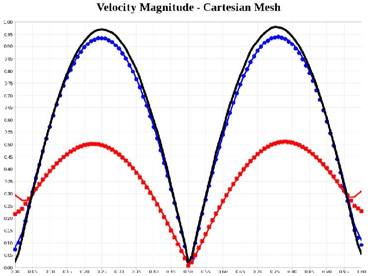
Speed up $(\theta = 1 \rightarrow \theta_{ij} = M_{ij}) = 480$

Numerical scheme	$IMEX(\theta = 1)$ (Mesh $50 * 50$)	$IMEX(\theta_{ij} = M_{ij})$ (Mesh $50 * 50$)
Number of iterations	43	56
CPU time (s)	3.75	5.77

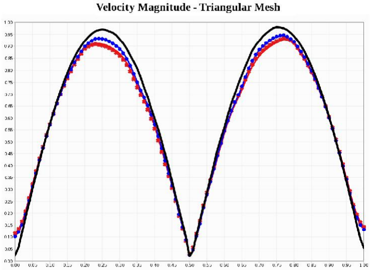
Speed up (explicit$\rightarrow$ implicit-explicit)$= 3.3$

Vortex in a box $(M\#0.026)$: Influence of the cell geometry

We plot a 1D-cut at $x = 0.5$ of the flow speed magnitude at time $T = 0.125s$.



Cartesian Mesh



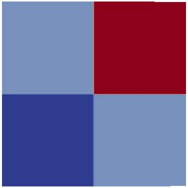
Triangular Mesh

2D-Riemann problem : test case

The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

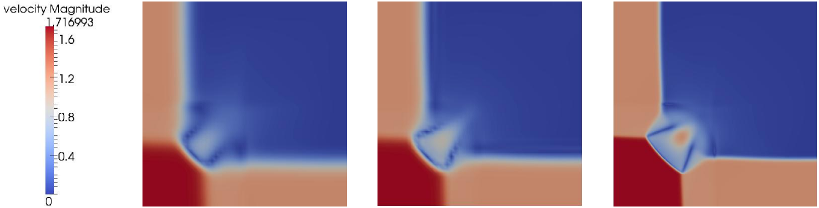
We consider the domain $\Omega = (0,1)^2$.
The initial condition corresponds to a 2D Riemann problem that consists of 4 shock waves. We impose Neumann boundary conditions.



This configuration leads to a Mach number that ranges from 10^{-5} to 3.15, so that we have both low Mach and order 1 Mach values.

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified explicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



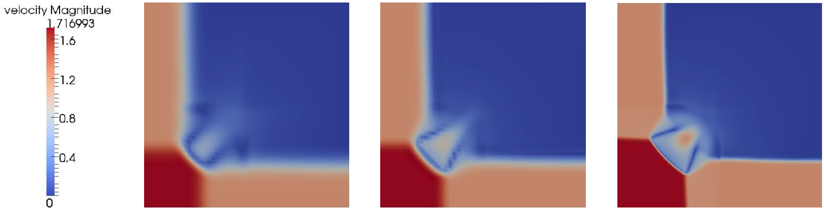
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

explicit scheme
$(\theta = 0)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified implicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



implicit-explicit scheme $(\theta = 1)$ Cartesian Mesh $50 * 50 cells$	implicit-explicit scheme $(\theta = 0)$ Cartesian Mesh $50 * 50 cells$	reference solution explicit scheme $(\theta = 1)$ Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: CPU time

Numerical scheme	$\text{EX}(\theta = 1)$ (Mesh $50 * 50$)	$\text{EX}(\theta = 0)$ (Mesh $50 * 50$)
Number of iterations	323	343
CPU time (s)	2.59	2.79

Speed up $(\theta = 1 \rightarrow \theta = 0) \approx 1$

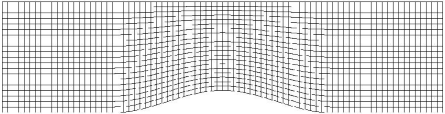
Numerical scheme	$\text{IMEX}(\theta = 1)$ (Mesh $50 * 50$)	$\text{IMEX}(\theta = 0)$ (Mesh $50 * 50$)
Number of iterations	216	218
CPU time (s)	10.28	10.33

Speed up (explicit$\rightarrow$ implicit-explicit)$= 0.25$

flow in a channel with bump

The fluid is equipped with a mixture of two perfect gas with different adiabatic coefficients equation of state : $\gamma_1 = 2$, $\gamma_2 = 1.4$.

We consider for the domain a channel with a 20% sinusoidal bump.



The initial condition corresponds to a constant state

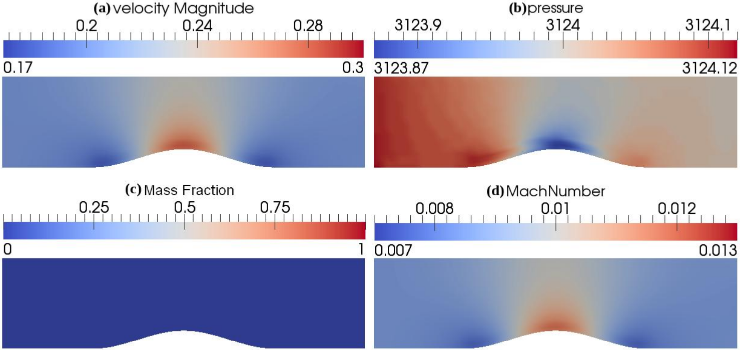
$$(\rho, Y, p, u, v) = (7.81, 0, 3124, 0, 0).$$

We impose inlet/outlet and Wall boundary conditions.

This configuration leads to a subsonic flow for $u_{in} = 0.2$ and a transonic flow for $u_{in} = 12$.

flow in a channel with bump : subsonic flow

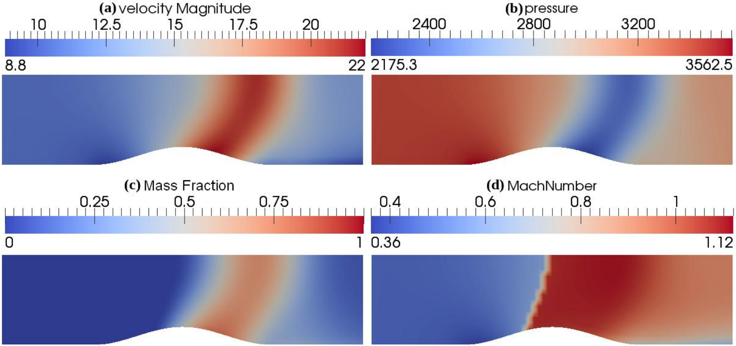
We plot the results obtained for the subsonic test case $(u_{in} = 0.2)$ on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = Lag$ and $\theta_{ij} = M_{ij}$



Flow speed animation

flow in a channel with bump : transonic flow

We plot the results obtained for the transonic test case ($u_{in} = 12$) on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = n$ and $\theta_{ij} = 0$



Flow speed animation

Publications

- C. Chalons, M. Girardin and S. Kokh, Large time step and asymptotic preserving numerical schemes for the gas dynamics equations with source terms, SIAM J. Sci. Comput., 35(6) (2013)
- C. Chalons, M. Girardin and S. Kokh, Operator-splitting-based asymptotic preserving scheme for the gas dynamics equations with stiff source terms, AIMS on Applied Mathematics, Proceedings of the 2012 International Conference on Hyperbolic Problems : Theory, Numerics, Applications, 8 (2014)
- C. Chalons, M. Girardin and S. Kokh, An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes, to appear in CICP (2016)
- C. Chalons, M. Girardin and S. Kokh, An all-regime Lagrange-Projection like scheme for 2D homogeneous models for two-phase flows on unstructured meshes, submitted to JCP
- M. Girardin, Méthodes numériques tout-régime et préservant l'asymptotique de type Lagrange-Projection. Application aux écoulements diphasiques en régime bas Mach, Thèse de l'Université Paris 6 (2014)

Numerical strategies

Several approaches can be envisaged to compute accurate solutions when $\epsilon << 1$

- Use and discretize the limit model (the nature of which changes)
- Couple the original and limit models at moving interfaces
- Design Asymptotic-Preserving schemes (consistency with the limit model when $\epsilon \to 0$ and with the original model when $\epsilon \to 0$, no coupling)
- Consider all-regime stability and consistency properties (ϵ is kept constant in order to compute accurate solutions also in intermediate regimes)